MAT3705 (Complex Analysis)

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Preface

This study guide is based on the prescribed book

> J.W. Brown and R.V. Churchill: Complex variables and Applications (8th edition), McGraw–Hill, New York, 2009,

and must be used in conjunction with this book. In the guide the prescribed book is called "the textbook", and the chapter numbers correspond with those in the textbook. Most of the guide consists of solutions to selected problems set in the textbook, and, since we make many references to various sections of the textbook, the guide will be of no use to you unless you have the textbook in front of you.

The study material consists of the following sections in the textbook:

Chapter 1: All sections. (Note that aside from section 11, the material in this chapter is mainly a revision of material dealt with in MAT1511 and MAT1503.)
Chapter 2: Sections 12–27. We will only study section 27 up to and including the formulation of the theorem on p. 84. (Note that in essence much of the material of sections 15, 16 and 18 is also covered in MAT2613 and MAT3711.)
Chapter 3: Study sections 29–35 only, and leave the section on inverse trigonometric and hyperbolic functions.
Chapter 4: Study all sections (37–54). (Note that sections 37–39 are largely a revision of concepts dealt with in MAT2615.)
Chapter 5: Study all sections (55–67). In section 66 the proofs of Theorems 1 and 2 may be left. (Note that aside from the fact that here we are dealing with series of complex rather than real numbers, the material in sections 55, 56 and 63 and much of the material in section 67 was dealt with in MAT2613 and hence may be viewed as revision.)
Chapter 6: Study sections 68, 69, 70, 72–77. (So leave section 71).
Chapter 7: Study sections 78–83 and sections 85–87. (So leave sections 84, 88 and 89.)

The material that is largely revision will not be greatly emphasized in this course, but should nevertheless be studied as it forms essential background for the rest of the course.

The guide consists mainly of solutions to some of the problems in the textbook. Use these solutions as follows:

- First study the relevant sections in the textbook.
- Then get the numbers of the problems solved in the guide, BUT DO NOT READ THE SOLUTIONS. Try and solve these problems on your own.
- Then compare your solutions with those in the guide.

When reading the textbook and this guide you should have paper and pencil at hand. There will be steps left out by the author which you must fill in as you read – it is not possible to write a mathematics book without leaving steps out and you should regard learning this "filling in " process as part of your mathematical training.

Finally keep in mind that no shortcut – no matter how slick – can ever serve as a substitute for hard work. In the words of Thomas Edison: "Genius is 1% inspiration and 99% perspiration".

Good luck with your studies!

LE Labuschagne

A road–map to MAT3705

This course aims at providing an introduction to the analysis of complex-valued functions defined on some subset of the complex plane. It investigates and weaves together the three strands of differentiability, Taylor and Laurent series, and integration theory. As the reader will realise the nature of complex analysis is such that it rests on a number of extremely deep and elegant principles and techniques. A lot of initial hard work is needed to establish these techniques and principles, but once in place, they prove to be an extremely powerful, almost indispensable, tool for solving a variety of problems in analysis. Initial effort at mastering and understanding these principles will be well rewarded. To fully appreciate the theory, you will need some geometric insight. If you try to learn this material "parrot fashion", you may just scrape a pass, but you will neither be able to appreciate the beauty of this theory, nor be able to exploit its power. In what follows we will attempt to give a brief resumé of the course, and the road we are going to walk in adding the powerful theory of complex analysis to our mathematical arsenal.

Chapter one of the textbook is mainly revisionary in nature and reviews the fundamentals of complex numbers in addition to introducing the conceptual framework within which we will proceed with deeper analysis in the subsequent chapters. The main function of chapter 1 is to set the scene for what follows and since much of this material is already covered in MAT1511 and MAT1503, you should not pause too long here.

Chapter two introduces and studies the idea of differentiability of complex functions, with cahpter three introducing and describing complex analogues of some common elementary functions. Though much more can be said, two overriding themes will emerge from your study of differentiability: Firstly the central role the Cauchy–Riemann equations play in identifying the points at which a given function is differentiable, and then later the close link of differentiability to power series. As you will see when investigating differentiability, the Cauchy–Riemann equations are an extremely powerful and useful tool in eliminating a large number of points where a given function cannot possibly be differentiable. However, they are not always as useful in finding points where the function actually is differentiable (see the discussion at the start of section 22 of the textbook).

Laplace's equation plays a very important role in Applied Mathematics. One very valuable serendipity of the complex theory of differentiability, is that it provides us with a large number of solutions of Laplace's equation. Specifically if a complex function is differentiable on some domain, then both its real and imaginary parts turn out to be solutions to Laplace's equation on that same domain.

With the theory of differentiability in place, we proceed to investigate integration theory for complex functions. The theory of complex integration is extremely elegant and once again closely linked to the concept of analyticity. In chapter four we initiate our investigation of complex integration by first of all considering the problem of existence of antiderivatives, and then (through the Cauchy–Goursat theorem) showing that analyticity of a function is sufficient to guarantee the existence of antiderivatives. Building on this foundation we may then derive the very powerful Cauchy Integral formulae.

Besides yielding other unexpected bonuses like Liouville's theorem, the Fundamental Theorem of Algebra, and the Maximum Modulus Principle, the Cauchy Integral formulae provide the tools we need to proceed to the next phase of our programme which is to show that any function which is analytic on a region except maybe at finitely many exceptional points, can effectively be broken up into powers of $(z - z_0)$ on that region. We develop these ideas in a mathematically precise way in the form of the theory of Taylor and Laurent series in chapter five. For example if we compare Taylor's theorem to the corollary on p. 215 of the textbook, it is clear that that a function is analytic at a point (i.e. differentiable is some neighbourhood of a point) if and only if it can be written as a power series at that point. Ultimately we will see that any function f which is differentiable in a neighbourhood of z_0, except perhaps at z_0 itself, can be written in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

around z_0. Therefore terms of the form $(z - z_0)^n$, prove to be the basic building blocks of such functions. Now let γ be a positively oriented circle in this neighbourhood centred at z_0. The easily established fact that

$$\int_\gamma (z - z_0)^n \, dz = \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 2\pi i & \text{if} \quad n = -1 \end{cases}$$

$(n \in \mathbb{Z})$ now proves to be rather more significant than anticipated. Arguing formally we have that

$$\int_\gamma f(z) \, dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z - z_0)^n \, dz = 2\pi i \, a_{-1}$$

The integral of any such function may therefore be computed by effectively breaking the function up into its component parts and integrating each component separately.

Having shown that any "nice" function may be broken up into powers of $(z - z_0)$ at each point z_0, we are finally in a position to establish the promised residue theory in chapter six. In short the result basically says that if a function f is of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

inside and on a positively oriented closed curve γ (with z_0 inside γ), then

$$\int_\gamma f(z) \, dz = 2\pi i a_{-1}.$$

For such functions the process of integration therefore consists of computing the coefficient a_{-1} (the so–called residue of f at z_0). As may be expected much of chapter six is then devoted to developing techniques for computing the coefficient a_{-1}. This simple yet elegant principle of integrating by computing residues enable us to effectively compute a wide range of diverse complex integrals.

Finally in chapter seven we turn to applications, showing how the power of complex analysis may be used to elegantly solve a number of highly non–trivial real integrals.

Contents

CHAPTER 1

Complex Numbers

All sections must be studied. The work in this chapter is mainly revisionary in the sense of revising much of the material on complex numbers dealt with in MAT1511 and MAT1503. Whereas sections 1–10 are more technical, the material in section 11 is more conceptual by nature. Some of these concepts may be new to you. Be sure you grasp their meaning as they will form the framework within which we will describe more advanced concepts and techniques in later chapters.

The set of all complex numbers is essentially a two-dimensional extension of the field of real numbers. By a *complex number* we mean a number comprising a real and imaginary part. It can be written in the form $a+ib$, where a and b are real numbers, and i is postulated to be the imaginary unit with the property $i^2 = -1$. The complex numbers clearly contain the real numbers – these may be identified with the complex numbers for which the imaginary part is zero. Extending the field of numbers from the reals to the complex numbers, will enable us to solve equations like $w^2 = -1$, that we weren't able to solve using only real numbers. Specifically with complex numbers, a solution exists to every polynomial equation of degree one or higher. However although we gain a lot in passing to complex numbers, we also lose something: in view of the fact that the field of complex numbers is two-dimensional, there is no sensible way in which to order complex numbers. In other words statements like $p \geq q$ that seem so natural when dealing with the reals, make no sense for complex numbers.

In this first chapter we revise the basic algebraic properties of complex numbers, we look at two ways of representing complex numbers (cartesian form and polar form), we look at the process of computing roots of complex numbers, and investigate the properties of the modulus and complex conjugation functions.

Solutions to selected problems

Below you will find solutions to some of the exercises in chapter 1. Be sure to attempt these yourself before you work through the solutions.

Exercise 1, §2, p. 5

Verify that
(a) $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)=-2i;$ (b) $(2,-3)(-2,1)=(-1,8);$

(c) $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1);$

Solution:
(a)

$$\begin{aligned}
&\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right)\\
=&\left(\sqrt{2}-i\right)+\left[(-i)\left(1-\sqrt{2}i\right)\right]\\
=&\left(\sqrt{2}-i\right)+\left[\left(0.1-(-1)\left(-\sqrt{2}\right)\right)+i\left((-1)1+0\left(-\sqrt{2}\right)\right)\right]\\
=&\left(\sqrt{2}-i\right)+\left(-\sqrt{2}-i\right)\\
=&\left(\sqrt{2}-\sqrt{2}\right)+i(-1-1)\\
=&-2i
\end{aligned}$$

(b)

$$\begin{aligned}
(2,-3)(-2,1) &= (2(-2)-(-3)1,\ (-3)(-2)+2.1)\\
&= (-1,8)
\end{aligned}$$

(c)

$$\begin{aligned}
(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) &= [(3,1)(3,-1)]\left(\frac{1}{5},\frac{1}{10}\right)\\
&= (3.3-1(-1),1.3+3(-1))\left(\frac{1}{5},\frac{1}{10}\right)\\
&= (10,0)\left(\frac{1}{5},\frac{1}{10}\right)\\
&= \left(10.\frac{1}{5}-0.\frac{1}{10},0.\frac{1}{5}+10.\frac{1}{10}\right)\\
&= (2,1)
\end{aligned}$$

Exercise 4, §2, p. 5
Verify that each of the two numbers $z=1\pm i$ satisfies the equation

$$z^2-2z+2=0.$$

Solution:
$z=1+i$ yields

$$\begin{aligned}
(1+i)^2-2(1+i)+2 &= ((1-1)+i(1+1))-2(1+i)+2\\
&= 2i-(2+2i)+2\\
&= 0.
\end{aligned}$$

Similarly if $z = 1 - i$ then

$$(1-i)^2 - 2(1-i) + 2 = -2i - (2-2i) + 2 = 0.$$

Exercise 5, §2, p. 5

Prove that multiplication is commutative, as stated at the beginning of Sec. 2.

Solution:

Let $z_1 = x_1 + iy$ and $z_2 = x_2 + iy_2$ where $x_1,\ x_2,\ y_1$ and y_2 are real. Then using the fact that multiplication of real numbers is commutative, we can show that

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \\ &= (x_2x_1 - y_2y_1) + i(x_2y_1 + y_2x_1) \\ &= (x_2 + iy_2)(x_1 + iy_1) \\ &= z_2 z_1. \end{aligned}$$

Exercise 7, §2, p. 5

Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

Solution:

$$\begin{aligned} z(z_1 + z_2 + z_3) &= z((z_1 + z_2) + z_3) && \text{by (2) of section 2} \\ &= z(z_1 + z_2) + zz_3 && \text{by (3) of section 2} \\ &= (zz_1 + zz_2) + zz_3 && \text{by (3) of section 2} \\ &= zz_1 + zz_2 + zz_3 && \text{by (2) of section 2} \end{aligned}$$

Exercise 9, §2, p. 5

(a) Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

Solution:

(a) Let (u, v) be any complex number such that $(x, y) + (u, v) = (x, y)$ for all $(x, y) \in \mathbb{C}$. Then surely

$$\begin{aligned} (0,0) &= (0,0) + (u,v) && \text{by assumption} \\ &= (u,v) && \text{by (4) of section 2.} \end{aligned}$$

(b) Let (u, v) be given such that $(x, y)(u, v) = (x, y)$ for all $(x, y) \in \mathbb{C}$. Then

$$\begin{aligned} (1,0) &= (1;0)(u,v) && \text{by assumption} \\ &= (u,v)(1,0) && \text{by (1) of section 2} \\ &= (u,v) && \text{by (4) of section 2.} \end{aligned}$$

Exercise 1, §3, p. 8

Verify that

(a) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = -\dfrac{2}{5};$ (b) $\dfrac{5}{(1-i)(2-i)(3-i)} = \dfrac{i}{2};$

(c) $(1-i)^4 = -4.$

Solution:

(a)

$$\begin{aligned}\frac{1+2i}{3-4i}+\frac{2-i}{5i} &= \frac{(1+2i)\,5i+(3-4i)\,(2-i)}{(3-4i)\,5i}\\ &= \frac{(-10+5i)+(2-11i)}{20+15i}\\ &= \frac{(-8-6i)}{(20+15i)}\\ &= \frac{(-2)\,(4+3i)}{5\,(4+3i)}\\ &= -\frac{2}{5}\end{aligned}$$

(b)

$$\begin{aligned}\frac{5}{(1-i)\,(2-i)\,(3-i)} &= \frac{5}{[(2-1)+i\,(-2-1)]\,(3-i)}\\ &= \frac{5}{(1-3i)\,(3-i)}\\ &= \frac{5}{((3-3)+i\,(-9-1))}\\ &= -\frac{5}{10i}\times\frac{10i}{10i}\\ &= \frac{50i}{100}\\ &= \frac{i}{2}\end{aligned}$$

(c)

$$\begin{aligned}(1-i)^4 &= [(1-i)\,(1-i)]^2\\ &= [(1.1-(-1)\,.\,(-1))+i\,((-1)\,1+1\,(-1))]^2\\ &= [-2i]^2\\ &= (0.0-(-2)\,(-2))+i\,(0.\,(-2)+(-2)\,0)\\ &= -4\end{aligned}$$

Exercise 4, §3, p. 8

Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1z_2)\,z_3=0$ and use a similar result (Sec. 3) involving two factors.

Solution:

Suppose $z_1z_2z_3=0$. By the discussion at the start of section 3, it is clear that if $(z_1z_2)\,z_3=0$ then one of z_1z_2 and z_3 is zero. If therefore $z_3\neq 0$ we must then have $z_1z_2=0$ in which case one of z_1 and z_2 must then be zero. Therefore if $z_1z_2z_3=0$, then at least one of $z_1,\ z_2$ and z_3 is zero.

Exercise 2, §4, p. 12

Verify inequalities (3), Sec. 4, involving $\Re z,\ \Im z$, and $|z|$.

Solution:
Let $z = \Re(z) + i\Im(z)$. Now since $\Im(z)^2 \geq 0$ we have that

$$\Re(z)^2 \leq \Re(z)^2 + \Im(z)^2 = |z|^2 .$$

Taking square roots now yields

$$|\Re(z)| = \sqrt{\Re(z)^2} \leq |z| .$$

Since by definition

$$|\Re(z)| = \begin{cases} -\Re(z) & \text{if} \quad \Re(z) < 0 \\ \Re(z) & \text{if} \quad \Re(z) \geq 0 \end{cases}$$

we also have

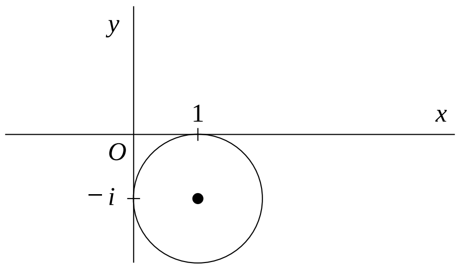
$$\Re(z) \leq |\Re(z)| .$$

In a similar fashion we can show that

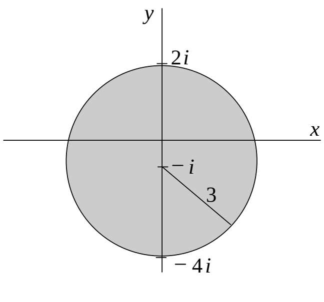
$$\Im(z) \leq |\Im(z)| \leq |z| .$$

Exercise 4 (a) & (b), §4, p. 12
In each case sketch the set of points determined by the given condition:
(a) $|z - 1 + i| = 1$; (b) $|z + i| \leq 3$;

Solution:
(a) Here $1 = |z - 1 + i| = |z - (1 - i)|$. Thus this denotes the circle with centre $(1 - i)$ and radius 1.



(b) $3 \geq |z + i| = |z - (-i)|$ This is the locus of all points within a distance of no more than 3 units from $(-i)$.



Exercise 5, §4, p. 12
Using the fact that $|z_1 - z_2|$ is the distance between two points z_1 and z_2, give a geometric argument that
(a) the equation $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;
(b) the equation $|z - 1| = |z + i|$ represents the line through the origin whose slope is -1.

Solution:

(a) $|z-4i|+|z+4i|=10$ is the set of points $z=(x,y)$ for which the sum of the distance from $4i=(0,4)$ and from $-4i=(0,-4)$ is precisely 10 (a constant). Since 10 is larger than the distance between $(0,4)$ and $(0,-4)$, (8 units) this yields an ellipse with foci at $(0,4)$ and $(0,-4)$.

(b) $|z-1|=|z+i|$ is the set of all points equi–distant from $1=(1,0)$ and $-i=(0,-1)$. The locus of points is therefore a straight line. Note in particular that both $(0,0)$ and $\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{1}{2}\left((1,0)+(0,-1)\right)$ are equidistant from $(1,0)$ and $(0,-1)$ and hence on this line. The slope of the line is therefore $\frac{\left(0-\left(-\frac{1}{2}\right)\right)}{\left(0-\frac{1}{2}\right)}=-1$.

Exercise 2, §5, p. 14

In each case sketch the set of points determined by the given condition:

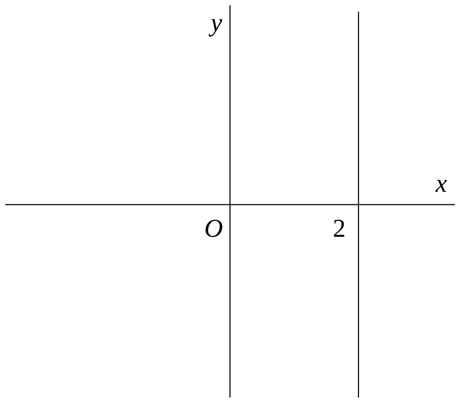
(a) $\Re\left(\overline{z}-i\right)=2$; (b) $|2\overline{z}+i|=4$

Solution:

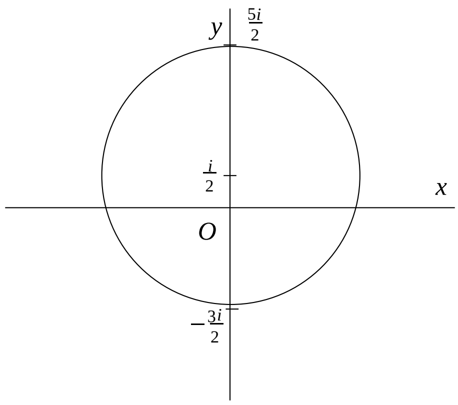
(a) With $z=x+iy\,(x,y\in\mathbb{R})$ we get

$$\begin{aligned}\Re\left(\overline{z}-i\right)&=\Re\left(x-i\left(y+1\right)\right)\\&=x\\&=\Re z.\end{aligned}$$

Thus $\Re\left(\overline{z}-i\right)=2$ is the straight line $\Re\left(z\right)=2$.



(b) Since $\overline{2z-i}=2\overline{z}+i$, we have that $|2z-i|=\left|\overline{2z-i}\right|=|2\overline{z}+i|$. So $|2\overline{z}+i|=4$, is the same as $|2z-i|=4$. But this holds if and only if $|z-(i/2)|=2$. So this is the circle of radius 2 centred at $i/2$.



Exercise 3, §5, p. 15
Verify properties (3) and (4) of $\overline{z}$ in Sec. 5.

Solution:
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $x_1,\ x_2,\ y_1$ and y_2 are real. Then $\overline{z}_1 = x_1 - iy_1$ and $\overline{z}_2 = x_2 - iy_2$. Therefore using the elementary properties of addition and multiplication we see that

$$\begin{aligned}
\overline{z_1 - z_2} &= \overline{((x_1 + iy_1) - (x_2 + iy_2))} \\
&= \overline{(x_1 - x_2) + i(y_1 - y_2)} \\
&= (x_1 - x_2) - i(y_1 - y_2) \\
&= (x_1 - iy_1) - (x_2 - iy_2) \\
&= \overline{z}_1 - \overline{z}_2
\end{aligned}$$

and that

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} \\
&= \overline{(x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)} \\
&= (x_1x_2 - y_1y_2) - i(y_1x_2 + x_1y_2) \\
&= (x_1x_2 - (-y_1)(-y_2)) + i((-y_1)x_2 + x_1(-y_2)) \\
&= (x_1 - iy_1)(x_2 - iy_2) \\
&= \overline{z}_1\overline{z_2}.
\end{aligned}$$

Exercise 5, §5, p. 15
Verify property (9) of moduli in Sec. 5.

Solution:
Note that

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right|^2 &= \left(\tfrac{z_1}{z_2}\right)\overline{\left(\tfrac{z_1}{z_2}\right)} && \text{by (7) of section 5} \\
&= \left(\tfrac{z_1}{z_2}\right)\left(\tfrac{\overline{z}_1}{\overline{z}_2}\right) && \text{by (5) of section 5} \\
&= \tfrac{z_1\overline{z}_1}{z_2\overline{z}_2} \\
&= \left(\tfrac{|z_1|}{|z_2|}\right)^2 && \text{by (7) of section 5.}
\end{aligned}$$

Now take square roots to see that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Exercise 9, §5, p. 15
By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and then using inequality (8), Sec. 4, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}.$$

Solution:
First note that

$$\begin{aligned}
\left|z^4 - 4z^2 + 3\right| &= \left|\left(z^2 - 3\right)\left(z^2 - 1\right)\right| \\
&= \left|z^2 - 3\right|\left|z^2 - 1\right| \\
&\geq \left||z|^2 - 3\right|\left||z|^2 - 1\right|.
\end{aligned}$$

Thus if $|z| = 2$ then $\left|z^4 - 4z^2 + 3\right| \geq (4-3)(4-1) = 3$, whence

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^4 - 4z^2 + 3|} \leq \frac{1}{3}.$$

Exercise 11, §5, p. 15

Use mathematical induction to show that when $n = 2,\ 3, \ldots$.

(a) $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_n$;

(b) $\overline{z_1 z_2 \cdots z_n} = \overline{z}_1 \overline{z}_2 \cdots \overline{z}_n$.

Solution:

(a) By (2) in section 5, $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ for all $z_1,\ z_2 \in \mathbb{C}$. Now suppose that for some fixed $k \geq 2$ we have that

$$\overline{z_1 + z_2 + z_3 + \cdots + z_k} = \overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_k$$

for all $z_1,\ z_2, \ldots,\ z_k \in \mathbb{C}$. Then given any $z_1,\ z_2, \ldots, z_k, z_{k+1}$, it follows that

$$\begin{aligned} & \overline{z_1 + z_2 + z_3 + \cdots + z_k + z_{k+1}} \\ = \ & \overline{(z_1 + z_2 + \cdots + z_k) + z_{k+1}} \\ = \ & \overline{z_1 + z_2 + \cdots + z_k} + \overline{z}_{k+1} \quad \text{by (2) of section 5} \\ = \ & \overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_k + \overline{z}_{k+1} \quad \text{(by the induction hypothesis).} \end{aligned}$$

Thus by induction

$$\overline{z_1 + z_2 + \cdots + z_n} = \overline{z}_1 + \overline{z}_2 + \cdots + \overline{z}_n \quad \text{for all } n \geq 2.$$

(b) This follows by a similar argument using (4) of section 5 instead of (2).

Exercise 14, §5, p. 15

Using expressions (6), Sec. 5, for $\Re z$ and $\Im z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

Solution:

Suppose $z = x + iy$ where $x,\ y \in \mathbb{R}$. Then by (6) of section 5

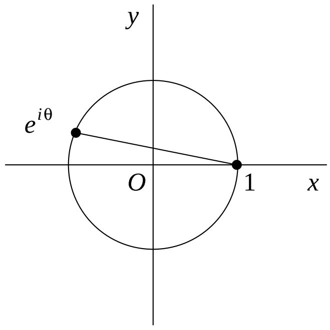
$$\begin{aligned} x^2 - y^2 = 1 \quad & \Leftrightarrow \quad \Re(z)^2 - \Im(z)^2 = 1 \\ & \Leftrightarrow \quad \left(\frac{z + \overline{z}}{2}\right)^2 - \left(\frac{z - \overline{z}}{2i}\right)^2 = 1 \\ & \Leftrightarrow \quad \frac{(z + \overline{z})^2}{4} + \frac{(z - \overline{z})^2}{4} = 1 \\ & \Leftrightarrow \quad \left(z^2 + 2z\overline{z} + \overline{z}^2\right) + \left(z^2 - 2z\overline{z} + \overline{z}^2\right) = 4 \\ & \Leftrightarrow \quad z^2 + \overline{z}^2 = 2 \end{aligned}$$

Exercise 4, §8, p. 22

Solve the equation $\left|e^{i\theta} - 1\right| = 2$ for $\theta \, (0 \leq \theta < 2\pi)$ and verify the solution geometrically.

Solution:

Rewrite $\left|e^{i\theta} - 1\right| = 2$ as $|(\cos\theta - 1) + i\sin\theta|^2 = 4$, which reduces to $\cos\theta = -1$. Since θ is to be in the interval $0 \leq \theta < 2\pi$, it follows that $\theta = \pi$. This solution of the equation $\left|e^{i\theta} - 1\right| = 2$ is geometrically evident if we recall that $e^{i\theta}$ lies on the circle $|z| = 1$ and that $\left|e^{i\theta} - 1\right|$ is the distance between the points $e^{i\theta}$ and 1. See the figure below.



Exercise 5, §8, p. 23

By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(a) $i\left(1-\sqrt{3}i\right)\left(\sqrt{3}+i\right) = 2\left(1+\sqrt{3}i\right)$; (b) $5i/(2+i) = 1+2i$;

(c) $(-1+i)^7 = -8(1+i)$; (d) $\left(1+\sqrt{3}i\right)^{-10} = 2^{-11}\left(-1+\sqrt{3}i\right)$.

Solution:

(a) Clearly $i = e^{i\pi/2}$.

For $1-\sqrt{3}i$ we have $r = \sqrt{1+3} = 2$. Thus we need θ such that $1-\sqrt{3}i = 2(\cos\theta + i\sin\theta)$, i.e. $\cos\theta = \frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$. We may let $\theta = -\frac{\pi}{3}$. Then

$$\begin{aligned} 1-\sqrt{3}i &= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ &= 2e^{-i\pi/3}. \end{aligned}$$

For $\sqrt{3}+i$, $r = \sqrt{3+1} = 2$. Select φ such that $\sqrt{3}+i = 2(\cos\varphi + i\sin\varphi)$, i.e. $\cos\varphi = \frac{\sqrt{3}}{2}$, $\sin\varphi = \frac{1}{2}$. We may let $\varphi = \frac{\pi}{6}$ whence $\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$. Thus

$$\begin{aligned} i\left(1-\sqrt{3}i\right)\left(\sqrt{3}+i\right) &= e^{\frac{i\pi}{2}}.2e^{-i\frac{\pi}{3}}.2e^{i\frac{\pi}{6}} \\ &= 4e^{i\left(\frac{\pi}{3}\right)} \\ &= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 2\left(1+\sqrt{3}i\right). \end{aligned}$$

(b) Clearly $5i = 5e^{i\frac{\pi}{2}}$. Now for $2+i$, $r = \sqrt{2^2+1} = \sqrt{5}$. Thus if we select $\theta \in \left[0, \frac{\pi}{2}\right]$ so that $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$, then we will have

$$2+i = \sqrt{5}\cos\theta + i\sqrt{5}\sin\theta = \sqrt{5}e^{i\theta}.$$

(Since here $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{1}{2}$, $\theta = \arctan\left(\frac{1}{2}\right)$ will do the trick). Then

$$\begin{aligned}
\frac{5i}{2+i} &= \frac{5e^{i\frac{\pi}{2}}}{\sqrt{5}e^{i\theta}} \qquad (\text{where } \theta = \arctan\frac{1}{2}) \\
&= \sqrt{5}e^{i(\pi/2-\theta)} \\
&= \sqrt{5}\left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right) \\
&= \sqrt{5}\left(\sin\theta + i\cos\theta\right) \qquad \text{by trigonometric identities} \\
&= \sqrt{5}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i\right) \qquad \text{by the way } \theta \text{ was chosen} \\
&= 1+2i.
\end{aligned}$$

(c) For $-1+i$ we have $|-1+i| = \sqrt{2}$. Thus to write $-1+i$ in polar form we need to find θ with $-1+i = \sqrt{2}\left(\cos\theta + i\sin\theta\right)$. Then $\cos\theta = -\frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$. Clearly $\theta = \pi - \frac{\pi}{4}$ will do the trick, i.e.

$$(-1+i) = \sqrt{2}e^{i\frac{3\pi}{4}}.$$

Then

$$\begin{aligned}
(-1+i)^7 &= \left(\sqrt{2}\right)^7\left(e^{i\frac{3\pi}{4}}\right)^7 \\
&= 2^{\frac{7}{2}}e^{i\frac{21\pi}{4}} \\
&= 2^3 2^{\frac{1}{2}}e^{i\left(5\pi+\frac{\pi}{4}\right)} \\
&= 8\sqrt{2}\left(\cos\left(5\pi+\frac{\pi}{4}\right) + i\sin\left(5\pi+\frac{\pi}{4}\right)\right) \\
&= 8\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \\
&= -8\left(1+i\right).
\end{aligned}$$

(d) We need to select θ so that

$$\begin{aligned}
1+\sqrt{3}i &= \left|1+\sqrt{3}i\right|\left(\cos\theta + i\sin\theta\right) \\
&= 2\cos\theta + i2\sin\theta
\end{aligned}$$

(i.e. $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$). We may therefore set $\theta = \frac{\pi}{3}$ to get $1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$. Then

$$\begin{aligned}
\left(1+\sqrt{3}i\right)^{-10} &= 2^{-10}\left(e^{i\frac{\pi}{3}}\right)^{-10} \\
&= 2^{-10}\left(e^{i\left(-10\frac{\pi}{3}\right)}\right) \\
&= 2^{-10}e^{i\left(-3\pi-\frac{\pi}{3}\right)} \\
&= 2^{-10}\left(\cos\left(-3\pi-\frac{\pi}{3}\right) + i\sin\left(-3\pi-\frac{\pi}{3}\right)\right) \\
&= 2^{-10}\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) \\
&= 2^{-11}\left(-1+\sqrt{3}i\right).
\end{aligned}$$

Exercise 7, §8, p. 23

Let z be a nonzero complex number and n a negative integer ($n = -1, -2, \ldots$). Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions $z^m = r^m e^{im\theta}$ and $z^{-1} = (1/r)\,e^{i(-\theta)}$, verify that $\left(z^m\right)^{-1} = \left(z^{-1}\right)^m$ and hence that the definition $z^n = \left(z^{-1}\right)^m$ in Sec. 7 could have been written alternatively as $z^n = \left(z^m\right)^{-1}$.

