Tutorial Letter 201/1/2015

Statistical Inference III

STA3702

Semester 1

Department of Statistics

SOLUTIONS TO ASSIGNMENT 01

BAR CODE

QUESTION 1 **[Total marks=40]**

(a) $MME : X^2/\sigma^2 \sim \chi_1^2 \implies E(X^2/\sigma^2) = 1 \implies E(X^2) = \sigma^2 \implies MME$ of σ^2 is $S^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$. **(4)**

$MLE : L(\sigma^2) = \prod_{i=1}^n f(x_i|\sigma^2) = (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)$ and $l(\sigma^2) = \ln L(\sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2 \implies l'(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n x_i^2 \implies$ the MLE of σ^2 is $\hat{\sigma}^2$ which solves $-\frac{n}{\hat{\sigma}^2} + \frac{1}{\hat{\sigma}^4}\sum_{i=1}^n x_i^2 = 0$ and the solution is $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$ which is also the MME of σ^2. **(12)**

(b) $\frac{nS^2}{\sigma^2} \sim \chi_n^2$ since for $i = 1, 2, ..., n$ the $X_i^2/\sigma^2 \sim \chi_1^2$ and independent. **(2)**

(c) $E(S^2) = \frac{1}{n}\sum_{i=1}^n E(X_i^2) = \frac{n}{n}\sigma^2 = \sigma^2 \implies S^2$ is an unbiased estimator of σ^2. Furthermore, from part (b)

$2n = Var\left(\frac{nS^2}{\sigma^2}\right) = \frac{n^2}{\sigma^4}Var(S^2) \implies Var(S^2) = 2n\frac{\sigma^4}{n^2} = \frac{2}{n}\sigma^4 \to 0$ as $n \to \infty$. Hence S^2 is a consistent estimator of σ^2. **(4+5)**

(d) $E(S^{*2}) = \frac{1}{n-1}\sum_{i=1}^n E(X_i^2) = \frac{n}{n-1}\sigma^2 \to \sigma^2$ as $n \to \infty$. This means the bias goes to zero as $n \to \infty$. Furthermore, from part (b)

$2n = Var\left(\frac{(n-1)S^{*2}}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4}Var(S^{*2}) \implies Var(S^{*2}) = 2n\frac{\sigma^4}{(n-1)^2} = \frac{2n}{(n-1)^2}\sigma^4 \to 0$ as $n \to \infty$. Hence S^{*2} is also consistent estimator of σ^2. **(4+5)**

(e) $MSE(S^2) = Bias^2(S^2) + Var(S^2) = 0 + \frac{2}{n}\sigma^4$ and

$MSE(S^{*2}) = Bias^2(S^{*2}) + Var(S^{*2}) = \left(\frac{n}{n-1}\sigma^2 - \sigma^2\right)^2 + \frac{2n}{(n-1)^2}\sigma^4 = \frac{2n+1}{(n-1)^2}\sigma^4 > MSE(S^2)$. Hence S^2 is a better estimator of σ^2. **(4)**

QUESTION 2 **[Total marks=17]**

(a) The MLE of σ^4 is $(S^2)^2$ by the invariance property of $MLE's$. **(2)**

(b) $l''\left(\sigma^2\right) = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^n x_i^2 = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}nS^2 \implies I_n(\sigma^2) = -E[l''\left(\sigma^2\right)] = E\left[\frac{1}{\sigma^4}\frac{nS^2}{\sigma^2} - \frac{n}{2\sigma^4}\right] = \frac{n}{2\sigma^4}$ since $E\left(\frac{nS^2}{\sigma^2}\right) = n$. **(6)**

(c) $g(\sigma^2) = E(S^2) = \sigma^2 \implies g'\left(\sigma^2\right) = 1 \implies CRLB = \frac{[g'(\sigma^2)]^2}{I_n(\sigma^2} = \frac{2}{n}\sigma^4 = Var(S^2)$. **(6)**