Solution:

Here $z = re^{i\theta}$ is any nonzero complex number and n a negative integer ($n = -1$, $-2, \ldots$). Also, $m = -n = 1, 2, \ldots$. By writing

$$\left(z^m\right)^{-1} = \left(r^m e^{im\theta}\right)^{-1} = \frac{1}{r^m} e^{i(-m\theta)}$$

and

$$\left(z^{-1}\right)^m = \left[\frac{1}{r} e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{i(-m\theta)} = \frac{1}{r^m} e^{i(-m\theta)},$$

we see that $\left(z^m\right)^{-1} = \left(z^{-1}\right)^m$. Thus the definition $z^n = \left(z^{-1}\right)^m$ can also be written as $z^n = \left(z^m\right)^{-1}$.

Exercise 8, §8, p. 23

Prove that two nonzero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c}_2$.

Suggestion: Note that

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = \exp\left(i\theta_1\right)$$

and [see Exercise 2(b)]

$$\exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \overline{\exp\left(i\frac{\theta_1 - \theta_2}{2}\right)} = \exp\left(i\theta_2\right).$$

Solution:

First of all, given two nonzero complex numbers z_1 and z_2, suppose that there are complex numbers c_1 and c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1 \overline{c}_2$. Since

$$|z_1| = |c_1|\,|c_2| \quad \text{and} \quad |z_2| = |c_1|\,|\overline{c_2}| = |c_1|\,|c_2|,$$

it follows that $|z_1| = |z_2|$.

Suppose, on the other hand, that we know only that $|z_1| = |z_2|$. We may write

$$z_1 = r_1 \exp\left(i\theta_1\right) \quad \text{and} \quad z_2 = r_1 \exp\left(i\theta_2\right).$$

If we introduce the numbers

$$c_1 = r_1 \exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \quad \text{and} \quad c_2 = \exp\left(i\frac{\theta_1 - \theta_2}{2}\right),$$

we find that

$$c_1 c_2 = r_1 \exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \exp\left(i\frac{\theta_1 - \theta_2}{2}\right) = r_1 \exp\left(i\theta_1\right) = z_1$$

and

$$c_1 \overline{c}_2 = r_1 \exp\left(i\frac{\theta_1 + \theta_2}{2}\right) \exp\left(-i\frac{\theta_1 - \theta_2}{2}\right) = r_1 \exp(i\theta_2) = z_2.$$

That is,

$$z_1 = c_1 c_2 \quad \text{and} \quad z_2 = c_1 \overline{c}_2.$$

Exercise 10, §8, p. 23

Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$ (b) $\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$.

Solution:
We know from de Moivre's formula that

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta,$$

or

$$\cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

That is,

$$\left(\cos^3\theta - 3\cos\theta\sin^2\theta\right) + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right) = \cos 3\theta + i\sin 3\theta.$$

By equating real parts and then imaginary parts here, we arrive at the desired trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$

Exercise 11, §8, p. 24

(a) Use the binomial formula (Sec. 3) and de Moivre's formula (Sec. 7) to write

$$\cos n\theta + i\sin n\theta = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n = 1, 2, \ldots).$$

Then define the integer m by means of the equations

$$m = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd,} \end{cases}$$

and use the above sum to obtain the expression [compare Exercise 10(a)]

$$\cos n\theta = \sum_{k=0}^{m}\binom{n}{2k}(-1)^k\cos^{n-2k}\theta\sin^{2k}\theta \qquad (n = 1, 2, \ldots).$$

(b) Write $x = \cos\theta$ and suppose that $0 \le \theta \le \pi$, in which case $-1 \le x \le 1$. Point out how it follows from the final result in part (a) that each of the functions

$$T_n(x) = \cos\left(n\cos^{-1}x\right) \qquad (n = 0, 1, 2, \ldots)$$

is a polynomial of degree n in the variable x.

Solution:

(a) By the Binomial theorem

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k \qquad (n \in \mathbb{N}).$$

Combining this with de Moivre's formula it follows that

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k.$$

Since $(i)^{4m} = 1,\ (i)^{4m+1} = i,\ (i)^{4m+2} = -1$ and $(i)^{4m+3} = -i$ for each $m \in \mathbb{Z}$, it follows that taking the real part of the above sum is the same as summing over only even values of k. Hence if m denotes the largest integer less than or equal to $n/2$, we get that

$$\begin{aligned}
\cos n\theta &= \Re\left(\cos n\theta + i\sin n\theta\right)\\
&= \Re\left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\left(i\sin\theta\right)^{k}\right)\\
&= \sum_{p=0}^{m}\binom{n}{2p}\cos^{n-2p}\theta . i^{2p}\sin^{2p}\theta\\
&= \sum_{p=0}^{m}\binom{n}{2p}(-1)^{p}\cos^{n-2p}\theta\sin^{2p}\theta .
\end{aligned}$$

(Here we have replaced even values of k by $2p$.)

(b) Let $0 \le \theta \le \pi$ and let $x = \cos\theta$. Then $\sin\theta \ge 0$, and so $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$. Hence

$$\begin{aligned}
T_n(x) &= \cos(n\arccos x)\\
&= \cos(n\theta)\\
&= \sum_{p=0}^{m}\binom{n}{2p}(-1)^{p}\cos^{n-2p}\theta\sin^{2p}\theta\\
&\quad (m \text{ as in part (a)})\\
&= \sum_{p=0}^{m}\binom{n}{2p}(-1)^{p}x^{n-2p}\left(\sqrt{1-x^2}\right)^{2p}\\
&= \sum_{p=0}^{m}\binom{n}{2p}(-1)^{p}x^{n-2p}\left(1-x^2\right)^{p}.
\end{aligned}$$

Now for any $1 \le p \le m$

$$\begin{aligned}
x^{n-2p}\left(1-x^2\right)^{p} &= x^{n-2p}\sum_{r=0}^{p}\binom{p}{r}(-1)^{r}x^{2r}\\
&= \sum_{r=0}^{p}\binom{p}{r}(-1)^{r}x^{n-2(p-r)}
\end{aligned}$$

is a polynomial of degree n. (To see that this really is of degree n and not something smaller, note that for this polynomial the nth term is the term corresponding to $p = r$; that is $(-1)^p x^n$.)

If we substitute the above formula into the expression for $T_n(x)$, it follows that T_n is a polynomial of degree n with nth term

$$\sum_{p=0}^{m}\binom{n}{2p}(-1)^{p}\left[(-1)^{p}x^{n}\right] = \sum_{p=0}^{m}\binom{n}{2p}x^{n}.$$

Exercise 1, §10, p. 29

Find the square roots of (a) $2i$; (b) $1-\sqrt{3}i$, and express them in rectangular coordinates.

Solution:

(a) Since $2i = 2\exp\left[i\left(\frac{\pi}{2} + 2k\pi\right)\right]$ $(k = 0, \pm1, \pm2, \ldots)$, the desired roots are

$$(2i)^{\frac{1}{2}} = \sqrt{2}\exp\left[i\left(\frac{\pi}{4} + k\pi\right)\right] \qquad (k = 0, 1)$$

That is,

$$c_0 = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1 + i$$

and

$$c_1 = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)e^{i\pi} = -c_0 = -\left(1 + i\right),$$

c_0 being the principal root.

(b) Observe that $1 - \sqrt{3}i = 2\exp\left[i\left(-\frac{\pi}{3} + 2k\pi\right)\right]$ $(k = 0, \pm1, \pm2, \ldots)$. Hence

$$\left(1 - \sqrt{3}i\right)^{\frac{1}{2}} = \sqrt{2}\exp\left[i\left(-\frac{\pi}{6} + k\pi\right)\right]. \qquad (k = 0, 1)$$

The principal root is

$$c_0 = \sqrt{2}e^{-i\frac{\pi}{6}} = \sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{\sqrt{3} - i}{\sqrt{2}},$$

and the other root is

$$c_1 = \left(\sqrt{2}e^{-i\frac{\pi}{6}}\right)e^{i\pi} = -c_0 = -\frac{\sqrt{3} - i}{\sqrt{2}}.$$

Exercise 5, §10, p. 30

(a) Let a denote any fixed real number, and show that the two square roots of $a + i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right),$$

where $A = \sqrt{a^2 + 1}$ and $\alpha = \text{Arg}(a + i)$.

(b) With the aid of the trigonometric identities

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}, \quad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2},$$

show that the square roots obtained in part (a) can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A + a} + i\sqrt{A - a}\right).$$

Solution:

(a) Let a denote any fixed real number. In order to find the two square roots of $a + i$ in exponential form, we write

$$A = |a + i| = \sqrt{a^2 + 1} \quad \text{and} \quad \alpha = \text{Arg}\,(a + i).$$

Since

$$a + i = A\exp\left[i\left(\alpha + 2k\pi\right)\right] \qquad (k = 0, \pm1, \pm2, \ldots)$$

we see that

$$(a + i)^{\frac{1}{2}} = \sqrt{A}\exp\left[i\left(\frac{\alpha}{2} + k\pi\right)\right]. \qquad (k = 0, 1)$$

That is, the desired square roots are

$$\sqrt{A}e^{i\frac{\alpha}{2}} \quad \text{and} \quad \sqrt{A}e^{i\frac{\alpha}{2}}e^{i\pi} = -\sqrt{A}e^{i\alpha/2}.$$

(b) Since $a+i$ lies above the real axis, we know that $0<\alpha<\pi$. Thus $0<\frac{\alpha}{2}<\frac{\pi}{2}$, and this tells us that $\cos\left(\frac{\alpha}{2}\right)>0$. Since $\cos\alpha=\frac{a}{A}$, it follows that

$$\cos\frac{\alpha}{2}=\sqrt{\frac{1+\cos\alpha}{2}}=\frac{1}{\sqrt{2}}\sqrt{1+\frac{a}{A}}=\frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}}$$

and

$$\sin\frac{\alpha}{2}=\sqrt{\frac{1-\cos\alpha}{2}}=\frac{1}{\sqrt{2}}\sqrt{1-\frac{a}{A}}=\frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}}.$$

Consequently,

$$\begin{aligned}\pm\sqrt{A}e^{i\frac{\alpha}{2}} &= \pm\sqrt{A}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right)=\pm\sqrt{A}\left(\frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}}+i\frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}}\right)\\ &= \pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a}+i\sqrt{A-a}\right).\end{aligned}$$

Exercise 6, §10, p. 30

Find the four roots of the equation $z^4+4=0$ and use them to factor z^4+4 into quadratic factors with real coefficients.

Solution

The four roots of the equation $z^4+4=0$ are the four fourth roots of the number -4. To find those roots, we write $-4=4\exp\left[i\left(\pi+2k\pi\right)\right]\quad(k=0\pm1,\pm2,\ldots)$. Then

$$(k=0,1,2,3)\qquad (-4)^{\frac{1}{4}}=\sqrt{2}\exp\left[i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)\right]=\sqrt{2}e^{i\frac{\pi}{4}}e^{ik\frac{\pi}{2}}$$

To be specific,

$$c_0=\sqrt{2}e^{i\frac{\pi}{4}}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=1+i,$$

$$\begin{aligned}c_1 &= c_0e^{i\frac{\pi}{2}}=(1+i)\,i=-1+i,\\ c_2 &= c_0e^{i\pi}=(1+i)\,(-1)=-1-i,\\ c_3 &= c_0e^{i\frac{3\pi}{2}}=(1+i)\,(-i)=1-i.\end{aligned}$$

This enables us to write

$$\begin{aligned}z^4+4 &= (z-c_0)\,(z-c_1)\,(z-c_2)\,(z-c_3)\\ &= \left[(z-c_1)\,(z-c_2)\right].\left[(z-c_0)\,(z-c_3)\right]\\ &= \left[(z+1)-i\right]\left[(z+1)+i\right].\left[(z-1)-i\right]\left[(z-1)+i\right]\\ &= \left[(z+1)^2+1\right].\left[(z-1)^2+1\right]\\ &= \left(z^2+2z+2\right)\left(z^2-2z+2\right).\end{aligned}$$

Exercise 7, §10, p. 309

Show that if c is any nth root of unity other than itself, then

$$1+c+c^2+\cdots+c^{n+1}=0.$$

Solution:

Let c be any nth root of unity other than itself. With the aid of the identity (Exercise 9, Sec. 8),

$$(z\neq1)\qquad 1+z+z^2+\cdots+z^{n-1}=\frac{1-z^n}{1-z}$$

we find that

$$1 + c + c^2 + \cdots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1 - 1}{1 - c} = 0.$$

Exercise 1–3, §11, p. 33

Sketch the following sets and determine which are domains:

(a) $|z - 2 + i| \le 1$; (b) $|2z + 3| > 4$;

(c) $\Im z > 1$; (d) $\Im z = 1$;

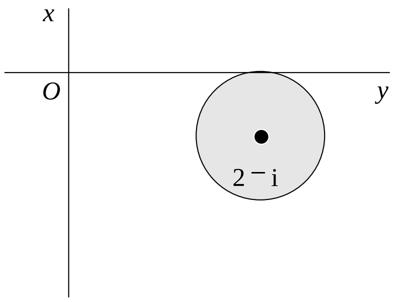
(e) $0 \le \operatorname{Arg} z \le \frac{\pi}{4}\,(z \ne 0)$; (f) $|z - 4| \ge |z|$.

Which sets in Exercise 1 are neither open nor closed?

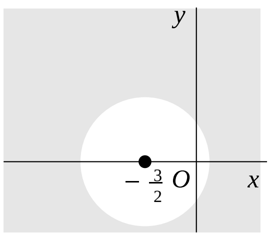
Which sets in Exercise 1 are bounded?

Solution:

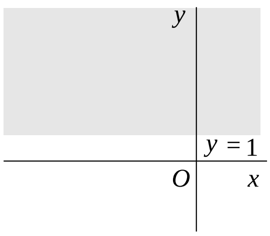
(a) Write $|z - 2 + i| \le 1$ as $|z - (2 - i)| \le 1$ to see that this is the set of points inside and on the circle centred at the point $2 - i$ with radius 1. It is closed and bounded. It is not a domain as it is not open.



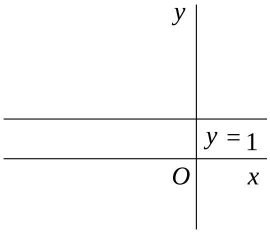
(b) Write $|2z + 3| > 4$ as $\left|z - \left(-\frac{3}{2}\right)\right| > 2$ to see that the set in question consists of all points exterior to the circle with center at $-\frac{3}{2}$ and radius 2. It is open and connected and hence a domain. It is not bounded.



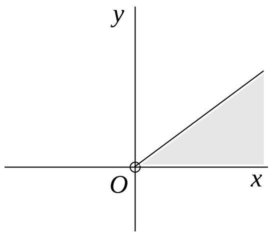
(c) Write $\Im z > 1$ as $y > 1$ to see that this is the half plane consisting of all points lying above the horizontal line $y = 1$. It is open and connected and hence a domain. It is not bounded.



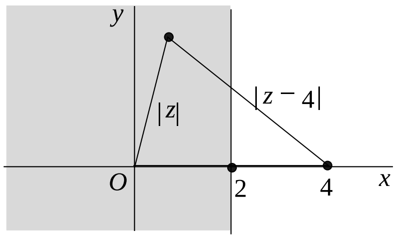
(d) The set $\Im z = 1$ is simply the horizontal line $y = 1$. It is closed, not open and hence not a domain. It is not bounded.



(e) The set $0 \leq \arg z \leq \frac{\pi}{4}\,(z \neq 0)$ is indicated below. It is not open and hence not a domain. It is also not closed since the boundary point 0 does not belong to the set (see (1) of section 6). It is not bounded.



(f) The set $|z-4| \geq |z|$ can be written in the form $(x-4)^2 + y^2 \geq x^2 + y^2$, which reduces to $x \leq 2$. The set is also geometrically evident since it consists of all points z such that the distance between z and 4 is greater than or equal to the distance between z and the origin. This set is closed but not open and so not a domain. It is not bounded.

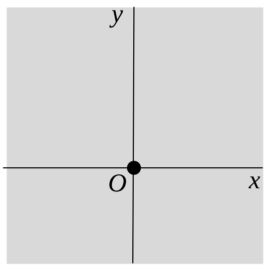


Exercise 4, §11, p. 33
In each case, sketch the closure of the set:

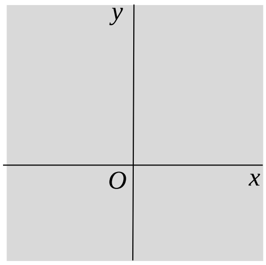
$$(a)\ -\pi < \arg z < \pi\,(z \neq 0)\,; \quad (b)\ |\Re z| < |z|\,; \quad (c)\ \Re\left(\frac{1}{z}\right) \leq \frac{1}{2}; \quad (d)\ \Re\left(z^2\right) > 0.$$

Solution:

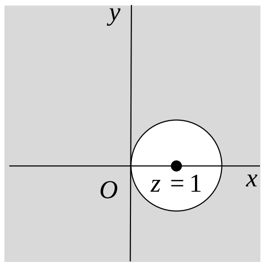
(a) The closure of the set $-\pi < \arg z < \pi\,(z \neq 0)$ is the entire plane.



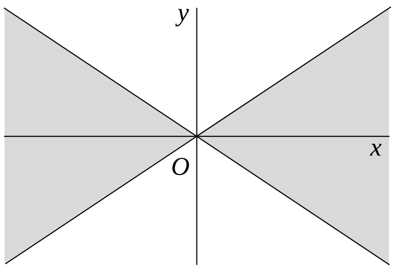
(b) We first write the set $|\Re z| < |z|$ as $|x| < \sqrt{x^2+y^2}$, or $x^2 < x^2 + y^2$. But this last inequality is the same as $y^2 > 0$, or $|y| > 0$. Hence the closure of the set $|\Re z| < |z|$ is the entire plane.



(c) Since $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$, the set $\Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$ can be written as $\frac{x}{x^2+y^2} \leq \frac{1}{2}$, or $\left(x^2 - 2x\right) + y^2 \geq 0$. Finally, by completing the square, we arrive at the inequality $(x-1)^2 + y^2 \geq 1^2$, which describes the circle, together with its exterior, that is centered at $z = 1$ with radius 1. The closure of this set is itself.



(d) Since $z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$, the set $\Re\left(z^2\right) > 0$ can be written as $y^2 < x^2$, or $|y| < |x|$. The closure of this set consists of the lines $y = \pm x$ together with the shaded region shown below.

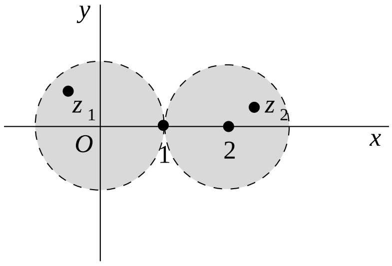


Exercise 5, §11, p. 33

Let S be the open set consisting of all points z such that $|z| < 1$ or $|z-2| < 1$. State why S is not connected.

Solution:

The set S consists of all points z such that $|z| < 1$ or $|z-2| < 1$, as shown below.



Since the point $z = 1$ is not in S, every polygonal line joining z_1 and z_2 must contain at least one point that is not in S. Thus it is clear that S is not connected.

CHAPTER 2

Analytic Functions

Study sections 12–27. (Section 26 should only be studied up to and including the formulation of the theorem on p. 84.) The thrust of sections 15, 16 and 18 is covered in more detail in MAT2613 and MAT3711. For this reason we will use but not greatly emphasize the material of these sections. However since this material is essential background for what follows, the student should still take care to acquaint himself with the ideas in these sections.

The proofs of the theorems in sections 26 need not be studied, although the student should nevertheless be familiar with both their formulation and application.

In the following discussion we will assume that $z = x + iy$ where $x,\ y \in \mathbb{R}$. Any complex function $f : \mathbb{C} \to \mathbb{C}$ may be decomposed into a real and imaginary part $f(z) = u(x, y) + iv(x, y)$ where u and v are real valued functions on $\mathbb{C}$. In fact since $\mathbb{C}$ is basically just a copy of $\mathbb{R}^2$, we may think of u and v as real–valued functions of two independent real variables. This simple little device of writing f as a combination of two such 2–variable real functions allows one to apply the results and techniques of real analysis to complex analysis. The essential idea behind this approach is to try and describe the nature and behaviour of the complex function f in terms of the behaviour of the real functions u and v. A classic example of where this was done with great success is provided by the Cauchy–Riemann equations (see sections 21 and 22). The theory surrounding these equations form much of the heart of the theory of differentiability for complex functions.

On a different note this device of describing f in terms of u and v also allows one to think of a complex function as a transformation of coordinates from the $z = x + iy = (x, y)$ coordinate system to the $f(z) = u(x, y) + iv(x, y) = (u, v)$ coordinate system. So yet another important way in which we can gain some understanding of the behaviour of a specific complex function f is to study its behaviour as a transformation of coordinates and to see how it transforms regions in the (x, y)–plane onto regions in the (u, v)–plane. We will not spend too much time on this aspect of transformation of regions by complex functions, but it is nevertheless an aspect of the theory that the student should be aware.

Solutions to selected problems

Solutions to some of the exercises in sections 12, 14, 18, 20, 23, 25 and 26 follow.

Exercise 1, §12, p. 37

For each of the functions below, describe the domain of definition that is understood:

(a) $f(z) = \dfrac{1}{z^2+1}$; (b) $f(z) = \text{Arg}\left(\frac{1}{z}\right)$;

(c) $f(z) = \dfrac{z}{z+\overline{z}}$; (d) $f(z) = \dfrac{1}{1-|z|^2}$.

Solution:

(a) $f(z) = \dfrac{1}{z^2+1}$ is defined for all z except where $z^2 = -1$, i.e. except where $z = \pm i$.

(b) $z \to \dfrac{1}{z}$ is not defined where $z = 0$. In addition $\text{Arg}(w)$ exists for every $w \neq 0$. Hence the composition $z \to \text{Arg}\left(\dfrac{1}{z}\right)$ is defined whenever $z \neq 0$.

(c) $f(z) = \dfrac{z}{z+\overline{z}}$ is not defined where $z + \overline{z} = 0$. But $z + \overline{z} = 2\Re(z)$. Hence f is here defined for all z with $\Re(z) \neq 0$.

(d) $f(z) = \dfrac{1}{1-|z|^2}$ is defined whenever $1 - |z|^2 \neq 0$, i.e. whenever $|z| \neq 1$.

Exercise 3, §12, p. 37

Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the fact (Sec. 5) that

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i}$$

to express $f(z)$ in terms of z, and simplify the result.

Solution:

With $z = x + iy$ we have $x = \dfrac{z+\overline{z}}{2}$ and $y = \dfrac{z-\overline{z}}{2i}$. Thus $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ becomes

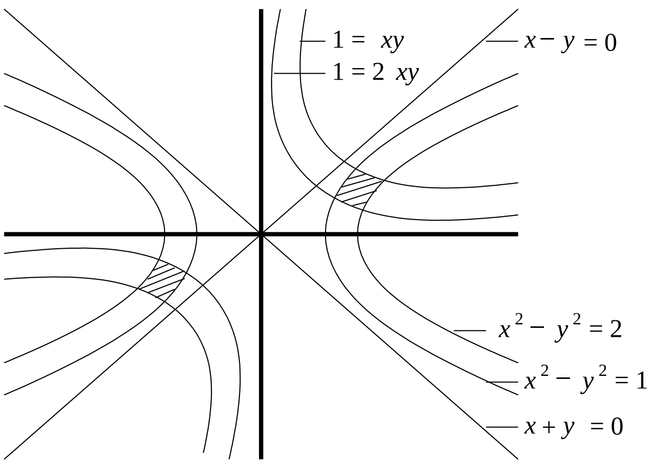
$$\begin{aligned} f(z) &= \left[\frac{1}{2}(z+\overline{z})\right]^2 - \left[\frac{1}{2i}(z-\overline{z})\right]^2 - 2\left[\frac{1}{2i}(z-\overline{z})\right] \\ &\quad + i\left([z+\overline{z}] - [z+\overline{z}]\left[\frac{1}{2i}(z-\overline{z})\right]\right) \\ &= \frac{1}{4}\left(z^2 + 2z\overline{z} + \overline{z}^2\right) + \frac{1}{4}\left(z^2 - 2z\overline{z} + \overline{z}^2\right) \\ &\quad - \frac{1}{i}(z-\overline{z}) + i(z+\overline{z}) - \frac{1}{2}\left(z^2 - \overline{z}^2\right) \\ &= \overline{z}^2 + i(z-\overline{z}) + i(z+\overline{z}) \\ &= \overline{z}^2 + 2iz. \end{aligned}$$

Exercise 1, §14, p. 44

By referring to Example 1, Sec. 13, find a domain in the z plane whose image under the transformation $w = z^2$ is the square domain in the w plane bounded by the lines $u = 1$, $u = 2$, $v = 1$ and $v = 2$. (See Fig. 2, Appendix 2.)

Solution:

By Example 1, Sec. 13, the mapping $w = z^2$ will map the hyperbolae $1 = 2xy$ and $2 = 2xy$ onto the lines $v = 1$ and $v = 2$. In Example 2, Sec. 11, we saw that $u = \Re(w) = x^2 - y^2$. Thus $w = z^2$ will also map hyperbolae of the form $c = x^2 - y^2$ onto lines of the form $c = u$. In particular it will map the region bounded by $1 = x^2 - y^2$, $2 = x^2 - y^2$ and $1 = 2xy$, $1 = xy$ onto the square bounded by $u = 1$, $u = 2$, $v = 1$, $v = 2$. However the region bounded by $1 = x^2 - y^2$, $2 = x^2 - y^2$ and $1 = 2xy$, $1 = xy$ consists of two separate domains each of which maps onto the square bounded by $u = 1$, $u = 2$ and $v = 1$, $v = 2$ (see the sketch). To see that each maps **onto** the whole square observe that we can still cover all possible values of $v = 2xy$ and $u = x^2 - y^2$ in the square if we insist that either $x, y > 0$ or $x, y < 0$.



Exercise 3, §14, p. 44

Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ is mapped by the transformation (a) $w = z^2$; (b) $w = z^3$; (c) $w = z^4$.

Solution:

Let $z = re^{i\theta}\,(0 \leq r,\ 0 \leq \theta)$. Since $z^n = r^n e^{in\theta}$ and since $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{4}$ if and only if $0 \leq r^n \leq 1$ and $0 \leq n\theta \leq n\frac{\pi}{4}$, it is clear that $z \to z^n$ maps the sector $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ into the sector $0 \leq \rho \leq 1$, $0 \leq \varphi \leq n\frac{\pi}{4}$. To see that the map is onto note that given any $w_0 = \rho_0 e^{i\varphi_0}$ with $0 \leq \rho_0 \leq 1$, $0 \leq \varphi_0 \leq n\frac{\pi}{4}$, we may set $r_0 = (\rho_0)^{\frac{1}{n}}$ and $\theta_0 = \frac{\varphi_0}{n}$. Then surely $0 \leq r_0 \leq 1$, $0 \leq \theta_0 \leq \frac{\pi}{4}$ and $z_0^n = w_0$, where $z_0 = r_0 e^{i\theta_0}$.

From the above it is clear that

(a) $z \to z^2$ maps the sector $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ onto the quarter–disc $0 \leq \rho \leq 1$, $0 \leq \varphi \leq \frac{\pi}{2}$;

(b) $z \to z^3$ maps the sector $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ onto the sector $0 \leq \rho \leq 1$, $0 \leq \varphi \leq \frac{3\pi}{4}$;

(c) $z \to z^4$ maps the sector $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ onto the region $0 \leq \rho \leq 1$, $0 \leq \varphi \leq \pi$, i.e. onto the upper half of the disc $|w| \leq 1$ where $w = \rho e^{i\varphi}$.

Exercise 5, §18, p. 55

Show that the limit of the function $f(z) = \left(\frac{z}{\overline{z}}\right)^2$ as z tends to 0 does not exist. Do this by letting nonzero points $z = (x, 0)$ and $z = (x, x)$ approach the origin. [Note that it is not sufficient to simply consider points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec. 15.]

Solution:

Let $z = x + iy$. Then

$$\left(\frac{z}{\overline{z}}\right)^2 = \left(\frac{x+iy}{x-iy}\right)^2 = \frac{x^2-y^2+i2xy}{x^2-y^2-i2xy}.$$

Now if $z = (x, 0) = x + i0$, then

$$\left(\frac{z}{\overline{z}}\right)^2 = \frac{x^2}{x^2} = 1.$$

If on the other hand $z = (x, x) = x + ix\,(y = x)$, then

$$\left(\frac{z}{\overline{z}}\right)^2 = -\frac{i2x^2}{i2x^2} = -1.$$

Thus along the line $z = (x, 0)$, $\left(\frac{z}{\overline{z}}\right)^2$ tends to 1 as $z \to 0$, whereas along the line $z = (x, x)$, $\left(\frac{z}{\overline{z}}\right)^2$ tends to -1 as $z \to 0$. Clearly then $\lim\limits_{z\to 0}\left(\frac{z}{\overline{z}}\right)^2$ can not exist.

Exercise 10, §18, p. 56

Use the theorem in Sec. 17 to show that

(a) $\lim\limits_{z\to\infty}\frac{4z^2}{(z-1)^2} = 4$; (b) $\lim\limits_{z\to 1}\frac{1}{(z-1)^3} = \infty$; (c) $\lim\limits_{z\to\infty}\frac{z^2+1}{z-1} = \infty$.

Solution:

(a)

$$\begin{aligned}\lim_{z\to\infty}\frac{4z^2}{(z-1)^2} &= \lim_{z\to 0}\frac{4\left(\frac{1}{z}\right)^2}{\left(\left(\frac{1}{z}\right)-1\right)^2} \quad \text{(by (2) of the theorem)}\\ &= \lim_{z\to 0}\frac{4}{(1-z)^2}\\ &= 4\end{aligned}$$

(b) Now $\lim\limits_{z\to 1}(z-1)^3 = 0$ and hence (by (1) of the theorem) $\lim\limits_{z\to 1}\frac{1}{(z-1)^3} = \infty$.

(c) Let $f(z) = \frac{z^2+1}{z-1}$. Then $f\left(\frac{1}{z}\right) = \frac{\left(\frac{1}{z}\right)^2+1}{\left(\frac{1}{z}\right)-1} = \frac{1+z^2}{z-z^2}$. Since

$$\begin{aligned}\lim_{z\to 0}\frac{1}{f\left(\frac{1}{z}\right)} &= \lim_{x\to 0}\frac{1}{(1+z^2)/(z-z^2)}\\ &= \lim_{z\to 0}\frac{z-z^2}{1+z^2}\\ &= 0\end{aligned}$$

it follows from (3) of the theorem that

$$\lim_{z\to\infty}\frac{z^2+1}{z-1} = \infty.$$

Exercise 11, §18, p. 56
With the aid of the theorem in Sec. 17 show that when

$$T(z) = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0),$$

(a) $\lim\limits_{z\to\infty} T(z) = \infty$ if $c = 0$;

(b) $\lim\limits_{z\to\infty} T(z) = \dfrac{a}{c}$ and $\lim\limits_{z\to -\frac{d}{c}} T(z) = \infty$ if $c \neq 0$.

Solution:

(a) Since here $c = 0$, we have $T(z) = \dfrac{1}{d}(az+b)$. In this case

$$\begin{aligned}\lim_{z\to 0} \frac{1}{T\left(\frac{1}{z}\right)} &= \lim_{z\to 0} \frac{1}{\frac{1}{d}\left(\frac{a}{z}+b\right)} \\ &= \lim_{z\to 0} \frac{dz}{a+bdz}.\end{aligned}$$

Note that $a \neq 0$ since by assumption $ad - bc \neq 0$ and $c = 0$. Thus

$$\lim_{z\to 0} \frac{1}{T\left(\frac{1}{z}\right)} = \lim_{z\to 0} \frac{dz}{a+bdz} = 0.$$

From (3) of the theorem we then have that $\lim\limits_{z\to\infty} T(z) = \infty$.

(b) Now let $c \neq 0$. Then

$$\begin{aligned}\lim_{z\to\infty} T(z) &= \lim_{z\to 0} T\left(\frac{1}{z}\right) \\ &= \lim_{z\to 0} \frac{a\left(\frac{1}{z}+b\right)}{c\left(\frac{1}{z}+d\right)} \\ &= \lim_{z\to 0} \frac{a+bz}{c+dz} \\ &= \frac{a}{c}.\end{aligned}$$

Moreover since

$$\begin{aligned}\lim_{z\to -\frac{d}{c}} \frac{1}{T(z)} &= \lim_{z\to -\frac{d}{c}} \frac{cz+d}{az+b} \\ &= \frac{c\left(-\frac{d}{c}\right)+d}{a\left(-\frac{d}{c}\right)+b} \quad \left(a\left(-\frac{d}{c}\right)+b \neq 0 \text{ since } ad-bc \neq 0\right) \\ &= 0,\end{aligned}$$

it follows that

$$\lim_{z\to -\frac{d}{c}} T(z) = \infty.$$

Exercise 13, §18, p. 56
Show that a set S is unbounded (Sec. 11) if and only if every neighbourhood of the point at infinity contains at least one point in S.

Solution:
A set S is unbounded

$\Leftrightarrow$ for every $R > 0$ we can find $w \in S$ so that w lies outside the circle $|z| = R$

$\Leftrightarrow$ for every $R > 0$ we can find $w \in S$ with $|w| > R$

$\Leftrightarrow$ for every $\varepsilon > 0$ we can find $w \in S$ with $|w| > \frac{1}{\varepsilon}$ (set $R = \frac{1}{\varepsilon}$)

$\Leftrightarrow$ every neighbourhood of ∞ contains at least one point of S.

Exercise 1, §20, p. 62
Use results in Sec. 20 to find $f'(z)$ when
(a) $f(z) = 3z^2 - 2z + 4$; (b) $f(z) = \left(1 - 4z^2\right)^3$;

(c) $f(z) = \dfrac{z-1}{2z+1}\left(z \neq -\dfrac{1}{2}\right)$; (d) $f(z) = \dfrac{\left(1+z^2\right)^4}{z^2}\ (z \neq 0)$.

Solutions:
(a) If $f(z) = 3z^2 - 2z + 4$ then $f'(z) = 6z - 2$.
(b) If $f(z) = \left(1 - 4z^2\right)^3$ then $f'(z) = 3\left(1 - 4z^2\right)^2 . (-8z) = -24z\left(1 - 4z^2\right)^2$.
(c) If $f(z) = \frac{z-1}{2z+1}\left(z \neq -\frac{1}{2}\right)$,then by the quotient rule

$$\begin{aligned} f'(z) &= \frac{(2z+1).1 - 2.(z-1)}{(2z+1)^2} \\ &= \frac{3}{(2z+1)^2} \quad \left(z \neq -\frac{1}{2}\right). \end{aligned}$$

(d) If $f(z) = \dfrac{\left(1+z^2\right)^4}{z^2}$ then by the quotient and chain rules

$$\begin{aligned} f'(z) &= \frac{z^2.\left[4\left(1+z^2\right)^3 .2z\right] - [2z].\left(1+z^2\right)^4}{z^4} \\ &= \frac{2z\left(1+z^2\right)^3\left(4z^2 - \left(1+z^2\right)\right)}{z^4} \\ &= \frac{2\left(1+z^2\right)^3\left(3z^2-1\right)}{z^3} \quad (z \neq 0). \end{aligned}$$

Exercise 8, §20, p. 63
Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when
(a) $f(z) = \overline{z}$; (b) $f(z) = \Re z$; (c) $f(z) = \Im z$.

Solution:
(a) Let $f(z) = \overline{z}$. Then with $w = f(z)$

$$\frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Now as $\Delta z = \Delta x + i\Delta y$ approaches 0 along the line $(\Delta x, 0)$, $\dfrac{\Delta w}{\Delta z} = \dfrac{\Delta x}{\Delta x} = 1$ approaches 1. On the other hand along the line $(0, \Delta y)$, $\dfrac{\Delta w}{\Delta z} = -\dfrac{i\Delta y}{i\Delta y}$ approaches -1. Thus for any z

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

fails to exist.
(b) Let $f(z) = \Re(z)$. Then with $w = f(z)$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta\left(\Re(z)\right)}{\Delta z} = \frac{\Delta x}{\Delta x + i\Delta y}.$$

Along the line $(\Delta x, 0)$ this tends to 1 as $\Delta z \to 0$ whereas along the line $(\Delta x, \Delta x)$ (i.e. $\Delta x = \Delta y$) this tends to

$$\frac{\Delta x}{\Delta x + i\Delta x} = \frac{1}{1+i} = \frac{1}{2} - i\left(\frac{1}{2}\right) \quad \text{as} \quad \Delta z \to 0.$$

Thus as before for any z

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

fails to exist.

(c) If $f(z) = \Im(z)$ then with $w = f(z)$

$$\frac{\Delta w}{\Delta z} = \frac{\Delta(\Im(z))}{\Delta z} = \frac{\Delta y}{\Delta x + i\Delta y}.$$

Along the line $(0, \Delta y)$ this tends to $\dfrac{1}{i} = -i$ as $\Delta z \to 0$, whereas along the line $\Delta x = \Delta y$ this tends to $\dfrac{\Delta x}{\Delta x + i\Delta x} = \dfrac{1}{1+i} = \dfrac{1}{2} - i\dfrac{1}{2}$ as $\Delta z \to 0$. As before this ensures that for any z

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

fails to exist.

Exercise 9, §20, p. 63

Let f denote the function whose values are

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or $(\Delta x, \Delta y)$, plane. Then show that $\dfrac{\Delta w}{\Delta z} = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in that plane. Conclude from these observations that $f'(0)$ does not exist. (Note that, to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane.)

Solution:

For $z = 0$ we have

$$\begin{aligned} \frac{f(0+\Delta z) - f(0)}{\Delta z} &= \frac{f(\Delta z) - 0}{\Delta z} \\ &= \frac{1}{\Delta z}\left(\frac{\overline{\Delta z}^2}{\Delta z}\right) \\ &= \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2 \\ &= \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right)^2 \\ &= \frac{\left(\Delta x^2 - \Delta y^2\right) - i2\Delta x\Delta y}{\left(\Delta x^2 - \Delta y^2\right) + i2\Delta x\Delta y}. \end{aligned}$$

On the real and imaginary axes (i.e. either $\Delta y = 0$ or $\Delta x = 0$) we get

$$\frac{f(0+\Delta z) - f(0)}{\Delta z} = 1.$$

(If $\Delta y = 0$, then $\dfrac{f(\Delta z) - f(0)}{\Delta z} = \dfrac{\Delta x^2}{\Delta x^2} = 1$ whereas if $\Delta x = 0$, then $\dfrac{f(\Delta z) - f(0)}{\Delta z} = \dfrac{-\Delta y^2}{-\Delta y^2} = 1$.) However on the line $\Delta x = \Delta y$ we get

$$\frac{f(\Delta z) - f(0)}{\Delta z} = \frac{-i2\Delta x^2}{i2\Delta x^2} = -1.$$

Thus there is no unique value we can ascribe to $\dfrac{f(\Delta z) - f(0)}{\Delta z}$ as $\Delta z \to 0$. Hence

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

fails to exist.

Exercise 1, §23, p. 71

Use the theorem in Sec. 21 to show that $f'(z)$ does not exist at any point if
(a) $f(z) = \overline{z}$; (b) $f(z) = z - \overline{z}$; (c) $f(z) = 2x + ixy^2$; (d) $e^x e^{-iy}$.

Solution:

Let $z = x + iy$.

(a) For $f(z) = \overline{z} = x - iy$ we have that $u(x, y) = x$ and $v(x, y) = -y$. Thus for all $z = (x, y)$, $u_x = 1$ and $v_y = -1$. Clearly the Cauchy–Riemann equations then fail to hold for every z. (We always have $u_x \neq v_y$.) Therefore f is nowhere differentiable.

(b) For $f(z) = z - \overline{z} = 2i\Im(z) = 2iy$ we have $u(x, y) = 0$ and $v(x, y) = 2y$. Therefore $u_x = 0$ and $v_y = 2$. As before the Cauchy–Riemann equations never hold since we always have $u_x \neq v_y$. Thus $f'(z)$ never exists.

(c) For $f(z) = 2x + ixy^2$ we have $u = 2x$ and $v = xy^2$. Therefore

$$u_x = 2, \quad u_y = 0, \quad v_x = y^2 \text{ and } v_y = 2xy.$$

The equation $u_y = -v_x$ will therefore hold precisely when $0 = -y^2$, i.e. when $y = 0$. However if $y = 0$ then

$$2 = u_x \neq v_y = 2xy = 0.$$

Therefore the equations $u_y = -v_x$ and $u_x = v_y$ can never hold simultaneously. Hence $f'(z)$ does not exist at any z.

(d) For $f(z) = e^x e^{-iy} = e^x \cos y - ie^x \sin y$ we have $u = e^x \cos y$, $v = -e^x \sin y$. Hence $u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = -e^x \sin y$ and $v_y = -e^x \cos y$. Recall that $e^x \neq 0$ for all x. Thus $e^x \cos y = u_x = v_y = -e^x \cos y$ will hold precisely when $\cos y = 0$, i.e. when $y = (2k+1)\frac{\pi}{2}$ $(k \in \mathbb{Z})$. On the other hand $-e^x \sin y = u_y = -v_x = e^x \sin y$ will hold precisely when $\sin y = 0$, i.e. when $y = k\pi$ $(k \in \mathbb{Z})$. Thus as before the equations $u_x = v_y$ and $u_y = -v_x$ can never hold simultaneously. Therefore $f'(z)$ fails to exist for each z.

Exercise 3, §23, p. 71

From results obtained in Secs. 21 and 22, determine where $f'(z)$ exists and find its value when
(a) $f(z) = \dfrac{1}{z}$; (b) $f(z) = x^2 + iy^2$; (c) $f(z) = z\Im z$.

Solution:

(a) Given $f(z) = \frac{1}{z}$ $(z \neq 0)$ we write

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x - iy}{x^2 + y^2}.$$

Thus here $u = \frac{x}{x^2 + y^2}$ and $v = -\frac{y}{x^2 + y^2}$. Now whenever $(x, y) \neq (0, 0)$ we have

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y$$

and

$$u_y = -\frac{2xy}{(x^2 + y^2)^2} = -v_x.$$

Moreover u_x, v_y, u_y, v_x as given above are defined and continuous on all of $\mathbb{C} - \{0\}$. Therefore by the theorem in Sec. 22 of the textbook $f'(z)$ exists whenever $z \neq 0$. At $z = 0$ $f(z)$ is not defined and hence $f'(0)$ does not exist. Using differentiation rules it is easy to see that $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$.

(b) Given $f(z) = x^2 + iy^2$ we have $u = x^2$ and $v = y^2$. Thus $u_x = 2x$, $u_y = 0$, $v_x = 0$, $v_y = 2y$. Clearly we always have $u_y = -v_x$ with $u_x = v_y$ holding precisely when $x = y$. In addition u_x, u_y, v_x and v_y are continuous on all of $\mathbb{C}$ and hence f is differentiable at all points $z = x + iy$ with $x = y$ and not differentiable at all other points. At a point $z = x + ix$ the derivative is

$$f'(x + ix) = u_x + iv_x = 2x.$$

(c) Given $f(z) = z\Im(z)$ we have $f(z) = xy + iy^2$ and hence $u = xy$ and $v = y^2$. Therefore $u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 2y$. Now $y = u_x = v_y = 2y$ will hold precisely when $y = 0$ with $x = u_y = -v_x = 0$ holding precisely when $x = 0$. Thus if $z \neq 0$ $f'(z)$ does not exist since then at least one of $u_x = v_y$ and $u_y = -v_x$ must fail. On the other hand if $z = 0$, we do have $u_x = v_y$ and $u_y = -v_x$. Since in addition we also have that u_x, u_y, v_x, v_y are continuous in a neighbourhood of $z = 0$ ($(x, y) = (0, 0)$), $f'(0)$ exists with

$$f'(0) = u_x(0, 0) + iv_x(0, 0) = 0.$$

Exercise 5, §23, p. 72

Show that when $f(z) = x^3 + i(1 - y)^3$, it is legitimate to write

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = i$.

Solution:

For $f(z) = x^3 + i(1 - y)^3$ we have $u = x^3$ and $v = (1 - y)^3$. Therefore $u_x = 3x^2$, $u_y = 0$, $v_x = 0$ and $v_y = -3(1 - y)^2$. Clearly $u_y = -v_x$ is always true with $u_x = v_y$ holding precisely $x^2 = -(1 - y)^2$, i.e. when $x^2 + (1 - y)^2 = 0$. Since $x^2 \geq 0$ and $(1 - y)^2 \geq 0$, $x^2 + (1 - y)^2 = 0$ can only be true if $x^2 = 0$ and $(1 - y)^2 = 0$, i.e. if $x = 0$ and $y = 1$. At all other points the Cauchy–Riemann equations fail to hold and hence $f'(z)$ fails to exist. Now the point $(x, y) = (0, 1)$ of course corresponds to $z = i$ and since u_x, u_y, v_x, v_y as given above satisfy the

Cauchy–Riemann equations at this point and are continuous in any neighbourhood of $z = i$, $f'(z)$ **does** exist at $z = i$ with

$$f'(z) = u_x(0,1) + iv_x(0,1) = 0.$$

Exercise 6, §23, p. 72

Let u and v denote the real and imaginary components of the function f defined by the equations

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & \text{when} \quad z \neq 0, \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at the origin $z = (0,0)$. [Compare Exercise 9, Sec. 20, where it is shown that $f'(0)$ nevertheless fails to exist.]

Solution:

Let $z = x + iy$. Then if $z \neq 0$,

$$\frac{\overline{z}^2}{z} = \frac{(x-iy)^2}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x\left(x^2-3y^2\right) + iy\left(y^2-3x^2\right)}{x^2+y^2}.$$

Therefore for

$$f(z) = \begin{cases} \dfrac{\overline{z}^2}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$$

we have

$$u(x,y) = \begin{cases} \dfrac{x\left(x^2-3y^2\right)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

and

$$v(x,y) = \begin{cases} \dfrac{y\left(y^2-3x^2\right)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

From first principles

$$\begin{aligned} \frac{\partial u}{\partial x}(0,0) &= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(\Delta x)^3 / (\Delta x)^2 - 0}{\Delta x} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y}(0,0) &= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} \\ &= 0. \end{aligned}$$

Similarly

$$\frac{\partial v}{\partial x}(0,0) = 0 \quad \text{and} \quad \frac{\partial v}{\partial y}(0,0) = 1.$$

Clearly $\dfrac{\partial u}{\partial x}(0,0) = \dfrac{\partial v}{\partial y}(0,0)$ and $\dfrac{\partial u}{\partial y}(0,0) = -\dfrac{\partial v}{\partial x}(0,0)$. However in spite of this fact we know from exercise 9, Section 20 of the textbook that $f'(0)$ does NOT exist.

This example shows that the Cauchy–Riemann equations are on their own not enough to guarantee differentiability of a function at a given point. We must also have that the derivatives u_x, u_y, v_x, v_y exist and are continuous in a neighbourhood of that point. In this present example the partial derivatives u_x, u_y, v_x and v_y are not continuous at $(0,0)$. For example by means of differentiation rules and what we've already shown it follows that

$$u_x(x,y) = \begin{cases} \dfrac{x^4 - 3y^4 + 6x^2y^2}{(x^2+y^2)^2} & \text{if} \quad (x,y) \neq (0,0) \\ 1 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

On the line $x = 0$ we have $u_x(0,y) = -3$ if $y \neq 0$. Thus as (x,y) tends to $(0,0)$ along this line we have $u_x \to -3$. Since $u_x(0,0) = 1 \neq -3$, the function u_x can not be continuous at $(0,0)$.

Exercise 7, §23, p. 72

Solve equations (2), Sec. 23, for u_x and u_y to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \qquad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Then use these equations and similar ones for v_x and v_y to show that, in Sec. 23, equations (4) are satisfied at a point z_0 if equations (6) are satisfied there. Thus complete the verification that equations (6), Sec. 23, are the Cauchy–Riemann equations in polar form.

Solution

From (2) in Sec. 23 of the textbook we know that

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad \text{(a)}$$

and

$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta. \qquad \text{(b)}$$

Therefore

$$\begin{aligned} u_r \cos\theta - u_\theta \frac{\sin\theta}{r} &= (u_x \cos\theta + u_y \sin\theta)\cos\theta \\ &\quad - (-u_x r \sin\theta + u_y r \cos\theta)\frac{\sin\theta}{r} \\ &= u_x\left(\cos^2\theta + \sin^2\theta\right) \\ &= u_x \end{aligned}$$

and

$$\begin{aligned} u_r \sin\theta + u_\theta \frac{\cos\theta}{r} &= (u_x \cos\theta + u_y \sin\theta)\sin\theta \\ &\quad + (-u_x r \sin\theta + y_y r \cos\theta)\frac{\cos\theta}{r} \\ &= u_y\left(\sin^2\theta + \cos^2\theta\right) \\ &= u_y. \end{aligned}$$

Similarly

$$v_r \cos\theta - v_\theta \frac{\sin\theta}{r} = v_x \quad \text{and} \quad v_r \sin\theta + v_\theta \frac{\cos\theta}{r} = v_y.$$

Therefore if

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad \frac{1}{r} u_\theta = -v_r$$

then

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \\ &= \left(\frac{1}{r} v_\theta\right)\cos\theta - (-v_r)\sin\theta \\ &= v_\theta \frac{\cos\theta}{r} + v_r \sin\theta \\ &= v_y \end{aligned}$$

and

$$\begin{aligned} u_y &= u_r \sin\theta + \left(u_\theta \frac{1}{r}\right)\cos\theta \\ &= \left(\frac{1}{r} v_\theta\right)\sin\theta - v_r \cos\theta \\ &= -\left[v_r \cos\theta - v_\theta \frac{\sin\theta}{r}\right] \\ &= -v_x. \end{aligned}$$

Since we already know from (6) of Sec. 23 of the textbook that

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad \frac{1}{r} u_\theta = -v_r$$

whenever

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

it follows that $u_r = \frac{1}{r} v_\theta$ and $\frac{1}{r} u_\theta = -v_r$ if and only if $u_x = v_y$ and $u_y = -v_x$.

Exercise 8, §23, p. 72

Suppose that a function $f(z) = u + iv$ is differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$. Use the expressions for u_x and v_x found in Exercise 7, together with the polar form (6), Sec. 23, of the Cauchy–Riemann equations, to show that $f'(z_0)$ can be written

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where u_r and v_r are evaluated at (r_0, θ_0).

Solution:

We know from section 21 of the textbook that $f'(z) = u_x + iv_x$ where–ever $f'(z)$ exists. But since

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \quad \text{and} \quad v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r}$$

(see exercise 7), it follows that

$$f'(z_0) = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} + i\left(v_r \cos\theta - v_\theta \frac{\sin\theta}{r}\right).$$

Now since $f'(z_0)$ exists, the Cauchy–Riemann equations hold. By exercise (7) above this is equivalent to

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad \frac{1}{r} u_\theta = -v_r.$$

Therefore

$$\begin{aligned} f'(z_\theta) &= u_r\cos\theta - \left(u_\theta\frac{1}{r}\right)\sin\theta + i\left(v_r\cos\theta - \left(v_\theta\frac{1}{r}\right)\sin\theta\right) \\ &= u_r\cos\theta + v_r\sin\theta + i\left(v_r\cos\theta - u_r\sin\theta\right) \\ &= \left(\cos\theta - i\sin\theta\right)\left(u_r + iv_r\right) \\ &= e^{-i\theta}\left(u_r + iv_r\right). \end{aligned}$$

Exercise 9, §23, p. 72

(a) With the aid of the polar form (6), Sec. 23, of the Cauchy–Riemann equations, derive the alternative form

$$f'(z_0) = \frac{-i}{z_0}\left(u_\theta + iv_\theta\right)$$

of the expression for $f'(z_0)$ found in Exercise 8.

(b) Use the expression for $f'(z_0)$ found in part (a) to show that the derivative of the function $f(z) = 1/z \ (z \neq 0)$ in Example 1, Sec. 23, is $f'(z) = -\dfrac{1}{z^2}$.

Solution:

(a) As in exercise (8) above it follows that

$$f'(z_0) = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} + i\left(v_r\cos\theta - v_\theta\frac{\sin\theta}{r}\right)$$

and

$$u_r = \frac{1}{r}v_\theta \quad \text{and} \quad \frac{1}{r}u_\theta = -v_r$$

where–ever $f'(z)$ exists. Hence $f'(z_0)$ may be written as

$$\begin{aligned} f'(z_0) &= \left(\frac{1}{r}v_\theta\right)\cos\theta - u_\theta\frac{\sin\theta}{r} + i\left(\left(-\frac{1}{r}u_\theta\right)\cos\theta - v_\theta\frac{\sin\theta}{r}\right) \\ &= \frac{1}{r}(-i)\left(\cos\theta - i\sin\theta\right)\left(u_\theta + iv_\theta\right) \\ &= -i\frac{1}{r}e^{-i\theta}\left(u_\theta + iv_\theta\right) \\ &= -\frac{i}{re^{i\theta}}\left(u_\theta + iv_\theta\right) \\ &= -\frac{i}{z_0}\left(u_\theta + iv_\theta\right) \\ \text{where} \quad z_\theta &= re^{i\theta}. \end{aligned}$$

(b) Let $z = re^{i\theta}$. Then

$$f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\left(\cos\theta - i\sin\theta\right).$$

Therefore $u = \frac{1}{r}\cos\theta$, $v = -\frac{1}{r}\sin\theta$. Clearly $u_\theta = -\frac{1}{r}\sin\theta$ and $v_\theta = -\frac{1}{r}\cos\theta$, and hence by 9(a)

$$\begin{aligned} f'(z) &= -\frac{i}{z}(u_\theta + iv_\theta) \\ &= -\frac{i}{z}\left(-\frac{1}{r}\sin\theta - i\frac{1}{r}\cos\theta\right) \\ &= -\frac{1}{z}\frac{1}{r}(\cos\theta - i\sin\theta) \\ &= -\frac{1}{z}\frac{1}{r}e^{-i\theta} \\ &= -\frac{1}{z}\frac{1}{re^{i\theta}} \\ &= -\frac{1}{z^2}. \end{aligned}$$

Exercise 1, §25, p. 77

Apply the theorem in Sec. 22 to verify that each of these functions is entire:

(a) $f(z) = 3x + y + i(3y - x)$; (b) $f(z) = \sin x \cosh y + i \cos x \sinh y$;

(c) $f(z) = e^{-y}\sin x - ie^{-y}\cos x$; (d) $f(z) = \left(z^2 - 2\right)e^{-x}e^{-iy}$.

Solution:

(a) For $f(z) = 3x + y + i(3y - x)$ we have $u = 3x + y$ and $v = 3y - x$. For all (x, y) we now have that $u_x = 3 = v_y$ and $u_y = 1 = -v_x$. Since in addition u_x, u_y, v_x, v_y are continuous on all of $\mathbb{C}$, f is entire (analytic on all of $\mathbb{C}$.)

(b) Given $f(z) = \sin x \cosh y + i \cos x \sinh y$ we have $u = \sin x \cosh y$ and $v = \cos x \sinh y$. For all (x, y) we now see that

$$u_x = \cos x \cosh y = v_y$$

and

$$u_y = \sin x \sinh y = -v_x.$$

In addition each of u_x, u_y, v_x, v_y are continuous on all of $\mathbb{C}$ and hence f is entire.

(c) Here $f = u + iv$ where $u = e^{-y}\sin x$ and $v = -e^{-y}\cos x$. But then

$$u_x = e^{-y}\cos x = v_y \quad \text{and} \quad u_y = -e^{-y}\sin x = -v_x$$

for each (x, y) with again each of u_x, u_y, v_x and v_y clearly continuous on all of $\mathbb{C}$. Therefore f is entire.

(d) Here

$$\begin{aligned} f(z) &= \left(z^2 - 2\right)e^{-x}e^{iy} \\ &= \left(\left(x^2 - y^2 - 2\right) + i2xy\right)e^{-x}(\cos y - i\sin y) \\ &= \left[\left(x^2 - y^2 - 2\right)e^{-x}\cos y + 2xye^{-x}\sin y\right] \\ &\quad + i\left[2xye^{-x}\cos y - \left(x^2 - y^2 - 2\right)e^{-x}\sin y\right] \end{aligned}$$

and hence

$$u = \left(x^2 - y^2 - 2\right)e^{-x}\cos y + 2xye^{-x}\sin y$$

and

$$v = 2xye^{-x}\cos y - \left(x^2 - y^2 - 2\right)e^{-x}\sin y.$$

Now

$$u_x = 2xe^{-x}\cos y - \left(x^2 - y^2 - 2\right)e^{-x}\cos y + 2ye^{-x}\sin y - 2xye^{-x}\sin y = v_y$$

and

$$u_y = -2ye^{-x}\cos y - \left(x^2 - y^2 - 2\right)e^{-x}\sin y + 2xe^{-x}\sin y + 2xye^{-x}\cos y = -v_x.$$

In addition all these first partials are continuous on all of $\mathbb{C}$, whence f is analytic on all of $\mathbb{C}$.

Exercise 4, §25, p. 77

In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:

(a) $f(z) = \dfrac{2z+1}{z(z^2+1)}$; (b) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$; (c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$.

Solution:

Each of the functions is a quotient of two polynomials. Since polynomials are analytic on all of $\mathbb{C}$, each of these functions will therefore be analytic where–ever the denominator is non–zero. At the zeros of the denominator the function is not defined and hence these points are singular points of the function.

(a) $f(z) = \dfrac{2^z+1}{z(z^2+1)}$ has singular points where $z\left(z^2+1\right) = 0$, i.e. where either $z = 0$ or $z = \pm i$.

(b) $f(z) = \dfrac{z^3+i}{z^2-3z+2}$ has singular points where $0 = z^2-3z+2 = (z-2)(z-1)$, i.e. where $z = 1, 2$.

(c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$ has singular points where either $z+2 = 0$ or $z^2+2z+2 = 0$, i.e. where either $z = -2$ or $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

Exercise 6, §25, p. 78

Use results in Sec. 23 to verify that the function

$$g(z) = \ln r + i\theta \quad (r > 0,\ 0 < \theta < 2\pi)$$

is analytic in the indicated domain of definition, with derivative $g'(z) = \dfrac{1}{z}$. Then show that the composite function $g\left(z^2+1\right)$ is an analytic function of z in the quadrant $x > 0,\ y > 0$, with derivative $2z/\left(z^2+1\right)$.

Suggestion: Observe that $\Im\left(z^2+1\right) > 0$ when $x > 0, y > 0$.

Solution:

For $g(z) = \ln r + i\theta$ $(r > 0,\ 2\pi > \theta > 0)$ where $z = re^{i\theta}$ we have that $u = \ln r$ and $v = \theta$. Clearly $u_r = \frac{1}{r}$, $v_\theta = 1$ and $u_\theta = v_r = 0$ on the region $(r > 0,\ 2\pi > \theta > 0)$. Thus

$$u_r = \frac{1}{r} = \frac{1}{r}v_\theta \quad \text{and} \quad \frac{1}{r}u_\theta = 0 = v_\theta$$

with in addition each of u_r, u_θ, v_θ, v_r continuous on the entire region. Therefore by the theorem in section 23 of the textbook $g'(z)$ exists in this region with

$$\begin{aligned} g'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\frac{1}{r} \\ &= \frac{1}{re^{i\theta}} \\ &= \frac{1}{z}. \end{aligned}$$

Now note that the polynomial z^2+1 is analytic everywhere. The composition $g(z^2+1)$ will therefore be analytic at all points z_0 for which z_0^2+1 lies in the region $(r > 0, 2\pi > \theta > 0)$. For any $z_0 = x_0 + iy_0$ we have that $z_0^2 + 1 = (x_0 + iy_0)^2 + 1 = (x_0^2 - y_0^2 + 1) + i2x_0y_0$ and hence that $\Im(z_0^2+1) = 2x_0y_0$. Thus if $x_0 > 0$, $y_0 > 0$ then $\Im(z_0^2+1) > 0$. Geometrically this means that z_0^2+1 then lies above the real axis, or rather that $|z_0^2+1| > 0$ and $0 < \mathrm{Arg}(z_0^2+1) < \pi$. From what we noted earlier $g(z_0^2+1)$ will therefore be differentiable at each such point. Thus $g(z^2+1)$ is differentiable, and hence analytic, on all of $x > 0$, $y > 0$. It now easily follows from the chain rule that $\frac{d}{dz}g(z^2+1) = [1/(z^2+1)].2z = 2z/(z^2+1)$ on this region.

Exercise 7, §25, p. 78

Let a function $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant in D if

(a) $f(z)$ is real–valued for all z in D;

(b) $|f(z)|$ is constant in D.

Suggestion: Use the Cauchy–Riemann equations and the theorem in Sec. 24 to prove part (a). To prove part (b), observe that $\overline{f(z)} = c^2/f(z)$ if $|f(z)| = c$, where $c \neq 0$; then use the main result in Example 3, Sec. 25.

Solution:

(a) Let f be analytic and real–valued on all of D. Since f is real–valued, $v = 0$, and since in addition f is analytic on D,

$$u_x = v_y = 0 \quad \text{and} \quad u_y = -v_x = 0.$$

But then

$$f'(z) = u_x + iv_x = 0$$

on all of D. Clearly f is then constant on D by the theorem in Sec. 24 of the textbook.

(b) If $|f(z)| = 0$ on all of D then surely $f(z) = 0$ on all of D in which case we are done. Hence suppose $|f(z)| = c$ where $c > 0$ on all of D. Then $f(z)$ has **no** zeros on D, and hence $c^2/f(z)$ will be analytic on D. But

$$\frac{c^2}{f(z)} = \frac{|f(z)|^2}{f(z)} = \frac{f(z)\overline{f(z)}}{f(z)} = \overline{f(z)},$$

and hence both f and $\overline{f}$ will then be analytic on D. By Example 3, Sec. 25, f must then be constant.

Exercise 2, §26, p. 81

Show that if v and V are harmonic conjugates of u in a domain D, then $v(x,y)$ and $V(x,y)$ can differ at most by an additive constant.

Solution:

If both v and V are harmonic conjugates of u in the domain D then both $u+iv$ and $u+iV$ are analytic on D. Thus it then follows from the Cauchy–Riemann equations that

$$v_x = -u_y = V_x \quad \text{and} \quad v_y = u_x = V_y.$$

Since now

$$\begin{aligned}\frac{\partial}{\partial x}(v-V) &= v_x - V_x = 0\\ \frac{\partial}{\partial y}(v-V) &= v_y - V_y = 0,\end{aligned}$$

we may argue as in the proof of the theorem in Sec. 24 of the textbook to show that $v-V$ is then constant.

Exercise 3, §26, p. 82

Show that if v is a harmonic conjugate of u in a domain D and also u is a harmonic conjugate of v, then $u(x,y)$ and $v(x,y)$ must be constant throughout D.