(d) Since $E(S^2) = \sigma^2$ and $Var(S^2) = CRLB$, this means S^2 is a $MVUE$ of σ^2 **(3)**

QUESTION 3 **[Total marks=25]**

(a) $\bar{X} = \frac{16}{6} = 2.6667.$ **(2)**

(b) $E(X) = \theta \implies MME$ of θ is $\tilde{\theta} = \bar{X} = 2.6667.$ **(4)**

(c) $L(\theta) = \prod_{i=1}^{6} p(x_i|\theta) = \frac{e^{-\theta}\theta^3}{3!} \times \frac{e^{-\theta}\theta^2}{2!} \times \frac{e^{-\theta}\theta^3}{3!} \times \frac{e^{-\theta}\theta^3}{3!} \times \frac{e^{-\theta}\theta^3}{3!} \times \frac{e^{-\theta}\theta^2}{2!} = \frac{e^{-6\theta}\theta^{16}}{96} \implies$ $l(\theta) = \ln L(\theta) = -\ln(96) + 16\ln(\theta) - 6\theta \implies l'(\theta) = \frac{16}{\theta} - 6 \implies$ the MLE of θ is $\hat{\theta}$ which solves $\frac{16}{\hat{\theta}} - 6 = 0$ and the solution is $\hat{\theta} = \frac{16}{6} = 2.6667$ **(8)**

(d) MME and MLE: $I_6(\tilde{\theta}) = I_6(\hat{\theta}) = -\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\theta=2.6667} = \left[\frac{16}{\theta^2}\right]_{\theta=2.6667} = 2.25$ since $\tilde{\theta} = \hat{\theta} = 2.6667.$ Hence $Var(\tilde{\theta}) = Var(\hat{\theta}) \approx \frac{1}{I_6(\tilde{\theta})} = \frac{1}{2.5} = 0.4444$ and $se(\tilde{\theta}) = se(\hat{\theta}) = \sqrt{0.4444} = 0.6667.$ **(8)**

(e) Neither since both are unbiased and have equal standard errors. **(3)**

QUESTION 4 **[Total marks=18]**

(a) $E(X) = \sum_{x=0}^{\infty} x\theta(1-\theta)^x = \frac{1-\theta}{\theta} \implies \theta = \frac{1}{1+E(X)}$ the MME of θ is $\tilde{\theta} = \frac{1}{1+\bar{X}}.$ **(4)**

(b) $L(\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \theta^n(1-\theta)^{\sum_{i=1}^{n} x_i} = \theta^n(1-\theta)^{n\bar{x}} \implies l(\theta) = \ln L(\theta) = n\ln(\theta) + n\bar{x}\ln(1-\theta) \implies l'(\theta) = \frac{n}{\theta} - \frac{n\bar{x}}{1-\theta} \implies$ the MLE of θ is $\hat{\theta}$ which solves $\frac{1}{\hat{\theta}} - \frac{\bar{x}}{\hat{\theta}} = 0$ and the solution is $\hat{\theta} = \frac{1}{1+\bar{x}}.$ **(6)**

(c) $l''(\theta) = -\frac{n}{\theta^2} - \frac{n\bar{x}}{(1-\theta)^2} \implies$

$I_n(\theta) = -nE[l''(\theta)] = \frac{n}{\theta^2} + \frac{n}{(1-\theta)^2}E(\bar{X}) = \frac{n}{\theta^2} + \frac{n}{(1-\theta)^2} \times \frac{1-\theta}{\theta} = \frac{n}{\theta^2(1-\theta)}.$

Now, for an unbiased estimator of θ, $g(\theta) = \theta$ and $g'(\theta) = 1$. Hence,

$$CRLB = \frac{[g'(\theta)]^2}{I_n(\theta)} = \frac{1}{I_n(\theta)} = \frac{1}{n}\theta^2(1-\theta).$$

(6)

TOTAL [100]