Solution:

If u and v are harmonic conjugates of each other then both $u+iv$ and $v+iu$ are analytic on D. Cauchy–Riemann equations must hold for both these functions whence

$$u_x = v_y, \quad u_y = -v_x$$

for $u+iv$ **and**

$$v_x = u_y, \quad v_y = -u_x$$

for $v+iu$ on all of D. It is not difficult to conclude from this that $u_x = u_y = v_x = v_y = 0$. As before by arguing as in the proof of the theorem in Sec.24 of the textbook we may then conclude from this that both u and v are constant throughout D.

Exercise 7, §26, p. 82

Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D, and consider the families of *level curves* $u(x,y) = c_1$ and $v(x,y) = c_2$, where c_1 and c_2 are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if $z_0 = (x_0, y_0)$ is a point in D which is common to two particular curves $u(x,y) = c_1$ and $v(x,y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

Suggestion: Note how it follows from the equation $u(x,y) = c_1$ and $v(x,y) = c_2$ that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

Solution:

Observe that the equation $u(x,y) = c_1$ implicitly defines a function y of x. To compute the derivative $\frac{dy}{dx}$ of this function we differentiate the equation $u(x,y) = c_1$ implicitly to get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0.$$

(We could also have used the chain rule for functions of 2 real variables to see that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{d}{dx}c_1 = 0.)$$

In any case at any (x,y) the slope of the line tangent to $u(x,y) = c_1$ is given by $m_1 = \frac{dy}{dx}$ where $\frac{\partial u}{\partial x} + m_1\frac{\partial u}{\partial y} = 0$. Similarly we can show that at any (x,y) on the curve $v(x,y)=c_2$ the slope of the line tangent to this curve at (x,y) is given by m_2 where $\dfrac{\partial v}{\partial x}+m_2\dfrac{\partial v}{\partial y}=0$.

Now let z_0 be a point on the intersection of $u(x,y)=c_1$ and $v(x,y)=c_2$ at which $f'(z_0)\neq 0$. Then

$$u_x(x_0,y_0)=v_y(x_0,y_0),\ u_y(x_0,y_0)=-v_x(x_0,y_0)$$

at z_0. Since $0\neq f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$, u_x and v_x can not both be zero at $z_0=x_0+iy_0$. Suppose $u_x(x_0,y_0)\neq 0$. Then

$$\begin{aligned}\frac{1}{m_1} &= -\frac{u_y(x_0,y_0)}{u_x(x_0,y_0)}\\ &= \frac{v_x(x_0,y_0)}{v_y(x_0,y_0)}\\ &= -m_2.\end{aligned}$$

This proves that the tangent lines to $u(x,y)=c_1$ and $v(x,y)=c_2$ at $z_0=x_0+iy_0$ are orthogonal. (To see this recall that two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ are orthogonal precisely when $m_1m_2=-1$.)

CHAPTER 3

Elementary Functions

Study only sections 29-35. Complex inverse trigonometric and hyperbolic functions will not be dealt with in this course.

Our primary objective in this chapter is to introduce complex analogs of the most important elementary functions, to describe the basic properties of these functions, and to gain some experience in working with these functions. The complex cousins of the well-known elementary real functions, are both more interesting and more tricky. For example whereas the real-valued version of sin is bounded, the same is not true of the complex version. Where for each positive real number x there is a unique real number $w = \log(x)$ for which $x = e^w$, for any given non-zero complex number z, there are infinitely many possible values we can ascribe to $\log(z)$. The complex version of the logarithm is therefore what we may call a *multi-valued* function. To produce a complex version of the logarithm which ascribes one single value to each element of its domain, we need to restrict the possible values of $\log(z)$ in a natural way, by taking a so-called *branch cut.* These and other aspects are explained in this chapter.

Solutions to selected problems

Solutions to some of the exercises in sections 29, 31–35 follow.

Exercise 1, §29, p. 92
Show that

(a) $\exp(2 \pm 3\pi i) = -e^2$; (b) $\exp\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}\,(1+i)$;

(c) $\exp(z + \pi i) = -\exp z$.

Solution
(a)

$$\begin{aligned} \exp(2 \pm 3\pi i) &= e^2 e^{\pm 3\pi i} \\ &= e^2 \left(\cos(\pm 3\pi i) + i \sin(\pm 3\pi i)\right) \\ &= e^2 \left(\cos 3\pi \pm i \sin 3\pi\right) \\ &= e^2 (-1 \pm i0) \\ &= -e^2 \end{aligned}$$

(b)

$$\begin{aligned} \exp\left(\frac{2+\pi i}{4}\right) &= \exp\left(\frac{1}{2} + i\frac{\pi}{4}\right) \\ &= e^{\frac{1}{2}}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) \\ &= \sqrt{e}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \\ &= \sqrt{\frac{e}{2}}\,(1+i) \end{aligned}$$

(c)

$$\begin{aligned} \exp(z + \pi i) &= \exp z \cdot \exp \pi i \\ &= \exp z\,(\cos \pi + i \sin \pi) \\ &= -\exp z \end{aligned}$$

Exercise 3, §29, p. 92
Prove that the function $\exp \bar{z}$ is not analytic anywhere.

Solution
Let $z = x + iy$. Then $\bar{z} = x - iy$ and

$$\begin{aligned} \exp(\bar{z}) &= e^x e^{-iy} \\ &= e^x \left(\cos(-y) + i \sin(-y)\right) \\ &= e^x \cos y - i e^x \sin y. \end{aligned}$$

Thus here

$$\begin{aligned} u &= \Re(\exp(\bar{z})) = e^x \cos y \\ v &= \Im(\exp(\bar{z})) = -e^x \sin y. \end{aligned}$$

Therefore

$$\begin{aligned} u_x &= e^x \cos y \\ v_x &= -e^x \sin y \\ u_y &= -e^x \sin y \\ v_y &= -e^x \cos y. \end{aligned}$$

Now since $e^x \neq 0$ for all x,

$$e^x \cos y = u_x = v_y = -e^x \cos y$$

can only hold when $\cos y = -\cos y$, i.e. when $\cos y = 0$. Similarly

$$-e^x \sin y = u_y = -v_x = e^x \sin y$$

can only hold when $\sin y = 0$. Since $\cos y$ and $\sin y$ can never simultaneously be zero, it follows that the Cauchy–Riemann equations never hold and hence that $\exp(\bar{z})$ is nowhere differentiable.

Exercise 5, §29, p. 92
Write $|\exp(2z+i)|$ and $\left|\exp\left(iz^2\right)\right|$ in terms of x and y. Then show that

$$\left|\exp(2z+i) + \exp\left(iz^2\right)\right| \leq e^{2x} + e^{-2xy}.$$

Solution
On setting $z = x + iy$ we see that

$$\begin{aligned} |\exp(2z+i)| &= |\exp(2x + i(2y+1))| \\ &= e^{2x} \end{aligned}$$

and

$$\begin{aligned} \left|\exp\left(iz^2\right)\right| &= \left|\exp\left(i\left(\left(x^2-y^2\right)+i2xy\right)\right)\right| \\ &= \left|\exp\left(-2xy + i\left(x^2-y^2\right)\right)\right| \\ &= e^{-2xy}. \end{aligned}$$

Hence

$$\begin{aligned} \left|\exp(2z+i) + \exp\left(iz^2\right)\right| &\leq |\exp(2z+i)| + \left|\exp\left(iz^2\right)\right| \\ &= e^{2x} + e^{-2xy}. \end{aligned}$$

Exercise 8, §29, p. 92
Find all values of z such that
(a) $e^z = -2$;
(b) $e^z = 1 + \sqrt{3}i$;
(c) $\exp(2z-1) = 1$.

Solution
(a) In polar form $-2 = 2(\cos\pi + i\sin\pi) = 2e^{i\pi}$. Therefore if

$$e^x e^{iy} = e^z = -2 = 2e^{i\pi}$$

then by the statement at the top of p. 25 of the textbook we must have

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \quad (n \in \mathbb{Z}).$$

Therefore $e^z = -2$ if and only if $z = \ln 2 + i(2n+1)\pi \ (n \in \mathbb{Z})$.

(b) We first write $1+\sqrt{3}i$ in polar form. Here $re^{i\theta} = 1+\sqrt{3}i$ where $r = \left|1+\sqrt{3}i\right| = 2$ and θ is chosen so that

$$\cos\theta + i\sin\theta = \frac{1}{r}\left(1+\sqrt{3}i\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$

i.e. $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$. Clearly $\theta = \frac{\pi}{3}$ will suffice. Hence $1+\sqrt{3}i = 2e^{i\pi/3}$. Again by the statement at the top of p. 25 of the textbook

$$e^x e^{iy} = e^z = 1+\sqrt{3}i = 2e^{i\pi/3}$$

if and only if $e^x = 2$ (i.e. $x = \ln 2$) and $y = \frac{\pi}{3} + 2n\pi \ (n \in \mathbb{Z})$. Thus $e^z = 1+\sqrt{3}i$ precisely when $z = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$, $n \in \mathbb{Z}$.

(c) With $z = x+iy$, $\exp(2z-1) = \exp((2x-1)+i2y)$. Thus as before

$$e^{2x-1}e^{i2y} = \exp(2z-1) = 1 = 1e^{i0}$$

if and only if

$$e^{2x-1} = 1 \quad \text{and} \quad 2y = 2n\pi, \ \ n \in \mathbb{Z}.$$

Consequently $\exp(2z-1) = 1$ if and only if $x = \frac{1}{2}$ (or equivalently $2x - 1 = \ln 1 = 0$) and $y = n\pi \ (n \in \mathbb{Z})$, that is $z = \frac{1}{2} + in\pi \ (n \in \mathbb{Z})$.

Exercise 9, §29, p. 92

Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm1, \pm2, \ldots)$. (Compare Exercise 4, Sec. 28.)

Solution

Observe that

$$\begin{aligned} \exp(iz) &= \exp(i(x+iy)) \\ &= \exp(-y+ix) \\ &= e^{-y}(\cos x + i\sin x). \end{aligned}$$

Thus

$$\begin{aligned} \overline{\exp(iz)} &= e^{-y}\overline{(\cos x + i\sin x)} \\ &= e^{-y}(\cos x - i\sin x) \\ &= e^{-y}(\cos(-x) + i\sin(-x)) \\ &= e^{-y}e^{-ix}. \end{aligned}$$

On the other hand

$$\begin{aligned} \exp(i\bar{z}) &= \exp(i(x-iy)) \\ &= \exp(y+ix) \\ &= e^y e^{ix}. \end{aligned}$$

Again by the statement at the top of p. 25 of the textbook

$$\begin{aligned} \overline{\exp(iz)} = \exp(i\bar{z}) &\Leftrightarrow e^{-y}e^{-ix} = e^y e^{ix} \\ &\Leftrightarrow e^{-y} = e^y \quad \text{and} \quad -x = x + 2n\pi \ (n \in \mathbb{Z}) \\ &\Leftrightarrow y = 0 \quad \text{and} \quad x = n\pi \ (n \in \mathbb{Z}) \\ &\Leftrightarrow z = n\pi \ (n \in \mathbb{Z}). \end{aligned}$$

Exercise 10, §29, p. 92

(a) Show that if e^z is real, then $\Im z = n\pi \ (n = 0, \pm1, \pm2, \ldots)$.

(b) If e^z is pure imaginary, what restriction is placed on z?

Solution

(a) $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$ is real if and only if $\sin y = 0$ if and only if $y = \Im z = n\pi \ (n \in \mathbb{Z})$.

(b) Similarly $e^z = e^x(\cos y + i\sin y)$ is pure imaginary if and only if $\cos y = 0$ if and only if $y = \Im z = (2n+1)\frac{\pi}{2} \ (n \in \mathbb{Z})$.

Exercise 11, §29, p. 92

Describe the behaviour of $\exp(x+iy)$ as

(a) x tends to $-\infty$;

(b) y tends to ∞.

Solution

(a) By (7) of Section 29 of the textbook

$$|\exp(x+iy)| = e^x.$$

As $x \to -\infty$ it therefore follows that $|\exp(x+iy)| = e^x \to 0$ and hence that

$$\exp(x+iy) \to 0.$$

(b) For any fixed x and $w = \exp(x+iy)$ it follows from (7) of Section 29 of the textbook that

$$y \in \arg(w) \quad \text{with} \quad |w| = e^x$$

Thus as y increases, $\exp(x+iy)$ rotates anticlockwise around the circle centred at the origin with radius e^x.

Exercise 13, §29, p. 92

Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in some domain D. State why the functions

$$\begin{aligned} U(x,y) &= e^{u(x,y)}\cos v(x,y) \\ V(x,y) &= e^{u(x,y)}\sin v(x,y) \end{aligned}$$

are harmonic in D and why $V(x,y)$ is, in fact, a harmonic conjugate of $U(x,y)$.

Solution

Since $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D and $\exp(z) = e^z$ analytic on all of $\mathbb{C}$, the composition

$$\begin{aligned} \exp \circ f(z) &= e^{f(z)} \\ &= e^{u(x,y)}e^{iv(x,y)} \\ &= e^{u(x,y)}(\cos(v(x,y)) + i\sin(v(x,y))) \end{aligned}$$

is analytic in D. Thus by Theorems 1 and 2 of Section 26 of the textbook it follows that

$$U(x,y) = \Re(\exp \circ f(z)) = e^{u(x,y)}\cos(v(x,y))$$

and

$$V(x,y) = \Im(\exp \circ f(z)) = e^{u(x,y)}\sin(v(x,y))$$

are both harmonic in D and that $V(x,y)$ is a harmonic conjugate of $U(x,y)$.

Exercise 1, §31, p. 97

Show that

(a) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$;

(b) $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

Solution

(a) For $-ei$ we clearly have $\text{Arg}(-ei) = -\frac{\pi}{2}$. Hence

$$\begin{aligned} \text{Log}\,(-ei) &= \ln|-ei| + i\text{Arg}\,(-ei) \\ &= \ln e - i\frac{\pi}{2} \\ &= 1 - i\frac{\pi}{2}. \end{aligned}$$

(b) To find $\text{Arg}(1-i)$ we need to select θ in the fourth quadrant with $-\pi < \theta \le \pi$ so that $\tan\theta = -1$. Clearly $\text{Arg}(1-i) = -\frac{\pi}{4}$ whence

$$\begin{aligned} \text{Log}\,(1-i) &= \ln|1-i| + i\text{Arg}\,(1-i) \\ &= \ln 2^{\frac{1}{2}} - i\frac{\pi}{4} \\ &= \frac{1}{2}\ln 2 - i\frac{\pi}{4}. \end{aligned}$$

Exercise 3, §31, p. 97

Show that

(a) $\text{Log}(1+i)^2 = 2\text{Log}(1+i)\,;$

(b) $\text{Log}(-1+i)^2 \neq 2\text{Log}(-1+i)\,.$

Solution

(a) In polar form $(1+i) = \sqrt{2}e^{i\pi/4}$. Therefore

$$\begin{aligned} \text{Log}\,(1+i) &= \ln 2^{\frac{1}{2}} + i\frac{\pi}{4} \\ &= \frac{1}{2}\ln 2 + i\frac{\pi}{4}. \end{aligned}$$

Moreover $(1+i)^2 = 2i = 2e^{i\pi/2}$ whence

$$\begin{aligned} \text{Log}\,(1+i)^2 &= \ln 2 + i\frac{\pi}{2} \\ &= 2\left(\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right) \\ &= 2\text{Log}\,(1+i)\,. \end{aligned}$$

(b) In polar form $-1+i = \sqrt{2}e^{i\theta}$ where θ is selected in the second quadrant so that $\tan\theta = -1$, i.e. $\theta = \frac{3\pi}{4}$. Therefore

$$\begin{aligned} \text{Log}\,(-1+i) &= \ln 2^{\frac{1}{2}} + i\frac{3\pi}{4} \\ &= \frac{1}{2}\ln 2 + i\frac{3\pi}{4}. \end{aligned}$$

Furthermore $(-1+i) = -2i = 2e^{-i\pi/2}$. Therefore

$$\text{Log}\,(-1+i)^2 = \ln 2 - i\frac{\pi}{2}$$

whereas

$$2\text{Log}\,(-1+i) = \ln 2 + i\frac{3\pi}{2}.$$

Exercise 5 §31, p. 97
Show that

(a) the set of values of $\log\left(i^{1/2}\right)$ is $\left(n+\frac{1}{4}\right)\pi i$ $(n=0,\pm1,\pm2,\ldots)$ and that the same is true of $\frac{1}{2}\log i$;

(b) the set of values of $\log\left(i^2\right)$ is *not* the same as the set of values of $2\log i$.

Solution

(a) In polar form $i=e^{i\pi/2}$ whence

$$(*)\qquad i^{\frac{1}{2}}=e^{i\left(\frac{\pi}{4}+k\pi\right)}\quad k=0,1$$

and

$$\begin{aligned}\log(i) &= \ln 1+i\left(\frac{\pi}{2}+2n\pi\right)\\ &= i\left(\frac{\pi}{2}+2n\pi\right)\end{aligned}$$

$(n\in\mathbb{Z})$. Moreover by (*)

$$\begin{aligned}\log\left(i^{\frac{1}{2}}\right) &= \ln 1+i\left(\frac{\pi}{4}+n\pi\right)\\ &= i\left(\frac{\pi}{4}+n\pi\right)\quad (n\in\mathbb{Z}).\end{aligned}$$

Clearly $\frac{1}{2}\log(i)=\log\left(i^{\frac{1}{2}}\right)$.

(b) From (a) above we know that

$$\log i=i\left(\frac{\pi}{2}+2n\pi\right)\quad (n\in\mathbb{Z})$$

and hence that

$$2\log i=i(\pi+4n\pi)\quad (n\in\mathbb{Z}).$$

In polar form $i^2=-1=1e^{i\pi}$ whence

$$\begin{aligned}\log i^2 &= \ln 1+i(\pi+2n\pi)\\ &= i(\pi+2n\pi)\quad (n\in\mathbb{Z}).\end{aligned}$$

Clearly $\log\left(i^2\right)\neq 2\log i$.

Exercise 7, §31, p. 97
Find all roots of the equation $\log z=(\pi/2)\,i$.

Solution

$$\begin{aligned}\log(z)=(\pi/2)\,i \quad &\Leftrightarrow\quad \ln|z|+i\theta=i\left(\frac{\pi}{2}\right)\quad (\text{where } z=|z|\,e^{i\theta})\\ &\Leftrightarrow\quad \ln|z|=0,\ \ \frac{\pi}{2}\in\arg(z)\\ &\Leftrightarrow\quad |z|=1,\ \ \operatorname{Arg}(z)=\frac{\pi}{2}\\ &\Leftrightarrow\quad z=e^{i\pi/2}=i\end{aligned}$$

Exercise 9, §31, p. 97
Show that

(a) the function $\mathrm{Log}(z-i)$ is analytic everywhere except on the half line $y=1$ $(x \leq 0)$;
(b) the function
$$\frac{\mathrm{Log}\,(z+4)}{z^2+i}$$
is analytic everywhere except on the portion $x \leq -4$ of the real axis and at the points $\pm(1-i)\sqrt{2}$.

Solution

(a) By (5) of Sec. 31 of the textbook $\mathrm{Log}z$ is analytic everywhere except where either $z=0$ or $\mathrm{Arg}(z)=\pi$. In terms of Cartesian coordinates this means that $\mathrm{Log}z$ is analytic everywhere except on the non–positive real axis. By contrast $z \to z-i$ is a $1-1$ analytic map from $\mathbb{C}$ onto $\mathbb{C}$. Hence the composition $\mathrm{Log}(z-i)$ is analytic on all of $\mathbb{C}$ except where $z-i = x+i\,(y-1)$ lies on the non–positive real–axis, i.e. except where $x \leq 0$ and $y=1$.
(b) By a similar argument to that in (a) above we can show that $\mathrm{Log}(z+4)$ is analytic on all of $\mathbb{C}$ except where $z+4=(x+4)+iy$ is on the non–positive real axis, i.e. except where $x \leq -4$ and $y=0$. Therefore
$$\frac{\mathrm{Log}\,(z+4)}{z^2+i}$$
will fail to be analytic on this portion and also where $z^2+i=0$. Now since $z^2=-i=e^{-i\pi/2}$ if and only if $z^2+i=0$, the roots of this equation will be
$$z = e^{i\left(-\frac{\pi}{4}+k\pi\right)} = \begin{cases} \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} & (k=0) \\ -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} & (k=1)\,. \end{cases}$$
Thus $\mathrm{Log}(z+4)\,/\left(z^2+i\right)$ is analytic everywhere except on the portion $x \leq -4$ of the real axis, and where $z = \pm\frac{1}{\sqrt{2}}\,(1-i)$.

Exercise 11, §31, p. 98
Show that
$$\Re\left[\log\,(z-1)\right] = \frac{1}{2}\ln\left[(x-1)^2+y^2\right] \quad (z \neq 1)\,.$$
Why must this function satisfy Laplace's equation when $z \neq 1$?

Solution
Since
$$\log\,(z-1) = \ln|z-1| + i\arg\,(z-1)$$
we surely have
$$\begin{aligned} \Re\left[\log\,(z-1)\right] &= \ln|z-1| \\ &= \ln\left((x-1)^2+y^2\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\ln\left((x-1)^2+y^2\right). \end{aligned}$$
Now for any z with $z-1 \neq 0$ (i.e. $z \neq 1$) we can find a branch of $\log\,(z-1)$ such that the branch cut is analytic at z (see (4) of Sec. 31 of the textbook). By Theorem

1 of Section 26, the real part, that is $\frac{1}{2}\ln\left((x-1)^2+y^2\right)$, must be harmonic at each such z.

Exercise 1, §32, p. 100

Show that if $\Re z_1 > 0$ and $\Re z_2 > 0$, then

$$\text{Log}\,(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2.$$

Solution

Since $\Re(z_1) > 0$ and $\Re(z_2) > 0$, both z_1 and z_2 are in either the first or fourth quadrant. Consequently

$$(*) \qquad \begin{aligned} -\frac{\pi}{2} &< \text{Arg}\,(z_1) < \frac{\pi}{2}, \\ -\frac{\pi}{2} &< \text{Arg}\,(z_2) < \frac{\pi}{2}. \end{aligned}$$

With $\theta_1 = \text{Arg}(z_1)$, $\theta_2 = \text{Arg}(z_2)$

$$\begin{aligned} z_1 z_2 &= \left(|z_1|\, e^{i\theta_1}\right)\left(|z_2|\, e^{i\theta_2}\right) \\ &= |z_1 z_2|\, e^{i(\theta_1+\theta_2)}, \end{aligned}$$

where $-\pi < \theta_1 + \theta_2 < \pi$ from (*) above. Thus here

$$\text{Arg}\,(z_1 z_2) = \theta_1 + \theta_2 = \text{Arg}\,(z_1) + \text{Arg}\,(z_2)$$

whence

$$\begin{aligned} \text{Log}\,(z_1 z_2) &= \ln|z_1 z_2| + i\text{Arg}\,(z_1 z_2) \\ &= \ln|z_1|\,|z_2| + i\left(\text{Arg}\,(z_1) + \text{Arg}\,(z_2)\right) \\ &= \left(\ln|z_1| + \ln|z_2|\right) + i\left(\text{Arg}\,(z_1) + \text{Arg}\,(z_2)\right) \\ &= \left(\ln|z_1| + i\text{Arg}\,(z_1)\right) + \left(\ln|z_2| + i\text{Arg}\,(z_2)\right) \\ &= \text{Log}\,(z_1) + \text{Log}\,(z_1). \end{aligned}$$

Exercise 2, §32, p. 100

Show that, for any two nonzero complex numbers z_1 and z_2,

$$\text{Log}\,(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2 + 2N\pi i$$

where N has one of the values $0, \pm 1$. (Compare Exercise 1.)

Solution

For general nonzero complex numbers z_1, z_2 we have $-\pi < \text{Arg}(z_1) \leq \pi$ and $-\pi < \text{Arg}(z_2) \leq \pi$ whence

$$(**) \qquad -2\pi < \text{Arg}\,(z_1) + \text{Arg}\,(z_2) \leq 2\pi.$$

Now as before

$$z_1 z_2 = |z_1|\,|z_2| \exp\left(i\left(\text{Arg}\,(z_1) + \text{Arg}\,(z_2)\right)\right)$$

and hence

$$\text{Arg}\,(z_1 z_2) = \text{Arg}\,(z_1) + \text{Arg}\,(z_2) + 2N\pi$$

where $N \in \mathbb{Z}$ is chosen so that

$$-\pi < \text{Arg}\,(z_1) + \text{Arg}\,(z_2) + 2N\pi \leq \pi.$$

By (**) above we need either $N = 0$, 1 or -1. For such an N

$$\begin{aligned}\mathrm{Log}\,(z_1 z_2) &= \ln|z_1 z_2| + i\mathrm{Arg}\,(z_1 z_2)\\ &= (\ln|z_1| + \ln|z_2|) + i\,(\mathrm{Arg}\,(z_1) + \mathrm{Arg}\,(z_2)) + i2N\pi\\ &= (\ln|z_1| + i\mathrm{Arg}\,(z_1)) + (\ln|z_2| + i\mathrm{Arg}\,(z_2)) + i2N\pi\\ &= \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2) + i2N\pi.\end{aligned}$$

Exercise 1, §33, p. 104
Show that when $n = 0, \pm 1, \pm 2, \ldots,$

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right);$
(b) $(-1)^{1/\pi} = e^{(2n+1)i}$.

Solution

(a) In polar form $1 + i = \sqrt{2}e^{i\pi/4}$. Hence

$$\begin{aligned}\log\,(1+i) &= \ln 2^{\frac{1}{2}} + i\left(\frac{\pi}{4} + 2n\pi\right)\\ &= \frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2n\pi\right) \quad (n \in \mathbb{Z})\,.\end{aligned}$$

Therefore

$$\begin{aligned}(1+i)^i &= \exp\,(i\log\,(1+i))\\ &= \exp\left(-\left(\frac{\pi}{4} + 2n\pi\right) + i\frac{\ln 2}{2}\right)\\ &= \exp\left(-\left(\frac{\pi}{4} + 2n\pi\right)\right)\exp\left(i\frac{\ln 2}{2}\right) \quad (n \in \mathbb{Z})\,.\end{aligned}$$

(b) In polar form $-1 = 1e^{i\pi}$, whence

$$\begin{aligned}\frac{1}{\pi}\log\,(-1) &= \frac{1}{\pi}\,(\ln 1 + i\,(\pi + 2n\pi))\\ &= 0 + i\,(1+2n) \quad (n \in \mathbb{Z})\,.\end{aligned}$$

Therefore

$$(-1)^{\frac{1}{\pi}} = \exp\left(\frac{1}{\pi}\log\,(-1)\right) = \exp\,(i\,(2n+1)) \quad (n \in \mathbb{Z})\,.$$

Exercise 2, §33, p. 104
Find the principal value of

(a) i^i;
(b) $\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]^{3\pi i}$;
(c) $(1-i)^{4i}$.

Solution

(a) In polar form $i = 1e^{i\pi/2}$. Therefore

$$\mathrm{Log}\,(i) = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

The principal value of i^i is then given by

$$i^i = \exp\,(i\mathrm{Log}\,(i)) = \exp\left(i^2\frac{\pi}{2}\right) = \exp\left(-\frac{\pi}{2}\right).$$

(b) In polar form $\frac{e}{2}\left(-1-\sqrt{3}i\right)$ is of the form $\frac{e}{2}\left(-1-\sqrt{3}i\right)=\left|\frac{e}{2}\left(-1-\sqrt{3}i\right)\right|e^{i\theta}$ $= ee^{i\theta}$ where θ is an angle in the third quadrant with $\tan\theta=\frac{y}{x}=\sqrt{3}$. For our purposes we also need $-\pi<\theta\leq\pi$. Then $\theta=-\frac{2\pi}{3}$ and hence

$$\begin{aligned}\operatorname{Log}\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right] &= \ln e+i\left(-\frac{2\pi}{3}\right)\\ &= 1-i\frac{2\pi}{3}.\end{aligned}$$

The principal value of $\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]^{3\pi i}$ is therefore given by

$$\begin{aligned}\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]^{3\pi i} &= \exp\left(3\pi i\operatorname{Log}\left[\frac{e}{2}\left(-1-\sqrt{3}i\right)\right]\right)\\ &= \exp\left(3\pi i\left(1-i\frac{2\pi}{3}\right)\right)\\ &= \exp\left(2\pi^2+i3\pi\right)\\ &= \exp\left(2\pi^2\right)\left(\cos 3\pi+i\sin 3\pi\right)\\ &= -\exp\left(2\pi^2\right)\end{aligned}$$

(c) In polar form $(1-i)=\sqrt{2}e^{-i\frac{\pi}{4}}$. Thus

$$\begin{aligned}\operatorname{Log}(1-i) &= \ln 2^{\frac{1}{2}}+i\left(-\frac{\pi}{4}\right)\\ &= \frac{1}{2}\ln 2-i\frac{\pi}{4}.\end{aligned}$$

The principal value of $(1-i)^{4i}$ is therefore given by

$$\begin{aligned}(1-i)^{4i} &= \exp\left(4i\operatorname{Log}(1-i)\right)\\ &= \exp\left(\pi+i2\ln 2\right)\\ &= e^{\pi}\left(\cos\left(2\ln 2\right)+i\sin\left(2\ln 2\right)\right).\end{aligned}$$

Exercise 6, §33, p. 104

Show that if $z\neq 0$ and a is a real number, then $|z^a|=\exp\left(a\ln|z|\right)=|z|^a$, where the principal value of $|z|^a$ is to be taken.

Solution:

For any $z\neq 0$ the principal value of $|z|^a$ is given by $|z|^a=\exp\left(a\operatorname{Log}|z|\right)=\exp\left(a\ln|z|\right)$. Moreover

$$z^a=\exp\left(a\log z\right)=\exp\left(a\left(\ln|z|+i\arg(z)\right)\right).$$

If in addition a is real, then

$$\Re\left(a\left(\ln|z|+i\arg(z)\right)\right)=a\ln|z|.$$

It then follows from (7) of Sec. 29 of the textbook that

$$\begin{aligned}|z^a| &= \left|\exp\left(a\ln|z|+ia\arg(z)\right)\right|\\ &= \exp\left(a\ln|z|\right)\\ &= |z|^a\end{aligned}$$

in this case.

Exercise 7, §33, p. 104

Let $c=a+bi$ be a fixed complex number, where $c\neq 0,\pm1,\pm2,\ldots$, and note that i^c is multiple–valued. What restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

Solution:
In polar form $i = 1.e^{i\frac{\pi}{2}}$ and hence

$$\log(i) = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{\pi}{2} + 2n\pi\right) \quad (n \in \mathbb{Z}).$$

Therefore

$$\begin{aligned} i^c &= \exp(c \log(i)) \\ &= \exp\left((a+ib)\, i\left(\frac{\pi}{2} + 2n\pi\right)\right) \\ &= \exp\left(-b\left(\frac{\pi}{2} + 2n\pi\right) + ia\left(\frac{\pi}{2} + 2n\pi\right)\right). \end{aligned}$$

From (7) of Sec. 29 of the textbook it follows that

$$\begin{aligned} |i^c| &= e^{-b\left(\frac{\pi}{2} + 2n\pi\right)} \\ &= e^{-b\left(\frac{\pi}{2}\right)} e^{-2n\pi b} \qquad (n \in \mathbb{Z}). \end{aligned}$$

Now the only way these values can be the same for all $n \in \mathbb{Z}$ is if the $2n\pi b$'s are the same for all $n \in \mathbb{Z}$, i.e. if $\Im(c) = b = 0$. Thus $|i^c|$ has only one possible value whenever c is real (that is $\Im(c) = 0$).

Exercise 9, §33, p. 104
Assuming that $f'(z)$ exists, state the differentiation formula for $d\left[c^{f(z)}\right]/dz$.

Solution:
We may mimic the proof of (11) at the end of Section 33 in the textbook to get

$$\begin{aligned} \frac{d}{dz} c^{f(z)} &= \frac{d}{dz} e^{f(z)\log c} = e^{f(z)\log c} \frac{d}{dz}\left(f(z)\log c\right) \\ &= c^{f(z)}.f'(z)\log c. \end{aligned}$$

Exercise 7, §34, p. 108
In Sec. 34, use expressions (13) and (14) to derive expressions (15) and (16) for $|\sin z|^2$ and $|\cos z|^2$.

Solution
Given that

$$\begin{aligned} \sin z &= \sin x \cosh y + i \cos x \sinh y \\ \cos z &= \cos x \cosh y - i \sin x \sinh y \end{aligned}$$

it follows that

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x \left(1 + \sinh^2 y\right) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \left(\sin^2 x + \cos^2 x\right)\sinh^2 y \\ &= \sin^2 x + \sinh^2 y \end{aligned}$$

and

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x \left(1 + \sinh^2 y\right) + \sin^2 x \sinh^2 y \\ &= \cos^2 x + \left(\cos^2 x + \sin^2 x\right)\sinh^2 y \\ &= \cos^2 x + \sinh^2 y. \end{aligned}$$

Exercise 8 §34, p. 108

Point out how it follows from expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$ that

(a) $|\sin z| \geq |\sin x|$;
(b) $|\cos z| \geq |\cos x|$.

Solution

It follows directly from what we proved in exercise (7) above that

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y} \geq \sqrt{\sin^2 x} = |\sin x|$$

and similarly that $|\cos z| \geq |\cos x|$.

Exercise 9, §34, p. 109

With the aid of expressions (15) and (16) in Sec. 34 for $|\sin z|^2$ and $|\cos z|^2$, show that

(a) $|\sinh y| \leq |\sin z| \leq \cosh y$;
(b) $|\sinh y| \leq |\cos z| \leq \cosh y$.

Solution

Again using what we showed in exercise (7) above it follows that

$$\begin{aligned} |\sinh y| &= \sqrt{\sinh^2 y} \\ &\leq \sqrt{\sin^2 x + \sinh^2 y} = |\sin z| \\ &\leq \sqrt{\cos^2 x + \sin^2 x + \sinh^2 y} \\ &= \sqrt{1 + \sinh^2 y} \\ &= \sqrt{\cosh^2 y} \\ &= \cosh y. \end{aligned}$$

(The last equality follows from the fact that $\cosh y \geq 0$ for all y.) By a similar argument to the above we can also show that

$$|\sinh y| \leq |\cos z| \leq \cosh y.$$

Exercise 11, §34, p. 109

Show that neither $\sin \bar{z}$ nor $\cos \bar{z}$ is an analytic function of z anywhere.

Solution

If $z = x + iy$, then $\bar{z} = x - iy$. From equation (13) of Section 34 of the textbook it follows that

$$\begin{aligned} \sin \bar{z} &= \sin x \cosh(-y) + i \cos x \sinh(-y) \\ &= \sin x \cosh y - i \cos x \sinh y. \end{aligned}$$

Thus for $\sin \bar{z}$, $u = \Re(\sin \bar{z}) = \sin x \cosh y$ and $v = \Im(\sin \bar{z}) = -\cos x \sinh y$. Therefore

$$\begin{aligned} u_x &= \cos x \cosh y \\ v_x &= \sin x \sinh y \\ u_y &= \sin x \sinh y \\ v_y &= -\cos x \cosh y. \end{aligned}$$

Clearly $\cos x \cosh y = u_x = v_y = -\cos x \cosh y$ can only hold if $\cos x \cosh y = 0$. Since $\cosh y \neq 0$ this means that $\cos x = 0$, i.e. that $x = (2n+1)\frac{\pi}{2}$ $(n \in \mathbb{Z})$. Similarly $\sin x \sinh y = u_y = -v_x = -\sin x \sinh y$ can only hold if $\sin x \sinh y = 0$. Since $\sin x \neq 0$ when $x = (2n+1)\frac{\pi}{2}$ $(n \in \mathbb{Z})$, we must then have that $\sinh y = 0$, i.e. that $y = 0$. Thus by the Cauchy–Riemann equations the only points where $\sin \bar{z}$ may be differentiable are the isolated points $z_n = (2n+1)\frac{\pi}{2}$ $(n \in \mathbb{Z})$. For any $z \in \mathbb{C}$ the function $\sin \bar{z}$ can therefore never be differentiable in an entire neighbourhood of z, that is $\sin \bar{z}$ is never analytic.

For $\cos \bar{z}$ we follow a similar argument. Here

$$\cos \bar{z} = \widetilde{u} + i\widetilde{v}$$

where

$$\begin{aligned} \widetilde{u} &= \cos x \cosh y, \\ \widetilde{v} &= \sin x \sinh y. \end{aligned}$$

The equation $-\sin x \cosh y = \widetilde{u}_x = \widetilde{v}_y = \sin x \cosh y$ only holds if $\sin x = 0$, i.e. if $x = n\pi$ $(n \in \mathbb{Z})$. For these values of x, $\cos x \sinh y = \widetilde{u}_y = -\widetilde{v}_x = -\cos x \sinh y$ can only hold if $y = 0$. Thus the only points where $\cos \bar{z}$ may be differentiable are the isolated points $z = n\pi$ $(n \in \mathbb{Z})$. As before it follows from this that $\cos \bar{z}$ is nowhere analytic.

Exercise 13, §34, p. 109
With the aid of expressions (13) and (14) in Sec. 34, give direct verifications of the relations obtained in Exercise 12.

Solutions
From the equations we verified in exercise (11), it follows that

$$\sin \bar{z} = \sin x \cosh y - i \cos x \sinh y = \overline{\sin x \cosh y + i \cos x \sinh y}$$

and

$$\cos \bar{z} = \cos x \cosh y + i \sin x \sinh y = \overline{\cos x \cosh y - i \sin x \sinh y}.$$

On applying (13) and (14) of Section 34 of the textbook, it now follows that $\sin \bar{z} = \overline{\sin z}$ and $\cos \bar{z} = \overline{\cos z}$.

Exercise 14, §34, p. 109
Show that

(a) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;
(b) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

Solution
We use what we proved in exercise (13) above.

(a) Note that

$$\overline{\cos(iz)} = \cos\left(\overline{iz}\right) = \cos((-i)\bar{z}) = \cos(i\bar{z})$$

for all z since $\cos(-w) = \cos w$ for all w.

(b) Here

$$\overline{\sin(iz)} = \sin\left(\overline{iz}\right) = \sin\left((-i)\,\bar{z}\right) = -\sin(i\bar{z}).$$

Thus $-\sin(i\bar{z}) = \overline{\sin(iz)} = \sin(i\bar{z})$ can only hold if $\sin(i\bar{z}) = 0$, i.e. if $i\bar{z} = n\pi\ (n \in \mathbb{Z})$. But $i\bar{z} = n\pi$ if and only if $z = im\pi\ (m \in \mathbb{Z})$ (set $m = -n$), and hence we are done.

Exercise 15, §34, p. 109

Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and the imaginary parts of $\sin z$ and $\cosh 4$.

Solution

By (13) of Section 34 we will have $\sin z = \cosh 4$ whenever $\sin x \cosh y = \cosh 4$ and $\cos x \sinh y = 0$. To find all z for which $\sin z = \cosh 4$, we therefore need to solve the simultaneous equations

$$\sin x \cosh y = \cosh 4 \quad \text{and} \quad \cos x \sinh y = 0.$$

Now $\cos x \sinh y = 0 \Leftrightarrow$ either $y = 0$ or $x = (2k+1)\,\pi/2\ (k \in \mathbb{Z})$. However if $y = 0$ then $\cosh y = 1$ in which case

$$\sin x \cosh y = \sin x \leq 1 < \cosh 4.$$

Thus $y = 0$ does not yield a solution and so we must have $x = (2k+1)\,\pi/2\ (k \in \mathbb{Z})$. We now substitute these values into $\sin x \cosh y$ to get the required solution. Now if $x = (2k+1)(\pi/2)$ then

$$\sin x = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd.} \end{cases}$$

However since $\cosh y = \frac{1}{2}\left(e^{y} + e^{-y}\right) > 0$, the case $\sin x = -1$ must be excluded in the light of the fact that we will then have that

$$\sin x \cosh y = -\cosh y < 0 < \cosh 4.$$

Hence we must have that $x = (4n+1)\,\pi/2\ (n \in \mathbb{Z})$. (Here we have set $k = 2n$.) For such an x we get

$$\sin x \cosh y = \cosh y.$$

Now since $\cosh y = \cosh(-y)$, it follows that either $y = 4$ or $y = -4$ will be enough to ensure that $\cosh y = \cosh 4$. Thus we finally conclude that $\sin z = \cosh 4 \Leftrightarrow z = (4n+1)\,\pi/2 \pm i4\ (n \in \mathbb{Z})$.

Exercise 16, §34, p. 109

Find all roots of the equation $\cos z = 2$.

Solution

By (14) of Section 34 of the textbook $\cos z = 2$ yields

$$\cos x \cosh y - i \sin x \sinh y = 2.$$

Comparing real and imaginary parts we get

$$\cos x \cosh y = 2 \quad \text{and} \quad \sin x \sinh y = 0.$$

Now if $\sin x \sinh y = 0$ then either $x = k\pi\ (k \in \mathbb{Z})$ or $y = 0$. However if $y = 0$, then $\cos x \cosh 0 = \cos x \leq 1 < 2$, and hence this does not yield a solution. Clearly we must have

$$x = k\pi\ \ (k \in \mathbb{Z}).$$

Substituting this into $\cos x \cosh y$ yields $\cos(k\pi)\cosh y = (-1)^{k}\cosh y$. If k is odd we get no solution since then $\cos(k\pi)\cosh y = -\cosh y < 0 < 2$. If $k = 2n$ is even, then $\cos(k\pi)\cosh y = \cosh y = 2$ will hold whenever $y = \pm \operatorname{arccosh}(2)$. Therefore $\cos z = 2$ precisely when

$$z = 2n\pi \pm i \operatorname{arc}\cosh(2) \ \ (n \in \mathbb{Z}).$$

Alternative

$$\begin{aligned}
\cos z &= z \Leftrightarrow \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = 2 \\
&\Leftrightarrow \left(e^{iz}\right)^2 - 4e^{iz} + 1 = 0 \\
&\Leftrightarrow e^{iz} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} \\
&\Leftrightarrow e^{-y}e^{ix} = e^{iz} = \left(2 \pm \sqrt{3}\right) e^{i0} \\
&\Leftrightarrow e^{-y} = \left(2 \pm \sqrt{3}\right) \quad \text{and} \quad x = 2n\pi \ \ (n \in \mathbb{Z}) \\
& \text{(By the remark at the top of p. 25 of the textbook.)} \\
&\Leftrightarrow y = -\ln\left(2 \pm \sqrt{3}\right) \quad \text{and} \quad x = 2n\pi \ \ (n \in \mathbb{Z}) \\
&\Leftrightarrow z = 2n\pi \pm i \ln\left(2 + \sqrt{3}\right)
\end{aligned}$$

In the last equality we used the fact that

$$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)} = 2 + \sqrt{3}$$

and hence that

$$-\ln\left(2-\sqrt{3}\right) = \ln\left(\frac{1}{2-\sqrt{3}}\right) = \ln\left(2+\sqrt{3}\right).$$

Exercise 4, §35, p. 111

Write $\sinh z = \sinh(x+iy)$ and $\cosh z = \cosh(x+iy)$, and show how expressions (9) and (10) in Sec. 35 follow from identities (7) and (8), respectively, in that section.

Solution

$$\begin{aligned}
\sinh z &= \sinh(x+iy) \\
&= \sinh x \cosh(iy) + \cosh x \sinh(iy) \qquad ((7) \text{ of Sec. } 35) \\
&= \sinh x \cos y + i \cosh x \sin y \qquad ((3) \text{ of Sec. } 35)
\end{aligned}$$

$$\begin{aligned}
\cosh z &= \cosh(x+iy) \\
&= \cosh x \cosh(iy) + \sinh x \sinh(iy) \qquad ((8) \text{ of Sec. } 35) \\
&= \cosh x \cos y + i \sinh x \sin y \qquad ((3) \text{ of Sec. } 35)
\end{aligned}$$

Exercise 5, §35, p. 111

Verify expression (12), Sec. 35, for $|\cosh z|^2$.

Solution
Using the expressions obtained in exercise (4) above we get

$$\begin{aligned}|\cosh z|^2 &= \cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y\\ &= \left(1+\sinh^2 x\right)\cos^2 y + \sinh^2 x\sin^2 y\\ &= \sinh^2 x\left(\cos^2 y + \sin^2 y\right) + \cos^2 y\\ &= \sinh^2 x + \cos^2 y.\end{aligned}$$

Exercise 6, §35, p. 111
Show that $|\sinh x| \le |\cosh z| \le \cosh x$ by using
(a) identity (12), Sec. 35;
(b) the inequalities obtained in Exercise 9(b), Sec. 34.

Solution
(a) Using the identity obtained above we see that

$$\begin{aligned}\sinh^2 x &\le |\cosh z|^2 = \sinh^2 x + \cos^2 y\\ &\le \sinh^2 x + \cos^2 y + \sin^2 y\\ &= \sinh^2 x + 1\\ &= \cosh^2 x.\end{aligned}$$

Taking square roots and keeping in mind that $\cosh x < 0$, it follows that

$$|\sinh x| \le |\cosh z| \le \cosh x.$$

(b) Recall that $\cosh z = \cos(iz) = \cos(-y+ix)$. Thus on applying the results of exercise (9) of Sec. 34 of the textbook, it surely follows that

$$|\sinh x| \le |\cos(iz)| = |\cosh z| \le \cosh x.$$

Exercise 8, §35, p. 112
Give details showing that the zeros of $\sinh z$ and $\cosh z$ are as in statements (14) and (15) in Sec. 35.

Solution

$$\begin{aligned}\frac{1}{2}\left(e^z - e^{-z}\right) = \sin z = 0 &\Leftrightarrow e^z = e^{-z} \qquad (\times e^z)\\ &\Leftrightarrow e^{2x}e^{i2y} = e^{2z} = 1\\ &\Leftrightarrow 2x = 0,\ \ 2y = 2n\pi\ \ (n\in\mathbb{Z})\\ &\qquad \text{(see the top of p. 25 of the textbook)}\\ &\Leftrightarrow z = in\pi\ \ (n\in\mathbb{Z})\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\left(e^z + e^{-z}\right) = \cosh z = 0 &\Leftrightarrow e^z = -e^{-z} \qquad (\times e^x)\\ &\Leftrightarrow e^{2z} = -1\\ &\Leftrightarrow 2z = (2n+1)\pi i\ \ (n\in\mathbb{Z})\\ &\qquad \text{(see the example in Section 29 of the textbook)}\\ &\Leftrightarrow z = \left(\frac{\pi}{2} + n\pi\right)i\ \ (n\in\mathbb{Z})\end{aligned}$$

Exercise 9, §35, p. 112

Using the results proved in Exercise 8, locate all zeros and singularities of the hyperbolic tangent function.

Solution

We recall that both $\cosh z$ and $\sinh z$ are analytic on all of $\mathbb{C}$. In addition, from what we showed in exercise (8) above, it is clear that $\cosh z$ and $\sinh z$ are never simultaneously zero. Therefore

$$\tanh z = \frac{\sinh z}{\cosh z}$$

is analytic at all points z where $\cosh z \neq 0$, and zero where $\sinh z = 0$. Therefore $\tanh z$ has singularities where $\cosh z = 0$ (i.e. where $z = \left(\frac{\pi}{2} + n\pi\right) i \ (n \in \mathbb{Z})$) and zeros where $z = n\pi i \ (n \in \mathbb{Z})$.

Exercise 14, §35, p. 112

Why is the function $\sinh(e^z)$ entire? Write its real part as a function of x and y, and state why that function must be harmonic everywhere.

Solution

Both the functions $z \to e^z$ and $z \to \sinh z$ are analytic on all of $\mathbb{C}$ and hence so is their composition $\sinh(e^z)$. By Theorem 1 of Section 26 the textbook, $\Re(\sinh(e^z))$ must therefore be harmonic on all of $\mathbb{C}$. Now for any $w \in \mathbb{C}$ it follows from exercise (4) above that

$$\Re(\sinh w) = \sinh(\Re(w))\cos(\Im(w)).$$

In particular for $w = e^z$ we have

$$w = e^z = e^x \cos y + i e^x \sin y$$

and hence

$$\Re(\sinh(e^z)) = \sinh(e^x \cos y)\cos(e^x \sin y).$$

Exercise 15, §35, p. 112

Find all roots of the equation

(a) $\sinh z = i$;
(b) $\cosh z = \frac{1}{2}$;

Solution

(a)

$$\begin{aligned}
\sinh z = i \quad &\Leftrightarrow \quad \frac{1}{2}\left(e^z - e^{-z}\right) = i \quad (\times e^z) \\
&\Leftrightarrow \quad (e^z)^2 - 2ie^z - 1 = 0 \\
&\Leftrightarrow \quad e^z = \frac{2i \pm \sqrt{-4+4}}{2} = i \\
&\Leftrightarrow \quad e^z = e^{i\frac{\pi}{2}} \\
&\Leftrightarrow \quad z = i\left(\frac{\pi}{2} + 2n\pi\right) \ (n \in \mathbb{Z}).
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{2}\left(e^z + e^{-z}\right) = \cosh z = \frac{1}{2} \quad &\Leftrightarrow \quad e^z + e^{-z} = 1 \quad (\times e^z) \\
&\Leftrightarrow \quad (e^z)^2 - e^z + 1 = 0 \\
&\Leftrightarrow \quad e^z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}
\end{aligned}$$

In polar form

$$\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)=e^{i\frac{\pi}{3}}$$

and

$$\frac{1}{2}-i\frac{\sqrt{3}}{2}=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)=e^{-i\frac{\pi}{3}}.$$

By the remark at the top of p. 25 of the textbook

$$e^{x}e^{-iy}=e^{z}=\frac{1}{2}+i\frac{\sqrt{3}}{2}=e^{i\frac{\pi}{3}}$$

if and only if

$$z=0+i\left(\frac{\pi}{3}+2n\pi\right)\ (n\in\mathbb{Z}).$$

Similarly

$$e^{z}=\frac{1}{2}-i\frac{\sqrt{3}}{2}=e^{-\frac{i\pi}{3}}$$

if and only if

$$z=0+i\left(-\frac{\pi}{3}+2n\pi\right)\ (n\in\mathbb{Z}).$$

Therefore

$$\cosh z=\frac{1}{2}\Leftrightarrow z=i\left(2n\pi\pm\frac{\pi}{3}\right)\ (n\in\mathbb{Z}).$$

Exercise 16, §35, p. 112
Find all roots of the equation $\cosh z=-2$.
Suggestion: Compare this with Exercise 16, Sec. 34.

Solution
Since $\cosh z=\cos(iz)=-\cos(iz+\pi)$ by (4) of Sec. 35 and (11) of Sec. 34 of the textbook, it follows that

$$\cosh z=-2\Leftrightarrow\cos(iz+\pi)=2.$$

Comparing this with what we showed in exercise (18) of Section 34 in the textbook, it follows that

$$\begin{aligned}\cosh z=-2 \quad&\Leftrightarrow\quad iz+\pi=2k\pi\pm i\ln\left(2+\sqrt{3}\right)\\ &\Leftrightarrow\quad z=\pm\ln\left(2+\sqrt{3}\right)+i\left(2n+1\right)\pi\quad(n=-k\in\mathbb{Z}).\end{aligned}$$

CHAPTER 4

Integrals

Study all sections. Note that much of sections 37 – 39 amount to a revision of material covered more extensively in MAT2615. We will therefore freely use the concepts and results contained in these sections, but will not emphasize them greatly.

In this chapter we start our investigation of the theory of integrating complex functions along some given contour. Such integrals are easy enough to define, but finding elegant ways to compute them is another matter. Once again the concept of analyticity ends up playing a crucial role. The Cauchy-Goursat theorem tells us that if a function is analytic on some domain, then for any closed contour C in that domain, $\int_C f(z)\,dz = 0$. By the Anti-derivative Theorem, this is equivalent to saying that such a function has an antiderivative on that domain. Once we are assured of the existence of an anti-derivative for a certain class of functions, then for that class we can try to mimic the theory of integration as done in elementary calculus. However the ramifications of the Cauchy-Goursat theorem go far beyond the existence anti-derivatives. The consequences of this theorem include the very elegant Cauchy Integration Formulas (a very powerful integration tool), the Maximum Modulus Principle (which tells us where to look for the maximum of $|f(z)|$ on some bounded closed region), Liouville's theorem (which basically tells us how difficult it is for an analytic function to be bounded), and even the Fundamental Theorem of Algebra (which guarantees that every polynomial equation of degree one or higher has a solution). But let's get down to practicalities, and close this overview chapter 4 with some hints on using this powerful theory to compute integrals.

As alluded to above, the elegance with which we are able to integrate a complex function on some given contour, depends entirely on the extent to which that function is analytic. If the function is very bad and not at all analytic on or near the contour, then our only real option is to parametrise the contour and try to compute the integral from "first principles". If however the integrand, say f, does behave well in the sense of being analytic everywhere except for some isolated points, we have many more options available to us.

If the contour C is not closed and f is analytic on some domain containing C, we may then use antiderivatives to compute $\int_C f(z)\,dz$. If C is closed and f is analytic on and inside C, then the Cauchy-Goursat theorem tells us that $\int_C f(z)\,dz = 0$. (Note that to be able to apply the Cauchy-Gourat theorem to some function f, it is not necessary for f to have **no** singularities. All we need is for f to have no singularities on or inside C. For example although $f(z) = \frac{1}{z}$ has a singularity at $z = 0$, this point is outside $|z-2| = 1$ and so we still have

$$\int_{|z-2|=1} \frac{1}{z} dz = 0$$

by this theorem.)

If now f is of the form $f(z) = \frac{g(z)}{p(z)}$ where g is analytic inside and on C and p is a polynomial, we can use the Cauchy Integration Formulas to compute the integral

$\int_C f(z)\,dz$. The idea is as follows: If $\frac{g(z)}{p(z)}$ is a rational function we can use partial fractions to decompose it into a sum of simpler terms to which we may apply the integration formulae. However even if g is not a polynomial, we can still use partial fractions to decompose $\frac{1}{p(z)}$ into a sum of terms of the form $\frac{K}{(z-a)^n}$. On multiplying throughout by g, we end up with a formula expressing f as a sum of terms of the form $\frac{Kg(z)}{(z-a)^n}$. Once this is done, we can then use the Cauchy Integration Formulas to compute the integral of each $\frac{Kg(z)}{(z-a)^n}$, and add the results to get $\int_C f(z)\,dz$. For example if $f(z) = \frac{\sin z}{z(z-2)}$ and C is the positively oriented curve $|z| = 3$, we may use partial fractions to conclude that

$$\frac{1}{z(z-2)} = \frac{1}{2(z-2)} - \frac{1}{2z}$$

and then multiply throughout by $\sin z$, to get

$$\frac{\sin z}{z(z-2)} = \frac{\frac{\sin z}{2}}{z-2} - \frac{\frac{\sin z}{2}}{z}.$$

It now follows from the integration formulas that

$$\begin{aligned}\int_C \frac{\sin z}{z(z-2)}dz &= \int_C \frac{\frac{\sin z}{2}}{z-2} - \int_C \frac{\frac{\sin z}{2}}{z}dz \\ &= 2\pi i\left(\frac{\sin z}{2}\bigg|_{z=2}\right) - 2\pi i\left(\frac{\sin z}{2}\bigg|_{z=0}\right) \\ &= \pi i \sin 2.\end{aligned}$$

(If of course we were integrating over the positively oriented circle $|z| = 1$ instead of $|z| = 3$, we would not have needed partial fractions since in this case only the singularity $z = 0$ lies inside $|z| = 1$. In particular this means that $\frac{\sin z}{z-2}$, although not differentiable at $z = 2$, is nevertheless analytic inside and on $|z| = 1$. We can then directly see that

$$\int_{|z|=1} \frac{\sin z}{z(z-2)}dz = \int_{|z|=1} \frac{\frac{\sin z}{(z-2)}}{z}dz = 2\pi i \left.\frac{\sin z}{(z-2)}\right|_{z=0} = 0.)$$

But what if f is of the form $f(z) = \frac{g(z)}{h(z)}$ with both g and h analytic, but with h not a polynomial. (For example something like $\tan(z) = \frac{\sin(z)}{\cos(z)}$ is such a function.) If in this case we want to compute the integral $\int_C f(z)\,dz$ for some closed contour C, there is no easy way to reduce it to an application of the Cauchy Integral Formulas like we did above. So although such functions are very nice, the technology we develop in this chapter, cannot adequately deal with this class of functions. Our theory of integration therefore clearly needs a bit more development. However before we can further refine the theory of integration, we need the background of the theory of power series expansions. This we investigate in the next chapter, before returning to the theory of integration.

Solutions to selected problems

Solutions to selected problems in sections 38, 39, 42, 43, 45, 49, 52 and 54 follow.

Exercise 2, §38, p. 121

Evaluate the following integrals:
(a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$; (b) $\int_0^{-\pi/6} e^{i2t} dt$; (c) $\int_0^\infty e^{-zt} dt$ $(\Re z > 0)$.

Solution:

(a)

$$\begin{aligned}
\int_1^2 \left(\frac{1}{t} - i\right)^2 dt &= \int_1^2 \left[\left(\frac{1}{t^2} - 1\right) - i\frac{2}{t}\right] dt \\
&= \left(-\frac{1}{t} - t\right) - i2\ln t\Big|_1^2 \\
&= -\frac{1}{2} - i.2\ln 2
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\frac{\pi}{6}} e^{i2t} dt &= \frac{1}{i2} e^{i2t}\Big|_0^{\frac{\pi}{6}} \\
&= \frac{1}{i2}\left(\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - 1\right) \\
&= \frac{1}{i2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= \frac{\sqrt{3}}{4} + i\frac{1}{4}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^\infty e^{-zt} dt &= \lim_{b\to\infty} \int_0^b e^{-zt} dt \\
&= \lim_{b\to\infty} -\frac{1}{z} e^{-zt}\Big|_0^b \\
&= \lim_{b\to\infty} \frac{1}{z}\left(1 - e^{-zb}\right)
\end{aligned}$$

Note that $\left|e^{-zb}\right| = e^{-\Re(z)b} \to 0$ as $b \to \infty$ since by hypothesis $\Re(z) > 0$. Hence $\int_0^\infty e^{-zt} dt = \lim\limits_{b\to\infty} \frac{1}{z}(1 - e^{-zt}) = \frac{1}{z}$.

Exercise 4, §38, p. 121

According to definition (2), Sec. 38 of integrals of complex–valued functions of a real variable,

$$\int_0^\pi e^{(1+i)x} dx = \int_0^\pi e^x \cos x dx + i\int_0^\pi e^x \sin x dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then identifying the real and imaginary parts of the value found.

Solution:

$$\int_0^\pi e^x \cos x dx + i\int_0^\pi e^x \sin x dx = \int_0^\pi e^{(1+i)x} dx$$
$$= \frac{1}{1+i} e^{(1+i)x}\Big|_0^\pi$$
$$= \frac{1}{1+i}\left(e^\pi e^{i\pi} - 1\right)$$
$$= \left(\frac{1}{2} - i\frac{1}{2}\right)\left(-e^\pi - 1\right)$$

Hence

$$\int_0^\pi e^x \cos x dx = \Re\left(\left(\frac{1}{2} - i\frac{1}{2}\right)\left(-e^\pi - 1\right)\right)$$
$$= -\frac{1}{2}\left(e^\pi + 1\right)$$
$$\int_0^\pi e^x \sin x dx = \Im\left(\left(\frac{1}{2} - i\frac{1}{2}\right)\left(-e^\pi - 1\right)\right)$$
$$= \frac{1}{2}\left(e^\pi + 1\right).$$

Exercise 2, §39, p. 125

Let C denote the right–hand half of the circle $|z| = 2$, in the counterclockwise direction, and note that two parametric representations for C are

$$z = z(\theta) = 2e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

and

$$z = Z(y) = \sqrt{4 - y^2} + iy \quad (-2 \le y \le 2).$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan\frac{y}{\sqrt{4 - y^2}} \qquad \left(-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}\right).$$

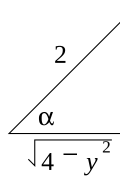
Also, show that this function ϕ has a positive derivative, as required in the conditions following equation (9), Sec. 39.

Solution:

With z and Z defined as before

$$z\left(\arctan\left(\frac{y}{\sqrt{4-y^2}}\right)\right)$$
$$= 2\exp\left(i\arctan\left(\frac{y}{\sqrt{4-y^2}}\right)\right)$$
$$= 2\left(\cos\left(\arctan\left(\frac{y}{\sqrt{4-y^2}}\right)\right) + i\sin\left(\arctan\left(\frac{y}{\sqrt{4-y^2}}\right)\right)\right)$$
$$= 2\left(\frac{\sqrt{4-y^2}}{2} + i\frac{y}{2}\right)$$
$$= \sqrt{4-y^2} + iy \qquad (-2 \le y \le 2).$$

Note that if $\alpha = \arctan\left(\frac{y}{\sqrt{4-y^2}}\right)$, i.e. $\tan\alpha = \frac{y}{\sqrt{4-y^2}}$, then



Finally note that with $f(y) = \frac{y}{\sqrt{4-y^2}}$, it follows from the chain rule for real functions that

$$\phi'(y) = \frac{d}{dy}\arctan(f(y)) = \frac{1}{1+f(y)^2}f'(y).$$

Now by elementary differentiation rules we can show that

$$f'(y) = \frac{4}{(4-y^2)^{\frac{3}{2}}}.$$

Substituting into $\phi'(y)$ we get

$$\phi'(y) = \frac{1}{1+\left(\frac{y^2}{4-y^2}\right)} \cdot \frac{4}{(4-y^2)^{\frac{3}{2}}} = \frac{1}{\sqrt{4-y^2}}.$$

Clearly $\phi'(y) > 0$ on $-2 < y < 2$. (Strictly speaking $\phi'(y)$ doesn't actually exist at $y = \pm 2$.)

Exercise 6, §39, p. 126

Let $y(x)$ be a real–valued function defined on the interval $0 \le x \le 1$ by means of the equations

$$y(x) = \begin{cases} x^3 \sin\left(\frac{\pi}{x}\right) & \text{when} \quad 0 < x \le 1, \\ 0 & \text{when} \quad x = 0. \end{cases}$$

(a) Show that the equation

$$z = x + iy(x) \qquad (0 \le x \le 1)$$

represents an arc C_1 that intersects the real axis at the points $z = 1/n\,(n = 1, 2, \ldots)$ and $z = 0$.

(b) Verify that the arc C_1 in part (a) is, in fact, a *smooth* arc. *Suggestion:* To establish the continuity of $y(x)$ at $x = 0$, observe that

$$0 \le \left|x^3 \sin\left(\frac{\pi}{x}\right)\right| \le x^3$$

when $x > 0$. A similar remark applies in finding $y'(0)$ and showing that $y'(x)$ is continuous at $x = 0$.

Solution:

(a) For $x = \frac{1}{n}$ we have

$$\begin{aligned} z\left(\frac{1}{n}\right) &= \frac{1}{n} + iy\left(\frac{1}{n}\right) \\ &= \frac{1}{n} + i\left(\frac{1}{n}\right)^3 \sin\left(\frac{\pi}{\left(\frac{1}{n}\right)}\right) \\ &= \frac{1}{n} + i\left(\frac{1}{n}\right)^3 \sin(n\pi) \\ &= \frac{1}{n}. \end{aligned}$$

For $x \neq \frac{1}{n}$ $(n \in \mathbb{Z})$, $\frac{\pi}{x} \neq n\pi$ and hence $\sin\left(\frac{\pi}{x}\right) \neq 0$ (that is $\Im(z(x)) \neq 0$). Thus for any x with $x \neq \frac{1}{n}$ $(n \in \mathbb{Z})$ $z(x)$, is not on the real axis. Clearly z intersects the real axis at precisely the points 0 and $\frac{1}{n}$ $(n \in \mathbb{Z})$.

(b) Since x^3, $\frac{1}{x}$, and $\sin x$ are all continuously differentiable for $x \neq 0$, it follows that $x^3 \sin\left(\frac{\pi}{x}\right)$ is continuously differentiable when $x \neq 0$. Clearly the same is then true of $z(x)$. It remains to show that z' exists and is continuous at 0. Now since $-h^2 \leq h^2 \sin\left(\frac{\pi}{h}\right) \leq h^2$ it is clear from the sandwich theorem that $h^2 \sin\left(\frac{\pi}{h}\right) \to 0$ as $h \to 0$ and hence that

$$\begin{aligned} z'(0) &= \lim_{h\to 0} \frac{z(h) - z(0)}{h} \\ &= \lim_{h\to 0} 1 + ih^2 \sin\left(\frac{\pi}{h}\right) \\ &= 1. \end{aligned}$$

For $x \neq 0$ we have $z'(x) = 1 + i\left(3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right)\right)$. Since $-3x^2 \leq 3x^2 \sin\left(\frac{\pi}{x}\right) \leq 3x^2$, $-\pi|x| \leq -\pi x \cos\left(\frac{\pi}{x}\right) \leq \pi|x|$ it follows from the sandwich theorem that $z'(x) = 1 + i\left(3x^2 \sin\left(\frac{\pi}{x}\right) - \pi x \cos\left(\frac{\pi}{x}\right)\right) \to 1 + i0 = z'(0)$. Thus z' is continuous at 0.

Exercise 1, 3 and 4, §42, p. 135

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)\, dz.$$

1. $f(z) = (z+2)/z$ and C is
   (a) the semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$;
   (b) the semicircle $z = 2e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$;
   (c) the circle $z = 2e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$.

3. $f(z) = \pi \exp(\pi \overline{z})$ and C is the boundary of the square with vertices at the points $0,\ 1,\ 1+i$, and i, the orientation of C being in the counterclockwise direction.

4. $f(z)$ is defined by the equation

$$f(z) = \begin{cases} 1 & \text{when } y < 0, \\ 4y & \text{when } y > 0, \end{cases}$$

and C is the arc from $z = -1-i$ to $z = 1+i$ along the curve $y = x^3$.

Solution of exercise 1:

(a) Observe that $dz = z'(\theta)\,d\theta = 2ie^{i\theta}d\theta$ and hence here

$$\begin{aligned}\int_C \frac{z+2}{z}dz &= \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}}2ie^{i\theta}d\theta \\ &= 2i\int_0^{\pi}\left(e^{i\theta}+1\right)d\theta \\ &= 2i\left(\frac{1}{i}e^{i\theta}+\theta\right)\Big|_0^{\pi} \\ &= 2i\left(\frac{1}{i}(-1-1)+\pi\right) \\ &= -4+i2\pi.\end{aligned}$$

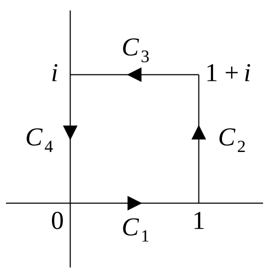
(b) By a similar argument to that in (a) it follows that in this case

$$\begin{aligned}\int_C \frac{z+2}{z}dz &= 2i\int_{\pi}^{2\pi}\left(e^{i\theta}+1\right)d\theta \\ &= 2i\left(\frac{1}{i}e^{i\theta}+\theta\right)\Big|_{\pi}^{2\pi} \\ &= 4+i2\pi.\end{aligned}$$

(c) Here the contour is just the join of the two contours considered in (a) and (b). Thus the integral turns out to be the sum of the integrals in (a) and (b). Hence

$$\int_C \frac{z+2}{z}dz = (-4+i2\pi)+(4+i2\pi) = i4\pi.$$

Solution of Exercise 3:



Here $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where $C_1,\ C_2,\ C_3$ and C_4 are as shown. These line segments may be parametrised as follows:

$C_1 : z_1(t) = t,\ 0 \le t \le 1$ $\qquad C_2 : z_2(t) = 1+it,\ 0 \le t \le 1$
$C_3 : z_3(t) = (1-t)+i,\ 0 \le t \le 1$ $\qquad C_4 : z_4(t) = i(1-t),\ 0 \le t \le 1$

Now

$$\begin{aligned}
\int_{C_1} \pi e^{\pi \overline{z}} dz &= \int_0^1 \pi e^{\pi t} dt = e^{\pi t}\Big|_0^1 = e^{\pi} - 1, \\
\int_{C_2} \pi e^{\pi \overline{z}} dz &= \int_0^1 \pi e^{\pi(1-it)} i dt \\
&= \pi e^{\pi} \int_0^1 e^{-i\pi t} i dt \\
&= -e^{\pi} \; e^{-i\pi t}\Big|_0^1 \\
&= -e \left(e^{-i\pi} - 1\right) \\
&= 2e^{\pi},
\end{aligned}$$

$$\begin{aligned}
\int_{C_3} \pi e^{\pi \overline{z}} dz &= \int_0^1 \pi e^{\pi((1-t)-i)} \left(-1\right) dt \\
&= -\pi e^{\pi} e^{-i\pi} \int_0^1 e^{-\pi t} dt \\
&= -e^{\pi} \left(e^{-\pi t}\Big|_0^1 \right) \\
&= e^{\pi} - 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_{C_4} \pi e^{\pi \overline{z}} dz &= \int_0^1 \pi e^{-\pi i(1-t)} \left(-i\right) dt \\
&= -i\pi e^{-i\pi} \int_0^1 e^{i\pi t} dt \\
&= e^{i\pi t}\Big|_0^1 \\
&= -2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_C \pi e^{\pi \overline{z}} dz &= \sum_{n=1}^{4} \int_{C_n} \pi e^{\pi \overline{z}} dz \\
&= \left(e^{\pi} - 1\right) + 2e^{\pi} + \left(e^{\pi} - 1\right) + (-2) \\
&= 4\left(e^{\pi} - 1\right).
\end{aligned}$$

Solution of Exercise 4:
Here

$$C : z = x + ix^3 \text{ where } -1 \leq x \leq 1.$$

Hence

$$\int_C f(z)\, dz = \int_{-1}^1 f\left(x + ix^3\right)\left(1 + i3x^2\right) dx.$$

Now on C, $y = x^3 > 0$ when $x > 0$ and $y = x^3 < 0$ when $x < 0$. Thus if $z \in C$ then

$$f(z) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 4x^3 & \text{if} \quad x > 0. \end{cases}$$

Therefore

$$\begin{aligned}\int_C f(z)\,dz &= \int_{-1}^{0} f\left(x+ix^3\right)\left(1+i3x^2\right)dx \\ &\quad + \int_{0}^{1} f\left(x+ix^3\right)\left(1+i3x^2\right)dx \\ &= \int_{-1}^{0}\left(1+i3x^2\right)dx + \int_{0}^{1} 4x^3\left(1+i3x^2\right)dx \\ &= \left(x+ix^3\right)\Big|_{-1}^{0} + \left(x^4+i2x^6\right)\Big|_{0}^{1} \\ &= (1+i)+1+i2 \\ &= 2+i3.\end{aligned}$$

Exercise 8, §42, p. 136

With the aid of the result in Exercise 3, Sec. 38, evaluate the integral

$$\int_C z^m \overline{z}^n dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

Solution:

The circle $|z| = 1$ may be parametrised by $z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. By exercise 2(b) of section 8 we have $\overline{z(\theta)} = \overline{e^{i\theta}} = e^{-i\theta}$. (To see this note that

$$\begin{aligned}\overline{e^{i\theta}} &= \overline{\cos\theta + i\sin\theta} \\ &= \cos\theta - i\sin\theta \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= e^{-i\theta}.)\end{aligned}$$

Hence

$$\begin{aligned}\int_C z^m \left(\overline{z}\right)^n dz &= \int_0^{2\pi} e^{im\theta}e^{-in\theta}ie^{i\theta}d\theta \\ &= i\int_0^{2\pi} e^{i(m-n+1)\theta}d\theta \\ &= \begin{cases} 2\pi i & \text{if} \quad m = n-1 \\ \dfrac{1}{(m-n+1)}\, e^{i(m-n+1)\theta}\Big|_0^{2\pi} & \text{if} \quad m \neq n-1 \end{cases} \\ &= \begin{cases} 2\pi i & \text{if} \quad m = n-1 \\ 0 & \text{if} \quad m \neq n-1. \end{cases}\end{aligned}$$

Exercise 10(b), §42, p. 136

Let C_0 denote the circle $|z - z_0| = R$, taken counterclockwise. Use the parametric representation $z = z_0 + R\,e^{i\theta}\,(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formulas:

(i) $\displaystyle\int_{C_o} \frac{dz}{z - z_0} = 2\pi i;$

(ii) $\displaystyle\int_{C_0} (z - z_0)^{n-1}\, dz = 0 \quad (n = \pm1, \pm2, \ldots).$

Solution:

(i)

$$\begin{aligned}\int_{C_0} \frac{dz}{z-z_0} &= \int_{-\pi}^{\pi} \frac{1}{(z_0 + R\,e^{i\theta}) - z_0} \left(iR\,e^{i\theta} d\theta\right) \\ &= i\int_{-\pi}^{\pi} d\theta = i2\pi\end{aligned}$$

(ii)

$$\begin{aligned}\int_{C_0} (z-z_0)^{n-1}\, dz &= \int_{-\pi}^{\pi} \left((z_0 + R\,e^{i\theta}) - z_0\right)^{n-1} \quad \left(iR\,e^{i\theta} d\theta\right) \\ &= i\int_{-\pi}^{\pi} R^n e^{in\theta} d\theta \\ &= \frac{1}{n}\, R^n e^{in\theta}\Big|_{-\pi}^{\pi} \\ &= \frac{1}{n} R^n \left(\left(e^{i\pi}\right)^n - \left(e^{-i\pi}\right)^n\right) \\ &= \frac{1}{n} R^n \left((-1)^n - (-1)^n\right) \\ &= 0\end{aligned}$$

Exercise 1, §43, p. 140

Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that

$$\left|\int_C \frac{dz}{z^2 - 1}\right| \leq \frac{\pi}{3}.$$

Solution:

The length of the arc of the circle $|z| = 2$ in the first quadrant is $\frac{1}{4}$ of the circumference, i.e. $\frac{1}{4}\left(\pi 2^2\right) = \pi$. In addition whenever $|z| = 2$, we have

$$\left|z^2 - 1\right| \geq |z|^2 - 1 = 3, \text{ i.e. } \frac{1}{|z^2 - 1|} \leq \frac{1}{3}.$$

Therefore by (1) in section 43 of the textbook

$$\left|\int_C \frac{1}{z^2 - 1} dz\right| \leq \frac{1}{3}\pi = \frac{\pi}{3}.$$

Exercise 3, §43, p. 140

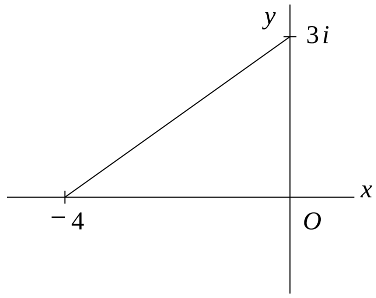
Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction, then

$$\left|\int_C \left(e^z - \overline{z}\right) dz\right| \leq 60.$$

Solution:

Here the length of C is the sum of the distances from 0 to $3i$, $3i$ to -4 and -4 to 0, that is

$$|0 - 3i| + |3i - (-4)| + |-4 - 0| = 12.$$



Now for any z on C we see from the sketch that $\Re(z) \leq 0$ and hence that $|e^z| = e^{\Re(z)} \leq e^0 = 1$. Also on C the point furthest away from the origin (i.e. the point where $|z-0|$ is a maximum) is -4. Therefore on C $|\overline{z}| = |z| = |z-0| \leq |-4-0| = 4$, whence $|e^z - \overline{z}| \leq |e^z| + |\overline{z}| \leq 5$. Consequently

$$\left|\int_C (e^z - \overline{z})\, dz\right| \leq 5 \times 12 = 60.$$

Exercise 5, §43, p. 141

Let C_R be the circle $|z| = R\,(R > 1)$, described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{\text{Log} z}{z^2} dz\right| < 2\pi\left(\frac{\pi + \ln R}{R}\right).$$

Solution:

The circumference of the circle C_R:$|z| = R$ is $2\pi R$. Recall that $-\pi < \text{Arg}(z) \leq \pi$. In addition since, $R > 1$, $\ln R > 0$. Therefore for any z with $|z| = R$ we have

$$\begin{aligned} |\text{Log}\,(z)| &= |\ln R + i\text{Arg}\,(z)| \\ &\leq |\ln R| + |\text{Arg}\,(z)| \\ &\leq \ln R + \pi. \end{aligned}$$

Consequently

$$\left|\int_{C_R} \frac{\text{Log}\,(z)}{z^2} dz\right| \leq \left(\frac{\ln R + \pi}{R^2}\right) 2\pi R = \frac{2\pi\,(\ln R + \pi)}{R}.$$

Exercise 8, §43, p. 141

Let C_N denote the boundary of the square formed by the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi$$

where N is a positive integer, and let the orientation of C_N be counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \geq |\sin x| \quad \text{and} \quad |\sin z| \geq |\sinh y|\,,$$

obtained in Exercises 8(a) and 9(a) of Sec. 34, show that $|\sin z| \geq 1$ on the vertical sides of the square and that $|\sin z| > \sinh(\pi/2)$ on the horizontal sides. Thus show that there is a positive constant A, *independent of* N, such that $|\sin z| \geq A$ for all points z lying on the contour C_N.

(b) Using the final result in part (a), show that

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \leq \frac{16}{(2N+1)\,\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

Solution:

(a) By means of the inequality $|\sin z| \geq |\sin x|$ it follows that on the lines $z = \pm\left(N+\frac{1}{2}\right)\pi + iy$ we have

$$\begin{aligned} |\sin z| &\geq \left|\sin\left(\pm\left(N+\frac{1}{2}\right)\pi\right)\right| \\ &= \left|\pm\sin\left(\left(N+\frac{1}{2}\right)\pi\right)\right| \\ &= \left|(-1)^N\right| \\ &= 1. \end{aligned}$$

Now note that $|\sin z| \geq |\sinh y|$. Therefore on the lines $z = x \pm i\left(N+\frac{1}{2}\right)\pi$ we have

$$\begin{aligned} |\sin z| &\geq \left|\sinh\left(\pm\left(N+\frac{1}{2}\right)\pi\right)\right| \\ &= \left|\pm\sinh\left(\left(N+\frac{1}{2}\right)\pi\right)\right| \\ &\geq \sinh\left(\frac{\pi}{2}\right) \qquad (\sinh \text{ is increasing on } [0,\infty)) \\ &> 1. \end{aligned}$$

Therefore $|\sin z| \geq 1$ on C_N, the perimeter of the square bounded by the lines

$$x = \pm\left(N+\frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N+\frac{1}{2}\right)\pi.$$

(b) Now for any z on C_N we surely have

$$|z| = \sqrt{x^2+y^2} \geq \max\{|x|,|y|\} = \left(N+\frac{1}{2}\right)\pi$$

and hence

$$\left|\frac{1}{z^2\sin z}\right| \leq \frac{1}{\left(N+\frac{1}{2}\right)^2\pi^2} = \frac{4}{(2N+1)^2\pi^2}.$$

In addition the length of C_N is

$$4\times 2\left[\left(N+\frac{1}{2}\right)\pi\right] = 4(2N+1)\pi.$$

Therefore

$$\begin{aligned} \left|\int_{C_N}\frac{dz}{z^2\sin z}\right| &\leq \frac{4}{(2N+1)^2\pi^2}\times 4(2N+1)\pi = \frac{16}{(2N+1)\pi}. \\ &\to 0 \qquad \text{as} \quad N\to\infty. \end{aligned}$$

It follows that $\int_{C_N}\frac{dz}{z^2\sin z} \to 0$ as $N\to\infty$.

Exercise 2, §45, p. 149

By finding an antiderivative, evaluate each of these integrals, where the path is an arbitrary contour between the indicated limits of integration:

(a) $\int_i^{\frac{i}{2}} e^{\pi z}dz$; (b) $\int_0^{\pi+2i}\cos\left(\frac{z}{2}\right)dz$; (c) $\int_1^3 (z-2)^3\,dz$.

Solution:

(a)

$$\int_i^{\frac{i}{2}} e^{\pi z} dz = \frac{1}{\pi} e^{\pi z}\Big|_i^{\frac{i}{2}} = \frac{1}{\pi}\left(e^{\frac{i\pi}{2}} - e^{i\pi}\right) = \frac{1}{\pi}(1+i)$$

(b)

$$\begin{aligned}\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz &= 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i} \\ &= 2\sin\left(\frac{\pi}{2}+i\right) - 0 \\ &= 2\left(\sin\left(\frac{\pi}{2}\right)\cosh 1 + i\cos\left(\frac{\pi}{2}\right)\sinh 1\right) \\ &= 2\cosh 1 \\ &= e + \frac{1}{e}\end{aligned}$$

(c)

$$\int_1^3 (z-2)^3\, dz = \frac{1}{4}(z-2)^4\Big|_1^3 = \frac{1}{4}\left(1^4 - (-1)^4\right) = 0$$

Exercise 5, §45, p. 149

Show that

$$\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where z^i denotes the principal branch

$$z^i = \exp(i\text{Log}z) \qquad (|z|>0,\ -\pi < \text{Arg } z < \pi)$$

and where the path of integration is any contour from $z=-1$ to $z=1$ that, except for its end points, lies above the real axis.

Suggestion: Use an antiderivative of the branch

$$z^i = \exp(i\log z) \qquad \left(|z|>0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the same power function.

Solution:

Except for the endpoints the entire contour lies in the region $0<\theta<\pi$, $|z|>0$. This region lies in both $(|z|>0,\ -\pi < \text{Arg}(z) < \pi)$ and $(|z|>0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2})$. Hence if $\log z$ is the branch $(|z|>0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2})$, then for any z on the given contour (except the endpoints ± 1) we will have that $\text{Log}(z) = \log(z)$ and hence that

$$\exp(i\text{Log}z) = \exp(i\log z).$$

Therefore replacing the one branch of z^i by the other in the integral will not change the value of the integral. We may therefore use the branch

$$z^i = \exp(i\log z) \quad \left(|z|>0,\ -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}\right).$$

For the same branch of $\log z$ the branch

$$z^{(1+i)} = \exp((1+i)\log z) \quad \left(|z|>0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}\right)$$

has the derivative

$$\frac{d}{dz}\left(z^{(1+i)}\right) = (1+i)\,z^i$$

on the region ($|z| > 0, -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$). Therefore $\frac{1}{1+i}\left(z^{(1+i)}\right) = \left(\frac{1}{2} - i\frac{1}{2}\right) z^{(1+i)}$ (with $z^{(1+i)}$) as above) is an antiderivative of z^i on a domain containing the entire contour from -1 to 1. In polar form $1 = e^{i2n\pi}$, $-1 = e^{i(\pi+2m\pi)}$. Selecting n and m so that $-\dfrac{\pi}{2} < 2n\pi < \dfrac{3\pi}{2}$ and $-\dfrac{\pi}{2} < \pi + 2m\pi < \frac{3\pi}{2}$, it follows that

$$\log(1) = \ln 1 + i0 = 0,\ \log(-1) = \ln 1 + i\pi = i\pi$$

and hence that

$$\begin{aligned}
\int_{-1}^{1} z^i dz &= \left(\frac{1}{2} - i\frac{1}{2}\right) z^{(1+i)}\Big|_{-1}^{1} \\
&= \frac{1}{2}(1-i)\exp((1+i)\log(z))\Big|_{-1}^{1} \\
&= \frac{1}{2}(1-i)(\exp(0) - \exp(-\pi + i\pi)) \\
&= \frac{1}{2}(1-i)\left(1 - e^{-\pi}e^{i\pi}\right) \\
&= \left(\frac{1+e^{-\pi}}{2}\right)(1-i).
\end{aligned}$$

Exercise 1, §49, p. 160

Apply the Cauchy–Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$; (b) $f(z) = ze^{-z}$; (c) $f(z) = \dfrac{1}{z^2+2z+2}$;

(d) $f(z) = \operatorname{sech} z$; (e) $f(z) = \tan z$; (f) $f(z) = \operatorname{Log}(z+2)$.

Solution:

Since

$$-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$$

it is clear that

$$\int_C f(z)\,dz = 0 \Leftrightarrow \int_{-C} f(z)\,dz = 0.$$

We may therefore assume that C is positively oriented in each of (a) – (f).

(a) The only point where $f(z) = \dfrac{z^2}{z-3}$ is not differentiable is $z = 3$. Therefore f is analytic inside and on $|z| = 1$, and so by Cauchy's theorem

$$\int_C \frac{z^2}{z-3} dz = 0.$$

(b) $z \to ze^{-z}$ is an entire function and so

$$\int_C ze^{-z} dz = 0$$

by Cauchy's theorem.

(c) $f(z) = \dfrac{1}{z^2+2z+2}$ fails to be differentiable where $z^2+2z+2=0$, i.e. where $z = \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) = -1 \pm i$. However since $|-1 \pm i| = \sqrt{2} > 1$, both these points lie outside C and so by Cauchy's theorem

$$\int_C \frac{1}{z^2+2z+2} dz = 0.$$

(d) $f(z) = \text{sech} z = \dfrac{1}{\cosh z}$ fails to be differentiable where $\cosh z = 0$, i.e. where $z = i\left(n+\dfrac{1}{2}\right)\pi$ $(n \in \mathbb{Z})$. However $\left|i\left(n+\dfrac{1}{2}\right)\dfrac{\pi}{2}\right| \geq \dfrac{\pi}{2} > 1$ for each $n \in \mathbb{Z}$. That is all these points lie outside the circle C and so

$$\int_C \text{sech} z dz = 0$$

by Cauchy's theorem.

(e) $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ fails to be differentiable where $\cos z = 0$, that is where $z = \left(n+\dfrac{1}{2}\right)\pi$ $(n \in \mathbb{Z})$. Since $\left|\left(n+\dfrac{1}{2}\right)\pi\right| \geq \frac{\pi}{2} > 1$, all these points are outside the circle $|z|=1$ and hence

$$\int_C \tan z dz = 0.$$

(f) $z \to \text{Log} z$ fails to be differentiable on the line segment $y=0$, $x \leq 0$. (See (5) in section 31 of the textbook.) Hence $z \to \text{Log}(z+2)$ fails to be differentiable precisely where $y=0$ and $x+2 \leq 0$, i.e. where $y=0$, $x \leq -2$. Since all these points lie outside the circle C, it follows that

$$\int_C \text{Log}\,(z+2)\, dz = 0.$$

Exercise 2, §49, p. 161

Let C_1 denote the positively oriented circle $|z|=4$ and C_2 the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ (Fig. 63). With the aid of the corollary in Sec. 49, point out why

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

when (a) $f(z) = \dfrac{1}{3z^2+1}$; (b) $f(z) = \dfrac{z+2}{\sin(z/2)}$; (c) $f(z) = \dfrac{z}{1-e^z}$.

Solution:

By the corollary in section 49 of the textbook we will have that

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

whenever f is analytic on the closed region consisting of the contours C_1 and C_2, and all the points between them. For each of the functions in (a), (b) and (c) we therefore only need to show that all the points where these functions fail to be differentiable lie either inside C_2 or outside C_1.

(a) $f(z) = \dfrac{1}{3z^2+1}$ fails to be differentiable where $3z^2+1=0$, i.e. where $z = \pm i\frac{1}{\sqrt{3}}$. Both these points clearly lie inside C_2, that is in the region $-1 < x < 1$, $-1 < y < 1$.

(b) $f(z) = \dfrac{z+2}{\sin\left(\frac{z}{2}\right)}$ fails to be differentiable where $\sin\left(\frac{z}{2}\right) = 0$, that is where $z_n = 2n\pi\,(n \in \mathbb{Z})$. Now for $n = 0$, $z_0 = 0$ lies inside C_2. For $n \neq 0$ we have

$$|z_n| = |2n\pi| \geq 2\pi > 4,$$

that is z_n lies outside C_1 when $n \neq 0$.

(c) $f(z) = \dfrac{z}{1-e^z}$ fails to be differentiable where $e^z = 1$, that is where $z_n = i2n\pi\,(n \in \mathbb{Z})$. A similar argument to that used in (b) now reveals that $z_0 = 0$ lies inside C_2, whereas all the other points lie outside C_1.

Exercise 6, §49, p. 163

Let C denote the entire positively oriented boundary of the half disk $0 \leq r \leq 1,\ 0 \leq \theta \leq \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \qquad \left(r > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right)$$

of the multiple–valued function $z^{\frac{1}{2}}$. Show that

$$\int_C f(z)\,dz = 0$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which constitute C. Why does the Cauchy–Goursat theorem not apply here?

Solution:

The half–circle C_1 from 1 to -1 (with $|z| = 1$) which lies in the upper half–plane, lies entirely in the region $\left(|z| > 0,\ -\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}\right)$. By (4) of section 32 in the textbook the branch

$$\frac{2}{3}z^{\frac{3}{2}} = \frac{2}{3}\exp\left(\frac{3}{2}\log z\right) \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}\right)$$

is then an antiderivative of $z^{\frac{1}{2}}$ on this region. Therefore using this branch of $\frac{2}{3}z^{\frac{3}{2}}$, we conclude from the theorem in section 42 that

$$\begin{aligned}
\int_{C_1} z^{\frac{1}{2}}\,dz &= \left.\frac{2}{3}z^{\frac{3}{2}}\right|_1^{-1} \\
&= \frac{2}{3}\left(\exp\left(\frac{3}{2}\log(-1)\right) - \exp\left(\frac{3}{2}\log(1)\right)\right) \\
&= \frac{2}{3}\left(\exp\left(\frac{3}{2}(i\pi)\right) - \exp(0)\right) \\
&= -\frac{2}{3} - i\left(\frac{2}{3}\right).
\end{aligned}$$

(The values of $\log(-1)$ and $\log(1)$ were computed in the solution to exercise 5 of section 43.) For any z on the line segment $(|z| > 0,\ \arg(z) = \pi)$ we have

$$\log(z) = \ln|z| + i\pi.$$

Therefore for any z on the real axis between -1 and 0 we have

$$\begin{aligned} z^{\frac{1}{2}} &= \exp\left(\frac{1}{2}\log z\right) \\ &= \exp\left(\frac{1}{2}\ln|z| + i\frac{\pi}{2}\right) \\ &= \exp\left(\ln\sqrt{|z|}\right)\exp\left(i\frac{\pi}{2}\right) \\ &= i\sqrt{|z|}. \end{aligned}$$

If we parametrise this segment by $C_2 : z(t) = (t-1)\,,\ 0 \le t \le 1$, then

$$\int_{C_2} z^{\frac{1}{2}}dz = \int_{C_2} i\sqrt{|z|}dz = i\int_0^1 \sqrt{1-t}dt = -i\,\frac{2}{3}(1-t)^{\frac{3}{2}}\Big|_0^1 = i\frac{2}{3}.$$

Similarly for any z on the non–negative real axis we have $z^{\frac{1}{2}} = \sqrt{z}$, and hence if C_3 is the line segment $C_3 : z = t,\ 0 \le t \le 1$, from 0 to 1 we have

$$\int_{C_3} z^{\frac{1}{2}}dz = \int_0^1 \sqrt{t}dt = \frac{2}{3}t^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

Therefore

$$\int_C z^{\frac{1}{2}}dz = \sum_{n=1}^{3}\int_{C_n} z^{\frac{1}{2}}dz = 0.$$

Finally note that we may not use Cauchy's theorem since $z^{\frac{1}{2}}$ fails to be differentiable at the point 0 **on** the contour.

Exercise 7, §49, p. 163

Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\int_C \overline{z}dz.$$

Suggestion: Note that expression (4), Sec. 46, can be used here even though the function $f(z) = \overline{z}$ is not analytic anywhere (see Example 2, Sec. 19).

Solution:

Let R denote the interior of the contour C. Then with $u = x$ and $v = -y$, it follows from (4) in section 46 of the textbook that

$$\begin{aligned} \frac{1}{2i}\int_C \overline{z}dz &= \frac{1}{2i}\int_C (x - iy)\,dz \\ &= \frac{1}{2i}\int_C (u + iv)\,dz \\ &= \frac{1}{2i}\left[\iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA\right] \\ &= \frac{1}{2i}\left[\iint_R (-0-0)\,dA + i\iint_R (1-(-1))\,dA\right] \\ &= \iint_R dA \end{aligned}$$

as required.

Exercise 1, §52, p. 170

Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

(a) $\displaystyle\int_C \frac{e^{-z}dz}{z - (\pi i/2)}$; (b) $\displaystyle\int_C \frac{\cos z}{z\,(z^2+8)}dz$ (c) $\displaystyle\int_C \frac{zdz}{2z+1}$;

(d) $\displaystyle\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz \quad (-2 < x_0 < 2)$; (e) $\displaystyle\int_C \frac{\cosh z}{z^4}dz$.

Solution:

(a) Since e^{-z} is an entire function and $i\frac{\pi}{2}$ is inside C, we have that

$$\int_C \frac{e^{-z}}{z - i\frac{\pi}{2}}dz = 2\pi i e^{-(i\pi/2)} = 2\pi i\,(-i) = 2\pi$$

by the Cauchy integral formulae.

(b) Here $\cos z$ is entire with $\dfrac{1}{(z^2+8)}$ failing to be differentiable where $z = \pm i2\sqrt{2}$. However both these points clearly lie outside C and so $\dfrac{\cos z}{z^2+8}$ is analytic inside and on C. Therefore

$$\begin{aligned}\int_C \frac{\cos z}{z\,(z^2+8)}dz &= \int_C \frac{\cos z/\left(z^2+8\right)}{z}dz \\ &= 2\pi i \left(\frac{\cos z}{z^2+8}\right)\bigg|_{z=0} \\ &= i\left(\frac{\pi}{4}\right).\end{aligned}$$

(c) Since $-\frac{1}{2}$ lies inside C we have

$$\int_C \frac{z}{2z+1}dz = \int_C \frac{z/2}{\left(z - (-\frac{1}{2})\right)}dz = 2\pi i \left(\frac{z}{2}\right)\bigg|_{z=-\frac{1}{2}} = -i\frac{\pi}{2}.$$

(d) $\tan(z/2) = \dfrac{\sin\left(\frac{z}{2}\right)}{\cos\left(\frac{z}{2}\right)}$ fails to be differentiable where $\cos(z/2) = 0$, that is where $z = (2n+1)\pi$ $(n \in \mathbb{Z})$. However for any $n, |(2n+1)\pi| \geq \pi > 2$. Therefore all these points lie outside C and so $\tan(z/2)$ is analytic (differentiable) inside and on C. By contrast x_0 lies inside C. Thus

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = \frac{2\pi i}{1!}\left(\frac{d}{dz}\tan(z/2)\right)\bigg|_{z=x_0} = i\pi\sec^2(x_0/2).$$

(e) The function $\cosh z$ is entire and so

$$\int_C \frac{\cosh z}{z^4}dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}\cosh z\right)\bigg|_{z=0} = \frac{\pi i}{3}\sinh 0 = 0.$$

Exercise 3, §52, p. 171

Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}dz \quad (|w| \neq 3),$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

Solution:

The point $z = 2$ clearly lies inside the circle $C : |z| = 3$. Therefore

$$g(2) = \int_C \frac{2z^2 - z - 2}{z - 2} dz = 2\pi i \left(2z^2 - z - 2\right)\Big|_{z=2} = 8\pi i.$$

Now for any fixed w the function

$$z \to \frac{2z^2 - z - 2}{(z - w)}$$

fails to be analytic where $z = w$. Now if $|w| > 3$, this point clearly lies outside C. The above function is then analytic inside and on C, whence by Cauchy's theorem

$$g(w) = \int_C \frac{2z^2 - z - 2}{(z - w)} dz = 0$$

for all such w.

Exercise 5, §52, p. 171

Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z - z_0)^2}.$$

Solution:

If f is analytic inside and on a simple closed contour then so is f' by theorem 1 of section 52 in the textbook. Therefore on applying (6) in section 51 first to f with $n = 1$ and then to f' with $n = 0$, it follows that

$$\int_C \frac{f(z)}{(z - z_0)^2} dz = 2\pi i f'(z_0) = \int_C \frac{f'(z)}{(z - z_0)} dz$$

for every z_0 inside C. If now z_0 is outside C, both $f(z)/(z - z_0)^2$ and $f'(z)/(z - z_0)$ are analytic inside and on C. For such z_0 we then have by Cauchy's theorem that

$$\int_C \frac{f(z)}{(z - z_0)^2} dz = 0 = \int_C \frac{f'(z)}{(z - z_0)} dz.$$

The claim follows.

Exercise 7, §52, p. 171

Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that, for any real constant a,

$$\int_C \frac{e^{az}}{z} dz = 2\pi i.$$

Then write the integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta} \cos(a\sin\theta)\, d\theta = \pi.$$

Solution:

Let C be the unit circle $z = e^{i\theta}$ $(-\pi < \theta \leq \pi)$ centred at 0. Since e^{az} is an entire function, it follows that

$$\int_C \frac{e^{az}}{z} dz = 2\pi i e^{az}\Big|_{z=0} = 2\pi i.$$

In terms of the parametrisation $z = e^{i\theta}$ $(-\pi < \theta \leq \pi)$ the integral becomes

$$\begin{aligned} 2\pi i &= \int_C \frac{e^{az}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{a\exp(i\theta)}}{e^{i\theta}} ie^{i\theta} d\theta \\ &= i\int_{-\pi}^{\pi} e^{a\exp(i\theta)} d\theta \\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta + ia\sin\theta} d\theta \\ &= i\int_{-\pi}^{\pi} e^{a\cos\theta} \left(\cos(a\sin\theta) + i\sin(a\sin\theta)\right) d\theta \\ &= -\int_{-\pi}^{\pi} e^{a\cos\theta} \sin(a\sin\theta)\, d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta} \cos\theta\,(a\sin\theta)\, d\theta. \end{aligned}$$

Comparing imaginary parts we conclude that

$$\int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta = 2\pi.$$

Since

$$\begin{aligned} e^{a\cos(-\theta)} \cos(a\sin(-\theta)) &= e^{a\cos\theta} \cos(-a\sin\theta) \\ &= e^{a\cos\theta} \cos(a\sin\theta), \end{aligned}$$

the integrand is an even function and so

$$\begin{aligned} \int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta &= \frac{1}{2}\int_{-\pi}^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\, d\theta \\ &= \pi. \end{aligned}$$

Exercise 9, §52, p. 172
Verify the expression

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)}{(s-z)^3} ds.$$

Solution:
We show how one may use the binomial formula and mathematical induction to prove (6) of Section 51. The specific case $n = 2$ then corresponds to the solution of the above problem.

Let C be a simple closed contour and z_0 a point inside C. Then by the theorem in section 50

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz.$$

Now suppose that for some integer $k \geq 0$ we have

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{k+1}} dz.$$

Now for $n = k+1$ if follows from this that

$$(\alpha)\begin{cases} \dfrac{f^{(k)}(z_0+\Delta z) - f^{(k)}(z_0)}{\Delta z} \\ = \dfrac{k!}{2\pi i}\displaystyle\int_C \left(\frac{1}{(z-z_0-\Delta z)^{k+1}} - \frac{1}{(z-z_0)^{k+1}} \right) \frac{f(z)}{\Delta z} dz \\ = \dfrac{k!}{2\pi i}\displaystyle\int_C \frac{(z-z_0)^{k+1} - ((z-z_0)-\Delta z)^{k+1}}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+1}} \frac{f(z)}{\Delta z} dz \\ = \dfrac{k!}{2\pi i}\displaystyle\int_C \frac{(z-z_0)^{k+1} - \sum_{m=0}^{k+1}\binom{k+1}{m}(-1)^{k+1-m}(\Delta z)^{k+1-m}(z-z_0)^m}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+1}} \frac{f(z)}{\Delta z} dz \\ = \dfrac{k!}{2\pi i}\displaystyle\int_C \frac{\sum_{m=0}^{k}\binom{k+1}{m}(-1)^{k-m}(\Delta z)^{k-m}(z-z_0)^m}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+1}} f(z)\, dz \\ = \dfrac{k!}{2\pi i}\displaystyle\sum_{m=0}^{k}\binom{k+1}{m}(-1)^{k-m}(\Delta z)^{k-m}\int_C \frac{f(z)\, dz}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+1-m}} \end{cases}$$

where Δz is small enough so that $z_0 + \Delta z$ is inside C. Let M be the maximum of $|f(z)|$ on C, L the length of C and d the shortest distance from z_0 to the points z of C. For any $z \in C$ and $0 < |\Delta z| < d$ we then have

$$|z - z_0| \geq d \qquad |z - z_0 - \Delta z| \geq |z - z_0| - |\Delta z| \geq d - |\Delta z|.$$

Given any m with $0 \leq m < k$ we then have

$$\left| (\Delta z)^{k-m} \int_C \frac{f(z)\, dz}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+1-m}} \right| \leq \frac{|\Delta z|^{k-m} ML}{(d-|\Delta z|)^{k+1} d^{k+1-m}} \to 0 \quad \text{as} \quad \Delta z \to 0.$$

Therefore in the above sum all the terms except the one with $m = k$ tend to 0 as $\Delta z \to 0$. To conclude the proof we show that for the integral in the $m = k$ term we have

$$\int_C \frac{f(z)\, dz}{(z-z_0-\Delta z)^{k+1}(z-z_0)} \to \int_C \frac{f(z)}{(z-z_0)^{k+2}} dz$$

as $\Delta z \to 0$. To see this note that

$$(\beta)\begin{cases} \int_C \left(\dfrac{1}{(z-z_0-\Delta z)^{k+1}(z-z_0)} - \dfrac{1}{(z-z_0)^{k+2}} \right) f(z)\, dz \\ = \int_C \dfrac{(z-z_0)^{k+1} - ((z-z_0)-\Delta z)^{k+1}}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+2}} f(z)\, dz \\ = \int_C \dfrac{(z-z_0)^{k+1} - \sum\limits_{r=0}^{k+1}\binom{k+1}{r}(-\Delta z)^{k+1-r}(z-z_0)^r}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+2}} f(z)\, dz \\ = -\sum\limits_{r=0}^{k}\binom{k+1}{r}(-\Delta z)^{k+1-r}\int_C \dfrac{f(z)\, dz}{(z-z_0-\Delta z)^{k+1}(z-z_0)^{k+2-r}}. \end{cases}$$

Now for any $0 \leq r \leq k$ we see that

$$\begin{aligned} & \left| (-\Delta z)^{k+1-r} \int_C \frac{f(z)\, dz}{(z - z_0 - \Delta z)^{k+1} (z - z_0)^{k+2-r}} \right| \\ \leq \ & \frac{|\Delta z|^{k+1-r} ML}{(d - |\Delta z|)^{k+1} d^{k+2-r}} \to 0 \end{aligned}$$

as $\Delta z \to 0$. Therefore the term in (β) above tends to zero. Applying this to (α) it follows that

$$\begin{aligned} & \frac{f^{(k)}(z_0 + \Delta z) - f^{(k)}(z_0)}{\Delta z} \\ \to \ & \frac{k!}{2\pi i} \binom{k+1}{k} \int_C \frac{f(z)}{(z - z_0)^{k+2}} dz \\ = \ & \frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+2}} dz \end{aligned}$$

as $\Delta z \to 0$. Therefore $f^{(k+1)}(z_0)$ exists and equals $\frac{(k+1)!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+2}} dz$. Thus by induction

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for all $n = 0, 1, 2, \ldots$.

Exercise 10, §52, p. 172

Let f be an entire function such that $|f(z)| \leq A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.

Suggestion: Use Cauchy's inequality (Sec. 52) to show that the second derivative $f''(z)$ is zero everywhere in the plane. Note that the constant M_R in Cauchy's inequality is less than or equal to $A(|z_0| + R)$.

Solution:

Let f be given such that $|f(z)| \leq A|z|$ for all z. Then for any z on the positively oriented circle $C_R : |z - z_0| = R$ we have $|f(z)| \leq A|z| = A|z + (z - z_0)| \leq A(|z_0| + |z - z_0|) = A(|z_0| + R)$. If in addition f is entire, Cauchy's integration formulae ensure that for each z_0

$$f^{(2)}(z_0) = \frac{2!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^3} dz.$$

Since the circumference of C_R is $2\pi R$, estimating the integral yields

$$\left| f^{(2)}(z_0) \right| \leq \frac{1}{\pi} \left(\frac{A|z_0| + R}{R^3} \right) 2\pi R.$$

As $R \to \infty$, the right–hand side tends to 0 thereby proving that $f^{(2)}(z_0) = 0$. Since z_0 was arbitrary this shows that $f^{(2)} = 0$, and hence that $f(z) = a_1 z + a_0$. Finally note that since $|f(z)| \leq A|z|$ for **all** z, $f(0)$ must be 0, that is $a_0 = 0$. Therefore $f(z) = a_1 z$.

Exercise 1, §54, p. 178

Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \Re[f(z)]$ has an upper bound; that is, $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane.

Suggestion: Apply Liouville's theorem (Sec. 53) to the function $g(z) = \exp[f(z)]$.

Solution:
Let $f(z) = u(x,y) + iv(x,y)$ be entire. Since e^z is also entire, so is $e^{f(z)}$. Now note that

$$\left|e^{f(z)}\right| = e^{u(x,y)}$$

by (7) in section 29. If therefore $u(x,y) \leq u_0$ for some u_0 and all z, $e^{f(z)}$ is a bounded entire function since then

$$\left|e^{f(z)}\right| = e^{u(x,y)} \leq e^{u_0}$$

for all z. By Liouville's theorem $e^{f(z)}$, and therefore also $\left|e^{f(z)}\right| = e^{u(x,y)}$, must then be constant. Thus $u(x,y) = \ln\left(e^{u(x,y)}\right)$ is constant.

Exercise 3, §54, p. 179
Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values (Sec. 54) to the function $g(z) = 1/f(z)$.

Solution:
If $f(z_0) = 0$ for some z_0 on the perimeter of R, then $|f(z_0)| = 0$ is surely the minimum of $|f|$ in which case we are done as $|f(z)| > 0$ for all z in the interior of R.

If $f(z) \neq 0$ on the perimeter of R as well, then $\frac{1}{f}$ is continuous on R and analytic on its interior since the same is true of f. By the maximum modulus principle $\left|\frac{1}{f}\right|$ then assumes its maximum on the perimeter of R and not the interior. Since

$$\max\left(\frac{1}{|f(z)|}\right) = \frac{1}{(\min|f(z)|)},$$

this proves that $|f|$ assumes its minimum on the perimeter of R and not the interior.

Exercise 6, §54, p. 179
Let $f(z) = u(x,y) + iv(x,y)$ be a function that is continuous in a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x,y)$ has a minimum value in R which occurs on the boundary of R and never in the interior. (See Exercise 3.)

Solution:
Since $f(z) = u(x,y) + iv(x,y)$ is continuous on the closed bounded region R and analytic on its interior, the same is true of $e^{f(z)}$. By (6) of section 29, $e^{f(z)} \neq 0$ on R. Therefore by what we showed in exercise 1 above $\left|e^{f(z)}\right| = e^{u(x,y)}$ assumes its minimum on the perimeter of R and not the interior. Since ln is increasing on $(0,\infty)$ and $e^{u(x,y)} > 0$, the same must then be true of $\ln\left(e^{u(x,y)}\right) = u(x,y)$.

Exercise 7, §54, p. 179
Let f be the function $f(z) = e^z$ and R the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$. Illustrate results in Sec. 54 and Exercise 6 by finding points in R where the component function $u(x,y) = \Re[f(z)]$ reaches its maximum and minimum values.

Solution:

For $f(z) = e^z$ we clearly have $\Re[f(z)] = e^x \cos y$. Now in the rectangular region $0 \leq x \leq 1$, $0 \leq y \leq \pi$ e^x increases from 1 to e as x varies from 0 to 1, and $\cos y$ decreases from 1 to -1 as y varies from 0 to π. Therefore $e^x \cos y$ assumes a maximum of e at the point $z = 1 + i0 = 1$ and a minimum of $-e$ at the point $z = 1 + i\pi$ on the perimeter of this rectangle.

CHAPTER 5

Series

Study all sections. Of this material sections 55, 56 and 63 are largely revision of material dealt with in MAT2613 and will therefore not be tested directly. The student should nevertheless be conversant with the concepts and results contained in these sections. In addition since Theorems 1 and 2 of section 66 follow fairly directly from Theorem 1 of section 65, the proofs of these two theorems need not be studied.

The basic goal of this chapter is to investigate the extent to which complex functions can be written as power series. The two theorems which form the cornerstones of this chapter are Taylor's theorem, and Laurent's theorem. Taylor's theorem establishes yet another very important fact regarding analytic functions. If a function is analytic in some region of the form $|z-a|<R$, then in that region the function can be written as a power series centred at a. But by the corollary to Theorem 1 in section 65, the converse is also true! Together these two results establish the fact that a function is analytic at a point if and only if it can be written as a power series in some neighbourhood of that point. The analytic functions are therefore precisely the class of functions which allow for power series expansions.

But what if we are interested in the behaviour of a function f at a given point a, and f is analytic all around this point, but not at the actual point we are interested in? Is there anything that can be said about such functions as far as power series expansions are concerned? Because f is not analytic at a itself, Taylor's theorem is no longer valid. However all is not lost. In the absence of Taylor's theorem, Laurent's theorem comes to our rescue. What this theorem tells us is that if f is analytic on a region of the form $r<|z-a|<R$ (which excludes the troublesome point a) then provided we allow negative powers of $z-a$, we can in that region still write f as a series of powers of $z-a$. Such an expansion in powers of $z-a$ we will then call a Laurent series for f in the region $r<|z-a|<R$.

When studying the behaviour of functions around isolated singular points it is this theory of Laurent series that proves to be especially helpful. So if we want to be able to study the behaviour of complex functions at such points, we clearly need a bit of practice in dealing with Laurent series. In this regard we suggest that example 3 of section 62 and the material in section 67 be studied with care as they demonstrate a wide range of techniques which prove to be very useful in the computation of Laurent series.

We close our overview of this chapter by presenting some hints on computing Laurent series of rational functions. If approached correctly then at least for rational functions, this should be a reasonably straightforward process. In short the idea runs as follows: If $f(z)=\frac{q(z)}{p(z)}$, where p and q are polynomials, needs to be expanded as a Laurent series centred at a (i.e. in powers of $z-a$) we can use Taylor's theorem to rewrite $q(z)$ in powers of $(z-a)$. Next we can use partial fractions to decompose $\frac{1}{p(z)}$ into a sum of terms of the form $\frac{K}{(z-b)^n}$. Each of these terms are then separately written in Laurent form and finally all the various expansions for $q(z)$ and the $\frac{K}{(z-b)^n}$'s combined algebraically to get the Laurent expansion for $f(z) = \frac{q(z)}{p(z)}$. So all that remains is to explain how to write terms like $\frac{1}{(z-b)^n}$ in Laurent–form.

However before doing so we note that a function f may have several Laurent expansions centred at $z = a$ depending on the number of singularities it has. A careful look Laurent's theorem reveals that if a particular form of the Laurent expansion in powers of $(z-a)$ holds at a point (say w_0) then this form will converge on the largest annulus of the form $S < |z-a| < R$ (where $0 \leq S < R$) which contains w_0 and on which f is still analytic! As soon as we leave the annulus and move beyond a point where f is **NOT** analytic, the expansion changes. For example if f has say two singularities at say z_0 and z_1 with say

$$0 < R_0 = |z_0 - a| < R_1 = |z_1 - a|$$

then f will be analytic on the annuli

$$|z-a| < R_0, \; R_0 < |z-a| < R_1 \quad \text{and} \quad R_1 < |z-a| < \infty,$$

but not on the circles $|z-a| = R_0$ and $|z-a| = R_1$ as such, since these contain z_0 and z_1. So by Laurent's Theorem on each of the annuli $|z-a| < R_0$, $R_0 < |z-a| < R_1$ and $R_1 < |z-a| < \infty$, f will have some Laurent expansion, however the expansion may be different on each of these sets!

We finally indicate how a term like $\frac{1}{z-b}$ (where $b \neq a$) may be written as a Laurent series centred at $z = a$. The expansions for $\frac{1}{(z-b)^2}$, $\frac{1}{(z-b)^3}$, ... may then be obtained from this one by differentiating the expansion for $\frac{1}{z-b}$. (Note for example that $\frac{1}{(z-b)^4} = -\frac{1}{6}\frac{d^3}{dz^3}\frac{1}{(z-b)}$.) Now we know that

$$\frac{1}{1-w} = 1 + w + w^2 + \ldots = \sum_{n=0}^{\infty} w^n \qquad (|w| < 1).$$

If therefore $\left|\frac{z-a}{b-a}\right| < 1$ (i.e. $|z-a| < |b-a|$) we may write

$$\frac{1}{z-b} = \frac{-1}{(b-a)} \frac{1}{1 - \left(\frac{z-a}{b-a}\right)}$$

and set $w = \left(\frac{z-a}{b-a}\right)$ to get

$$\frac{1}{z-b} = \frac{-1}{b-a} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(b-a)^n} = \sum_{n=0}^{\infty} \frac{-1}{(b-a)^{n+1}} (z-a)^n.$$

If on the other hand $\left|\frac{b-a}{z-a}\right| < 1$ (i.e. $|z-a| > |b-a|$) we set $w = \left(\frac{b-a}{z-a}\right)$ to get

$$\begin{aligned} \frac{1}{z-b} &= \frac{1}{z-a}\left(\frac{1}{1-\left(\frac{b-a}{z-a}\right)}\right) \\ &= \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{b-a}{z-a}\right)^n \\ &= \sum_{m=1}^{\infty}(b-a)^{m-1}\frac{1}{(z-a)^m}. \end{aligned}$$

Suppose for example that we are asked to compute the Laurent series of $\frac{z^2-8z}{(z-2)^2(z+1)}$ in each of the following regions: (i) $|z-1| < 1$, and (ii) $1 < |z-1| < 2$. So for each of these regions we must find an expansion in powers of $(z-1)$ that converges on the given region.

First of all note that by means of partial fractions we can see that

$$\frac{z^2-8z}{(z-2)^2(z+1)} = \frac{1}{z+1} - \frac{4}{(z-2)^2}.$$

We first consider the term $\frac{1}{z+1}$. If $|z-1|<2$ (equivalently $|\frac{z-1}{-2}|<1$), then

$$\begin{aligned}
\frac{1}{z+1} &= \frac{1}{2}\left(\frac{1}{1-(\frac{z-1}{-2})}\right)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{-2}\right)^n\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{n+1}}\\
&= \frac{1}{2} - \frac{(z-1)}{2^2} + \frac{(z-1)^2}{2^3}\cdots
\end{aligned}$$

If however $|z-1|>2$ (equivalently $|\frac{-2}{z-1}|<1$), then

$$\begin{aligned}
\frac{1}{z+1} &= \frac{1}{z-1}\left(\frac{1}{1-(\frac{-2}{z-1})}\right)\\
&= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{-2}{z-1}\right)^n\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{n+1}}\\
&= \frac{1}{(z-1)} - \frac{2}{(z-1)^2} + \frac{2^2}{(z-1)^3}\cdots
\end{aligned}$$

Now consider the term $\frac{1}{z-2}$. If $|z-1|<1$, then

$$\begin{aligned}
\frac{1}{z-2} &= (-1)\left(\frac{1}{1-(z-1)}\right)\\
&= (-1)\sum_{n=0}^{\infty}(z-1)^n\\
&= -1-(z-1)-(z-1)^2\cdots
\end{aligned}$$

and if $|z-1|>1$ (equivalently $|\frac{1}{z-1}|<1$), then

$$\begin{aligned}
\frac{1}{z-2} &= \frac{1}{z-1}\left(\frac{1}{1-(\frac{1}{z-1})}\right)\\
&= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n\\
&= \frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}\cdots
\end{aligned}$$

On differentiating these two expressions and changing the sign we get

$$\begin{aligned}\frac{1}{(z-2)^2} &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z-2}\right)\\ &= 1+2(z-1)+3(z-1)^2\ldots\\ &= \sum_{n=0}^{\infty}(n+1)(z-1)^n \quad \text{whenever} \quad |z-1|<1,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{(z-2)^2} &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z-2}\right)\\ &= \frac{1}{(z-1)^2}+\frac{2}{(z-1)^3}+\frac{3}{(z-1)^4}\ldots\\ &= \sum_{n=0}^{\infty}\frac{n+1}{(z-1)^{n+2}} \quad \text{whenever} \quad |z-1|>1.\end{aligned}$$

In the region $|z-1|<1$ it will of course also hold that $|z-1|<2$. So in this region

$$\begin{aligned}\frac{z^2-8z}{(z-2)^2(z+1)} &= \frac{1}{z+1}-\frac{4}{(z-2)^2}\\ &= \left(\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{n+1}}\right)-4\left(\sum_{n=0}^{\infty}(n+1)(z-1)^n\right)\\ &= (\frac{1}{2}-4)+(\frac{-1}{4}-8)(z-1)+(\frac{1}{8}-12)(z-1)^2\ldots\\ &= -3.5-8.25(z-1)-11.875(z-1)^2\ldots\end{aligned}$$

In the region $1<|z-1|<2$ we have $1<|z-1|$ AND $|z-1|<2$. So in this region

$$\begin{aligned}\frac{z^2-8z}{(z-2)^2(z+1)} &= \frac{1}{z+1}-\frac{4}{(z-2)^2}\\ &= \left(\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{2^{n+1}}\right)-4\left(\sum_{n=0}^{\infty}\frac{n+1}{(z-1)^{n+2}}\right)\\ &= \cdots-\frac{8}{(z-1)^3}-\frac{4}{(z-1)^2}+\frac{1}{2}-\frac{(z-1)}{2^2}+\frac{(z-1)^2}{2^3}\ldots\end{aligned}$$

Solutions to selected problems

Exercise 1, §56, p. 188
Use definition (2), Sec. 55, of limits to verify the limit of the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots)$$

found in Example 2, Sec. 55.

Solution:
For $z_n = -2 + i\frac{(-1)^n}{n^2}$ we have

$$|z_n - (-2)| = \left| i\frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}.$$

Given any $\varepsilon > 0$ we may select $N_\varepsilon \in \mathbb{N}$ so that $N_\varepsilon > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n \geq N_\varepsilon \Rightarrow \varepsilon > \frac{1}{N_\varepsilon^2} \geq \frac{1}{n^2} \Rightarrow |z_n - (-2)| = \frac{1}{n^2} < \varepsilon.$$

Hence $z_n \to -2$.

Exercise 2, §56, p. 188
Let θ_n denote the principal values of the arguments of the complex numbers

$$z_n = 2 + i\frac{(-1)^n}{n^2} \qquad (n = 1, 2, \ldots).$$

Point out why $\theta_n \to 0$, and compare this with Example 2, Sec. 55.

Solution:
For $n = 2k$ even, we have $z_{2k} = 2 + i\frac{1}{4k^2}$. This is a point in the first quadrant for which $\tan(\theta_{2k}) = \frac{1}{8k^2}$ (where $\theta_{2k} = \text{Arg}(z_{2k})$). Hence

$$\theta_{2k} = \arctan\left(\frac{1}{8k^2}\right) \to 0.$$

Similarly for $n = 2k - 1$ odd, $z_{2k-1} = 2 - i\frac{1}{(2k-1)^2}$ is a point in the fourth quadrant and hence here

$$\theta_{2k-1} = \arctan\left(\frac{-1}{2(2k-1)^2}\right) = \arctan\left(\frac{-1}{2(2k-1)^2}\right) \to 0$$

where $\theta_{2k-1} = \text{Arg}(z_{2k-1})$. Clearly (θ_n) then converges to 0.

Exercise 3, §56, p. 188
Use the inequality (see Sec. 4) $||z_n| - |z|| \leq |z_n - z|$ to show that

$$\text{if} \quad \lim_{n\to\infty} z_n = z, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |z|.$$

Solution:
Let (z_n) and z be given. By (5) in section 4 of the textbook

$$||z_n| - |z|| \leq |z_n - z| \qquad \text{for all } n \in \mathbb{N}.$$

Now if $\lim z_n = z$, then given $\varepsilon > 0$ we can find N_ε so that $|z_n - z| < \varepsilon$ whenever $n \geq N_\varepsilon$. But by the above inequality we then have that

$$||z|_n - |z|| < \varepsilon \qquad \text{whenever } n \geq N_\varepsilon.$$

Clearly this means that $\lim_n |z_n| = |z|$.

Exercise 4, §56, p. 188

Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula (10), Sec. 56. Then, with the aid of the theorem in Sec. 56, show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

when $0 < r < 1$. (Note that these formulas are also valid when $r = 0$.)

Solution:

For $z = re^{i\theta}$ (with $0 < r < 1$) we have $|z| = r < 1$. Hence by the formula in Section 56,

$$\sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1-re^{i\theta}}.$$

We proceed to compute the real and imaginary parts of the term $1/\left(1 - re^{i\theta}\right)$.

$$\begin{aligned}
\frac{1}{1-re^{i\theta}} &= \frac{1}{(1 - r\cos\theta) - ir\sin\theta} \times \frac{(1 - r\cos\theta) + ir\sin\theta}{(1 - r\cos\theta) + ir\sin\theta} \\
&= \frac{(1 - r\cos\theta) + ir\sin\theta}{(1 - r\cos\theta)^2 + r^2\sin^2\theta} \\
&= \frac{(1 - r\cos\theta) + ir\sin\theta}{1 - 2r\cos\theta + r^2\left(\cos^2\theta + \sin^2\theta\right)} \\
&= \frac{(1 - r\cos\theta)}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1 - 2r\cos\theta + r^2}.
\end{aligned}$$

Now since $\sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} r^n \left(\cos n\theta + i\sin n\theta\right) = \sum_{n=0}^{\infty} r^n \cos n\theta + i\sum_{n=0}^{\infty} r^n \sin n\theta$, it easily follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n \cos n\theta &= \Re\left(\sum_{n=0}^{\infty} r^n e^{in\theta}\right) \\
&= \Re\left(\frac{1}{1-re^{i\theta}}\right) \\
&= \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}.
\end{aligned}$$

Finally subtract the $n = 0$ term (ie. 1) from both sides to get the answer. Similarly

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

Exercise 1, §59, p. 195

Obtain the Maclaurin series representation

$$z\cosh\left(z^2\right) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (\,|z| < \infty).$$

Solution:

From example 3 of section 59 in the textbook we know that

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Therefore

$$z\cosh\left(z^2\right) = z\left(\sum_{n=0}^{\infty}\frac{\left(z^2\right)^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}.$$

Exercise 3, §59, p. 196
Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+(z^4/9)}.$$

Solution:
From the example in section 56 of the textbook we know that

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad \text{if} \quad |w|<1.$$

Therefore

$$\frac{1}{1+w} = \frac{1}{1-(-w)} = \sum_{n=0}^{\infty}(-w)^n = \sum_{n=0}^{\infty}(-1)\,w^n \quad \text{if} \quad |w|<1.$$

Consequently

$$\begin{aligned} f(z) &= \frac{z}{z^4+9} = \frac{z}{9}\left(\frac{1}{1+\frac{z^4}{9}}\right) \\ &= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\frac{z^{4n}}{9^n} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{9^{n+1}} \end{aligned}$$

where the series converges whenever $\left|\frac{z^4}{9}\right| < 1$, i.e. when $|z| < 9^{\frac{1}{4}} = \sqrt{3}$.

Exercise 6, §59, p. 196
Use representation (2), Sec. 59, for $\sin(z)$ to write the Maclaurin series for the function $f(z) = \sin\left(z^2\right)$, and point out how it follows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0 \quad (n = 0, 1, 2, \ldots).$$

Solution:
From example 2 in section 59 of the textbook we know that

$$\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \quad (|z|<\infty).$$

Therefore

$$\sin\left(z^2\right) = \sum_{n=0}^{\infty}(-1)^n\frac{\left(z^2\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+2}}{(2n+1)!}.$$

Now by Taylor's theorem the coefficients of the z^{4n} and z^{2n+1} terms in the Maclaurin series are exactly

$$\frac{f^{(4n)}(0)}{(4n)!} \quad \text{and} \quad \frac{f^{(2n+1)}(0)}{(2n+1)!} \qquad n = 0, 1, 2, \ldots .$$

But from the series expansion we just obtained it is clear that these coefficients must all be zero. This can only be the case if

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0$$

for all $n = 0, 1, 2, \ldots$.

Exercise 7, §59, p. 196

Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad \left(|z-i| < \sqrt{2}\right).$$

Suggestion: Start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

Solution:

We compute the Taylor series expansion of $\frac{1}{1-z}$ at $z = i$. Notice that

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\left(\frac{1}{1-(z-i)/(1-i)}\right).$$

Recall that $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ $(|w| < 1)$. Therefore on setting $w = (z-i)/(1-i)$ it follows that

$$\frac{1}{1-z} = \frac{1}{(1-i)}\left(\frac{1}{1-(z-1)/(1-i)}\right) = \frac{1}{(1-i)} \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^n} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

whenever $\left|\frac{z-i}{1-i}\right| < 1$, i.e. whenever $|z-i| < |1-i| = \sqrt{2}$.

Exercise 9, §59, p. 197

Use the identity $\sinh(z+\pi i) = \sinh z$, verified in Exercise 7(a), Sec 35., and the fact that $\sinh z$ is periodic with a period of $2\pi i$, to find the Taylor series for $\sinh z$ about the point $z = \pi i$.

Solution:

If we set $w = z - \pi i$, we have that

$$\sinh z = \sinh(w + \pi i) = -\sinh w$$

On applying (4) of Example 3 in Sec. 59 to $\sinh w$, it now follows that

$$\sinh z = -\sinh w = -\sum_{n=0}^{\infty} \frac{w^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{-1}{(2n+1)!} (z-\pi i)^{2n+1}.$$

Exercise 10, §59, p. 197

What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

Solution:

By Taylor's theorem the Taylor series centred at 0 (the Maclaurin series) for $\tanh z$ will exist on the largest disc of the form $|z| < R_0$ on which $\tanh z$ is still analytic. Therefore the largest circle within which the Maclaurin series will converge is the circle $|z| = R_0$ where R_0 is the distance from 0 to the nearest point where $\tanh z$ fails to be analytic. Now $\tanh z = \frac{\sinh z}{\cosh z}$ fails to be analytic where $\cosh z = 0$, i.e. where $z = (2k+1)\pi i$ $(k \in \mathbb{Z})$. The singularities closest to 0 are $\pm\pi i$ and so the Maclaurin series converges inside the circle $|z| = |\pm\pi i| = \pi$. The first three derivatives of $f(z) = \tanh z$ are

$$\begin{aligned} f^{(1)}(z) &= \operatorname{sech}^2 z,\ f^{(2)}(z) = 2\operatorname{sech}^2 z \tanh z \\ f^{(3)}(z) &= 4\operatorname{sech}^2 z \tanh^2 z + 2\operatorname{sech}^4 z. \end{aligned}$$

Therefore the coefficients of the first four terms of the Maclaurin series will be

$$\begin{aligned} f(0) &= \tanh 0 = 0, & \frac{f^{(1)}(0)}{1!} &= \frac{1}{1!}\operatorname{sech}^2 0 = 1 \\ \frac{f^{(2)}(0)}{2!} &= \frac{1}{2!}0 = 0; & \frac{f^{(3)}(0)}{3!} &= \frac{1}{3!}(4 \times 0 + 2) = \frac{1}{3}. \end{aligned}$$

with the first two nonzero terms in the series being $z + \frac{1}{3}z^3 + \ldots$.

Exercise 11, §59, p. 197

Show that when $z \neq 0$,

(a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \cdots$; (b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots$.

Solution:

(a) From example 1 of section 59 we have that $e^z = \sum_{n=0}^{\infty} \frac{1}{n!}z^n = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots$. Therefore

$$\begin{aligned} \frac{e^z}{z^2} &= \frac{1}{z^2}\left(1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{1}{3!}z + \frac{1}{4!}z^2 + \cdots \end{aligned}$$

(b) From example 2 of section 59, $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$. Therefore

$$\begin{aligned} \frac{\sin(z^2)}{z^4} &= \frac{1}{z^4}\left(\sin(z^2)\right) = \frac{1}{z^4}\left(\sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n-2}}{(2n+1)!} \\ &= \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots \end{aligned}$$

Exercise 12, §59, p. 197

Derive the expansions

(a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+3)!}$; (b) $z^3 \cosh\left(\dfrac{1}{z}\right) = \dfrac{z}{2} + z^3 + \sum_{n=1}^{\infty} \dfrac{1}{(2n+2)!} \cdot \dfrac{1}{z^{2n-1}}$

where $0 < |z| < \infty$.

Solution:

(a) From example 3 of section 59

$$\sinh z = z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \frac{1}{7!}z^7 + \cdots$$

and hence

$$\begin{aligned}\frac{\sinh z}{z^2} &= \frac{1}{z} + \frac{1}{3!}z + \frac{1}{5!}z^3 + \frac{1}{z!}z^5 + \cdots \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{1}{(2n+3)!} z^{2n+1}.\end{aligned}$$

(b) From example 3 of section 59

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \cdots .$$

Therefore

$$\begin{aligned} z^3 \cosh\left(\frac{1}{z}\right) &= z^3\left(1 + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{4!}\left(\frac{1}{z}\right)^4 + \frac{1}{6!}\left(\frac{1}{z}\right)^6 + \cdots\right) \\ &= z^3 + \frac{1}{2}z + \frac{1}{4!}\frac{1}{z} + \frac{1}{6!}\frac{1}{z^3} + \frac{1}{8!}\frac{1}{z^5} + \cdots \\ &= z^3 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!}\frac{1}{z^{2n-1}}.\end{aligned}$$

Exercise 1, §62, p. 205

Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

Solution:

From example 2 of section 59 $\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$ $(|w| < \infty)$. Therefore on setting $w = \frac{1}{z^2}$ we get

$$\begin{aligned} z^2 \sin\left(\frac{1}{z^2}\right) &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z^2}\right)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{4n}}. \qquad (0 < |z| < \infty)\end{aligned}$$

Exercise 2, §62, p. 205

Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right] \qquad (0 < |z+1| < \infty).$$

Solution:

For any $w \in \mathbb{C}$ we have $e^w = \sum\limits_{n=0}^{\infty} \frac{1}{n!} w^n$. Therefore $ee^z = e^{z+1} = \sum_{n=0}^{\infty} \frac{1}{n!}(z+1)^n$, whence

$$\begin{aligned}
\frac{e^z}{(z+1)^2} &= \frac{1}{(z+1)^2}\left[\frac{1}{e}\sum_{n=0}^{\infty}\frac{1}{n!}(z+1)^n\right] \\
&= \frac{1}{e}\left[\frac{1}{(z+1)^2}+\frac{1}{(z+1)}+\frac{1}{2!}+\frac{(z+1)}{3!}+\frac{(z+1)^2}{4!}+\cdots\right] \\
&= \frac{1}{e}\left[\frac{1}{(z+1)^2}+\frac{1}{(z+1)}+\sum_{n=0}^{\infty}\frac{1}{(n+2)!}(z+1)^n\right].
\end{aligned}$$

Exercise 4, §62, p. 206

Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

Solution:

We know from the example in section 56 that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad (|z| < 1)$$

Therefore for $0 < |z| < 1$ we surely have

$$\begin{aligned}
\frac{1}{z^2(1-z)} &= \frac{1}{z^2}\left(1+z+z^2+z^3+\cdots\right) \\
&= \frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\cdots \\
&= \frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty} z^n.
\end{aligned}$$

Now when $\left|\frac{1}{z}\right| < 1$ (i.e. when $1 < |z| < \infty$), it follows that

$$\begin{aligned}
\frac{1}{1-z} &= \left(-\frac{1}{z}\right)\left(\frac{1}{1-\left(\frac{1}{z}\right)}\right) = -\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right) \\
&= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots.
\end{aligned}$$

Therefore

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^3}-\frac{1}{z^4}-\frac{1}{z^5}-\cdots = -\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

Exercise 6, §62, p. 206
Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Solution:
To expand $\dfrac{z}{(z-1)(z-3)}$ in powers of $(z-1)$ we may write

$$\begin{aligned}\frac{1}{z-3} = \frac{1}{(z-1)-2} &= \left(-\frac{1}{2}\right)\left(\frac{1}{1-\left(\frac{z-1}{2}\right)}\right) \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}.\end{aligned}$$

whenever $\left|\dfrac{z-1}{2}\right| < 1$ (i.e. $|z-1| < 2$). Therefore for $0 < |z-1| < 2$ we get

$$\begin{aligned}\frac{z}{(z-1)(z-3)} &= \frac{z}{z-1}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}\right) \\ &= \left(1+\frac{1}{z-1}\right)\left(-\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}}\right) \\ &= -\frac{1}{2}\frac{1}{(z-1)} - \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}+\frac{1}{2^{n+2}}\right)(z-1)^n \\ &= -\frac{1}{2(z-1)} - \sum_{n=0}^{\infty}\frac{3}{2^{n+2}}(z-1)^n.\end{aligned}$$

Exercise 7, §62, p. 206
Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Solution:
For $\left|-z^2\right| = |z|^2 < 1$ (i.e. $|z| < 1$) we have that

$$\begin{aligned}\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} &= 1 + \left(-z^2\right) + \left(-z^2\right)^2 + \left(-z^2\right)^3 + \cdots \\ &= 1 - z^2 + z^4 - z^6 + z^8 - \cdots.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{z(1+z^2)} &= \frac{1}{z} - z + z^3 - z^5 + z^7 - \cdots \\ &= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}. \qquad (0 < |z| < 1)\end{aligned}$$

If on the other hand $\left|-\frac{1}{z^2}\right| < 1$ (i.e. $1 < |z| < \infty$), then

$$\frac{1}{1+z^2} = \frac{1}{z^2\left(1-\left(-\frac{1}{z^2}\right)\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}.$$

Thus

$$\frac{1}{z\left(1+z^{2}\right)}=\frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{z^{2n+2}}=\sum_{n=0}^{\infty}\frac{(-1)^{n}}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$$

whenever $1<|z|<\infty$.

Exercise 10, §62, p. 207

(a) Let z be any complex number, and let C denote the unit circle

$$w=e^{i\phi}\qquad(-\pi\leq\phi\leq\pi)$$

in the w plane. Then use that contour in expression (5), Sec. 60, for the coefficients in a Laurent series, adapted to such series about the origin in the w plane, to show that

$$\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]=\sum_{n=-\infty}^{\infty}J_{n}(z)\,w^{n}\qquad(0<|w|<\infty),$$

where

$$J_{n}(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left[-i\left(n\phi-z\sin\phi\right)\right]d\phi\qquad(n=0,\pm1,\pm2,\dots).$$

(b) With the aid of Exercise 5, Sec. 38, regarding certain definite integrals of even and odd complex–valued functions of a real variable, show that the coefficients in part (a) can be written

$$J_{n}(z)=\frac{1}{\pi}\int_{0}^{\pi}\cos\left(n\phi-z\sin\phi\right)d\phi\qquad(n=0,\pm1,\pm2,\dots).$$

Solution:

(a) Let C be the positively oriented unit circle $|w|=1$ parametrised by

$$w=e^{i\theta}\qquad-\pi<\theta\leq\pi.$$

For any fixed $z\in\mathbb{C}$ it then follows from the Theorem in section 60 that we can write the function

$$f_{z}(w)=\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]$$

as a Laurent series (in powers of w) at $w=0$

$$f_{z}(w)=\sum_{n=-\infty}^{\infty}J_{n}(z)\,w^{n}$$

where the coefficients $J_{n}(z)$ are given by

$$\begin{aligned}J_{n}(z)&=\frac{1}{2\pi i}\int_{C}\frac{f_{z}(w)}{w^{n+1}}dw\\&=\frac{1}{2\pi i}\int_{C}\frac{\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]}{w^{n+1}}dw.\end{aligned}$$

However for $w=e^{i\theta}$ we have

$$w-\frac{1}{w}=e^{i\theta}-e^{-i\theta}=2i\left[\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)\right]=2i\sin\theta$$

and so in terms of the given parametrisation we get

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]}{w^{n+1}}dw \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\exp\left[\frac{z}{2}(2i\sin\theta)\right]}{\left(e^{i\theta}\right)^{n+1}} ie^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta}e^{-in\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i[n\theta - z\sin\theta]}d\theta. \end{aligned}$$

(b) Let $J_n(z)$ be as in part (a). Notice that then

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i[n\theta - z\sin\theta]}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\cos(n\theta - z\sin\theta) - i\sin(n\theta - z\sin\theta)\right)d\theta \\ &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\cos(n\theta - z\sin\theta)\,d\theta - i\int_{-\pi}^{\pi}\sin(n\theta - z\sin\theta)\,d\theta\right]. \end{aligned}$$

Next observe that since

$$\begin{aligned} \cos(n(-\theta) - z\sin(-\theta)) &= \cos(-[n\theta - z\sin\theta]) \\ &= \cos(n\theta - z\sin\theta) \end{aligned}$$

and

$$\begin{aligned} \sin(n(-\theta) - z\sin(-\theta)) &= \sin(-[n\theta - z\sin\theta]) \\ &= -\sin(n\theta - z\sin\theta) \end{aligned}$$

the function $\theta \to \cos(n\theta - z\sin\theta)$ is even whereas $\theta \to \sin(n\theta - z\sin\theta)$ is odd. Therefore by exercise 5 of section 38

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\theta - z\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(n\theta - z\sin\theta)\,d\theta = 0.$$

Consequently

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta.$$

Exercise 2, §66, p. 219

By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z^2)} = \sum_{n=0}^{\infty}(n+1)z^n \qquad (|z|<1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1<|z-1|<\infty).$$

(Compare Example 2, Sec. 65.)

Solution:
We know that

$$\frac{1}{1-w}=\sum_{n=2}^{\infty}w^n \qquad |w|<1.$$

If now we differentiate term–wise then by Theorem 2 in section 65 we have

$$\frac{1}{(1-w)^2}=\frac{d}{dw}\left(\frac{1}{1-w}\right)=\sum_{n=0}^{\infty}nw^{n-1}=\sum_{n=0}^{\infty}(n+1)\,w^n$$

whenever $|w|<1$. If next we set $w=\dfrac{1}{1-z}$, it follows that

$$\left(\frac{z-1}{z}\right)^2=\frac{1}{\left(1-\left(\frac{1}{1-z}\right)\right)^2}=\sum_{n=0}^{\infty}(n+1)\frac{1}{(1-z)^n}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^n}$$

whenever $\left|\dfrac{1}{1-z}\right|<1$, i.e. when $1<|z-1|$. On dividing through by $(z-1)^2$ we see that

$$\frac{1}{z^2}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(z-1)^{n+2}}=\sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n}$$

(In the last equality we again used the fact that $(-1)^n=(-1)^{n+2}$.

Exercise 3, §66, p. 220
Find the Taylor series for the function

$$\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$$

about the point $z_0=2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2}=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (\,|z-2|<2).$$

Solution:
Whenever $\left|\dfrac{z-2}{2}\right|<1$ (i.e. when $|z-2|<2$), it follows that

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2}\left(\frac{1}{1+\left(\frac{z-2}{2}\right)}\right)=\frac{1}{2}\left(\frac{1}{1-\left(\frac{-(z-2)}{2}\right)}\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{-(z-2)}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n.\end{aligned}$$

From Theorem 2 in section 65 we now conclude that for all $|z-2|<2$ we have

$$\begin{aligned}\frac{1}{z^2} &= -\frac{d}{dz}\left(\frac{1}{z}\right)=-\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}n(z-2)^{n-1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(n+1)(z-2)^n.\end{aligned}$$

(In the last sum we made use of the fact that $(-1)^n=(-1)^{n+2}$.)

Exercise 5, §66, p. 220
Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{when} \quad z \neq \pm\pi/2 \\ -\dfrac{1}{\pi} & \text{when} \quad z = \pm\pi/2, \end{cases}$$

then f is an entire function.

Solution:
Since $\cos z$ is entire it is easy to see that $\dfrac{\cos z}{\left(z^2 - \left(\frac{\pi}{2}\right)^2\right)}$ is analytic (differentiable) whenever $z^2 - \left(\frac{\pi}{2}\right)^2 \neq 0$. To see that

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{if} \quad z \neq \pm\frac{\pi}{2} \\ -\dfrac{1}{\pi} & \text{if} \quad z = \pm\frac{\pi}{2} \end{cases}$$

is in fact analytic on all of $\mathbb{C}$, we therefore only need to check that it is differentiable at $z = \pm\frac{\pi}{2}$. To do this we expand $\cos z$ as a power series at $z = \frac{\pi}{2}$ (i.e. in powers of $\left(z - \frac{\pi}{2}\right)$). Recall from example 2 of section 59 that

$$\sin w = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!}$$

and from the top of p. 107 of section 34 that

$$\cos z = -\sin\left(z - \frac{\pi}{2}\right).$$

It therefore we set $w = \left(z - \frac{\pi}{2}\right)$ it follows that

$$\begin{aligned} \cos z &= -\sum_{n=0}^{\infty} (-1)^n \frac{\left(z - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} \\ &= -\left(z - \frac{\pi}{2}\right) + \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z - \frac{\pi}{2}\right)^5 + \cdots . \end{aligned}$$

But then

$$\begin{aligned} \frac{f(z) - f\left(\frac{\pi}{2}\right)}{z - \frac{\pi}{2}} &= \frac{1}{\left(z - \frac{\pi}{2}\right)}\left(\frac{\cos z}{\left(z + \frac{\pi}{2}\right)\left(z - \frac{\pi}{2}\right)} - \left(\frac{-1}{\pi}\right)\right) \\ &= \frac{1}{\pi\left(z + \frac{\pi}{2}\right)} + \frac{1}{\left(z + \frac{\pi}{2}\right)}\left[\frac{1}{3!}\left(z - \frac{\pi}{2}\right) - \frac{1}{5!}\left(z - \frac{\pi}{2}\right)^3 + \cdots\right] \\ &\to \frac{1}{\pi^2} \end{aligned}$$

as $z \to \dfrac{\pi}{2}$. Thus $f'\left(\dfrac{\pi}{2}\right)$ exists and equals $\dfrac{1}{\pi^2}$. To check differentiability at $z = -\dfrac{\pi}{2}$ note that

$$\begin{aligned} \cos z &= \cos(-z) = -\left(-z - \frac{\pi}{2}\right) + \frac{1}{3!}\left(-z - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(-z - \frac{\pi}{2}\right)^5 + \cdots \\ &= \left(z + \frac{\pi}{2}\right) - \frac{1}{3!}\left(z + \frac{\pi}{2}\right)^3 + \frac{1}{5!}\left(z + \frac{\pi}{2}\right)^5 - \cdots . \end{aligned}$$

Arguing as before it then follows that

$$\begin{aligned}\frac{f(z)-f\left(-\frac{\pi}{2}\right)}{z+\frac{\pi}{2}} &= \frac{1}{z+\frac{\pi}{2}}\left(\frac{\cos z}{\left(z-\frac{\pi}{2}\right)\left(z+\frac{\pi}{2}\right)}-\left(\frac{-1}{\pi}\right)\right)\\ &= \frac{1}{\pi\left(z-\frac{\pi}{2}\right)}-\frac{1}{\left(z-\frac{\pi}{2}\right)}\left[\frac{1}{3!}\left(z+\frac{\pi}{2}\right)-\frac{1}{5!}\left(z+\frac{\pi}{2}\right)^3+\cdots\right]\\ &\to -\frac{1}{\pi^2}\quad \text{as } z\to -\frac{\pi}{2}.\end{aligned}$$

Thus $f'\left(-\frac{\pi}{2}\right)$ exists and equals $-\frac{1}{\pi^2}$.

Exercise 6, §66, p. 220

In the w plane, integrate the Taylor series expansion (see Example 4, Sec. 59)

$$\frac{1}{w}=\sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1|<1)$$

along a contour interior to the circle of convergence from $w=1$ to $w=z$ to obtain the representation

$$\text{Log} z=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1|<1).$$

Solution:

From example 4 in section 59 we know that

$$\frac{1}{w}=\sum_{n=0}^{\infty}(-1)^n(w-1)^n \qquad (|w-1|<1).$$

Now by (5) in section 31, Log(z) is an anti–derivative of $\frac{1}{z}$ on the region ($|z|>0, -\pi<\text{Arg}(z)<\pi$). In particular the disc $|w-1|<1$ is entirely contained in this region. If therefore z is in this disc and we integrate from 1 to z along a contour inside this disc, it follows from (1) of section 45 and Theorem 1 of section 65 that

$$\begin{aligned}\text{Log} z &= \text{Log} w|_1^z=\int_1^z \frac{1}{w}dw\\ &= \int_1^z\sum_{n=0}^{\infty}(-1)^n(w-1)^n\,dw\\ &= \sum_{n=0}^{\infty}(-1)^n\left.\frac{(w-1)^{n+1}}{n+1}\right|_1^z\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n+1}}{n+1}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z-1|<1).\end{aligned}$$

Exercise 7, §66, p. 220

Use the result in Exercise 6 to show that if

$$f(z)=\begin{cases}\dfrac{\text{Log} z}{z-1} & \text{when } z\neq 1\\ 1 & \text{when } z=1,\end{cases}$$

then f is analytic throughout the domain $0<|z|<\infty$, $-\pi<\text{Arg}\, z<\pi$.

Solution:

Since Log$z$ is analytic on all of $(0 < |z| < \infty,\ -\pi < \text{Arg}(z) < \pi)$, it easily follows that $\text{Log}(z)/(z-1)$ is analytic on this entire region except where $z-1=0$, i.e. where $z=1$. To see that

$$f(z) = \begin{cases} \dfrac{\text{Log}z}{z-1} & \text{when} \quad z \neq 1 \\ 1 & \text{when} \quad z = 1, \end{cases}$$

is analytic on this entire region, we therefore only need to check that it is differentiable at $z=1$. Now by exercise 6 above we may write $f(z)$ as a power series whenever $0 < |z-1| < 1$. We get

$$f(z) = \frac{1}{z-1}\left(\sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^{n+1}}{n+1}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{n+1}.$$

Therefore for any such z we have that

$$\begin{aligned} \frac{f(z)-f(1)}{z-1} &= \frac{1}{z-1}\left[\left(\sum_{n=0}^{\infty}(-1)^n \frac{(z-1)^n}{n+1}\right) - 1\right] \\ &= -\frac{1}{2} + \frac{1}{3}(z-1) - \frac{1}{4}(z-1)^2 + \frac{1}{5}(z-1)^3 - \cdots \\ &\to -\frac{1}{2} \quad \text{as} \quad z \to 1. \end{aligned}$$

Therefore $f'(1)$ exists and equals $-\dfrac{1}{2}$.

Exercise 8, §66, p. 220

Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by the equations

$$g(x) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when} \quad z \neq z_0, \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when} \quad z = z_0 \end{cases}$$

is analytic at z_0.

Solution:

Since f is analytic at z_0 we can find some $R > 0$ so that f is analytic for all z with $|z-z_0| < R$. We show that g is differentiable throughout $|z-z_0| < R$.

For any $z \neq z_0$ with $|z-z_0| < R$, $g(z)$ is of the form $g(z) = \dfrac{f(z)}{(z-z_0)^{m+1}}$ where f is analytic at each such z. Clearly $g(z)$ is then also analytic (differentiable) at each such z.

To show that g is also differentiable at z_0 we first look at the Taylor series of f at z_0. By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Since by assumption $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, we therefore get

$$f(z) = c_{m+1}(z-z_0)^{m+1} + c_{m+2}(z-z_0)^{m+2} + cm_{+3}(z-z_0)^{m+3} + \cdots$$

where the c_n's are as before. Then

$$f(z_0+h) = c_{m+1}h^{m+1} + c_{m+2}h^{m+2} + \cdots$$

and hence

$$\begin{aligned}\frac{g(z_0+h)-g(z_0)}{h} &= \frac{1}{h}\left[\frac{f(z_0+h)}{(z_0+h-z_0)^{m+1}} - \frac{f^{m+1}(z_0)}{(m+1)!}\right] \\ &= \frac{1}{h}\left[(c_{m+1}+c_{m+2}h+\cdots)-c_{m+1}\right] \\ &= c_{m+2}+c_{m+3}h+c_{m+4}h^2+\cdots \\ &\to c_{m+2} \text{ as } h\to 0.\end{aligned}$$

Thus $g'(z_0)$ exists and equals

$$c_{m+2} = \frac{f^{(m+2)}(z_0)}{(m+2)!}.$$

Exercise 9, §66, p. 221

Suppose that a function $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

inside some circle $|z-z_0| = R$. Use Theorem 2 in Sec. 65, regarding term by term differentiation of such a series, and mathematical induction to show that

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k} (z-z_0)^k \qquad (n = 0, 1, 2, \ldots)$$

when $|z-z_0| < R$. Then, by setting $z = z_0$, show that the coefficients a_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the Taylor series for f about z_0. Thus give an alternative proof of Theorem 1 in Sec. 66.

Solution:

Suppose that for some $R > 0$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \qquad |z-z_0| < R.$$

It then follows from Theorem 2 of section 65 that

$$f'(z) = \sum_{k=0}^{\infty} (k+1)\, a_{k+1} (z-z_0)^k \qquad |z-z_0| < R.$$

Since $k+1 = \dfrac{(k+1!)}{k!}$, this can also be written as

$$f'(z) = \sum_{k=0}^{\infty} \frac{(k+1)!}{k!} a_{k+1} (z-z_0)^k \qquad |z-z_0| < R.$$

Now suppose that for some $m \in \mathbb{N}$ we have that

$$f^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} a_{k+m} (z-z_0)^k$$

for all $|z-z_0|<R$. On applying Theorem 2 of section 65 to $f^{(m)}$ above it then follows that

$$\begin{aligned} f^{(m+1)}(z) &= \frac{d}{dz}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}a_{k+m}(z-z_0)^k \\ &= \sum_{k=1}^{\infty}\frac{(k+m)!}{k!}ka_{k+m}(z-z_0)^{k-1} \\ &= \sum_{k=1}^{\infty}\frac{(k+m)!}{(k-1)!}a_{k+m}(z-z_0)^{k-1} \\ &= \sum_{k=0}^{\infty}\frac{(k+(1+m)!)}{k!}a_{k+(1+m)}(z-z_0)^k \end{aligned}$$

for all $|z-z_0|<R$. By induction it therefore follows that

$$f^{(n)}(z)=\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}a_{k+n}(z-z_0)^k \qquad |z-z_0|<R$$

for any $n\in\mathbb{N}$. If now we evaluate these series at z_0, only the $k=0$ term remains, whence

$$f^{(n)}(z_0)=\frac{n!}{0!}a_n=n!a_n \qquad n\in\mathbb{N}.$$

(By convention $0!=1$.) Therefore $a_n=\dfrac{f^{(n)}(z_0)}{n!}$ for each $n\in\mathbb{N}$, that is the coefficients in the series expansion $f(z)=\sum\limits_{k=0}^{\infty}a_k(z-z_0)^k$ are precisely the Taylor–coefficients of f at z_0.

Exercise 2, §67, p. 225
By writing $\csc(z)=\frac{1}{\sin(z)}$ and then using division, show that

$$\csc z=\frac{1}{z}+\frac{1}{3!}z+\left[\frac{1}{(3!)^2}-\frac{1}{5!}\right]z^3+ \qquad (0<|z|<\pi).$$

Solution:
Observe that $\csc z=\dfrac{1}{\sin z}$ with

$$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}=z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots.$$

Now if $0<|z|<\pi$, then $\sin z$ is non–zero and so in this case we may write

$$\csc z=\frac{1}{\sin z}=\frac{1}{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\frac{1}{7!}z^7+\cdots}.$$

By means of long–division

$$\begin{array}{r|l}
 & \frac{1}{z}+\frac{1}{3!}z+\left(\frac{1}{(3!)^2}-\frac{1}{5!}\right)z^3+\cdots \\
\hline
z-\frac{1}{3!}z^3+\frac{1}{5!}z^5-\cdots & 1 \\
 & \underline{1-\frac{1}{3!}z^2+\frac{1}{5!}z^4-\cdots} \\
 & \quad \frac{1}{3!}z^2-\frac{1}{5!}z^4+\cdots \\
 & \quad \underline{\frac{1}{3!}z^2-\left(\frac{1}{3!}\right)^2z^4+\frac{1}{3!5!}z^6-\cdots} \\
 & \quad \left[\left(\frac{1}{3!}\right)^2-\frac{1}{5!}\right]z^4+\cdots \\
 & \quad \underline{\left[\left(\frac{1}{3!}\right)^2-\frac{1}{5!}\right]z^4+\cdots} \\
 & \qquad\qquad \cdots
\end{array}$$

Therefore $\csc z = \dfrac{1}{z}+\dfrac{1}{3!}z+\left(\dfrac{1}{(3!)^2}-\dfrac{1}{5!}\right)z^3+\cdots$ if $0<|z|<\pi$.

Exercise 3, §67, p. 225

Use division to obtain the Laurent series representation

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \qquad (0<|z|<2\pi)\,.$$

Solution:

Recall that $e^z-1=\sum_{n=0}^{\infty}\dfrac{1}{n!}z^n-1=\sum_{n=1}^{\infty}\frac{1}{n!}z^n$. The zeros of e^z-1 are at all points z where $e^z=1$, i.e. $z=i2n\pi$ $(n\in\mathbb{Z})$. Thus if $0<|z|<2\pi$, then e^z-1 is non–zero and so in this case we may write

$$\frac{1}{e^z-1}=\frac{1}{\sum_{n=1}^{\infty}\frac{1}{n!}z^n}=\frac{1}{z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\frac{1}{4!}z^4+\cdots}.$$

By means of longdivision

$$\begin{array}{r|l}
 & \frac{1}{z}-\frac{1}{2}+\frac{1}{12}z-\frac{1}{720}z^3+\cdots \\
\hline
z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\frac{1}{4!}z^4+\cdots & 1 \\
 & \underline{1+\frac{1}{2!}z+\frac{1}{3!}z^2+\frac{1}{4!}z^3+\frac{1}{5!}z^4+\cdots} \\
 & \quad -\frac{1}{2}z-\frac{1}{6}z^2-\frac{1}{24}z^3-\frac{1}{120}z^4-\cdots \\
 & \quad \underline{-\frac{1}{2}z-\frac{1}{4}z^2-\frac{1}{12}z^3-\frac{1}{48}z^4-\cdots} \\
 & \qquad\quad \frac{1}{12}z^2+\frac{1}{24}z^3+\frac{1}{80}z^4+\cdots \\
 & \qquad\quad \underline{\frac{1}{12}z^2+\frac{1}{24}z^3+\frac{1}{72}z^4+\cdots} \\
 & \qquad\qquad\qquad -\frac{1}{720}z^4-\cdots
\end{array}$$

Thus $\dfrac{1}{e^{z-1}}=\dfrac{1}{z}-\dfrac{1}{2}+\dfrac{1}{12}z-\dfrac{1}{720}z^3+\cdots$ if $0<|z|<2\pi$.

Exercise 4, §67, p. 225

Use the expansion

$$\frac{1}{z^2\sinh z}=\frac{1}{z^3}-\frac{1}{6}\cdot\frac{1}{z}+\frac{7}{360}z+\cdots \qquad (0<|z|<\pi)$$

in Example 2, Sec. 67, and the method illustrated in Example 1, Sec. 62, to show that

$$\int_C\frac{dz}{z^2\sinh z}=-\frac{\pi i}{3}$$

where C is the positively oriented unit circle $|z|=1$.

Solution:

Observe that

$$\frac{1}{z^2\sinh z} = \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{7}{360}z + \cdots$$

for all $0 < |z| < \pi$. Since the positively oriented circle $C : |z| = 1$ falls within this region, it follows from Theorem 1 of section 65 that

$$\int_C \frac{1}{z^2\sinh z}dz = \int_C \frac{1}{z^3}dz - \frac{1}{6}\int_C \frac{1}{z}dz + \frac{7}{360}\int_C zdz + \cdots .$$

It then follows from exercise 10(b) of Sec. 42 of the textbook that

$$\begin{aligned}\int_C \frac{1}{z^2\sinh z}dz &= 0 - \frac{1}{6}2\pi i + 0 + 0 + \cdots \\ &= -\frac{\pi i}{3}.\end{aligned}$$

Exercise 8, §67, p. 227

The *Euler numbers* are the numbers E_n $(n = 0, 1, 2, \ldots)$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!}z^n \qquad (|z| < \pi/2).$$

Point out why this representation is valid in the indicated disk and why $E_{2n+1} = 0$ $(n = 0, 1, 2, \ldots)$. Then show that

$$E_0 = 1, \qquad E_2 = -1, \qquad E_4 = 5, \qquad \text{and} \qquad E_6 = -61.$$

Solution:

Observe that the singularities of $\text{sech}z = \frac{1}{\cosh z}$ are at all points z where $\cosh = 0$, i.e. where $z = \left(\frac{\pi}{2} + n\pi\right)i$ $(n \in \mathbb{Z})$
(see (15) in section 35). However none of these points are contained in the disk $|z| < \frac{\pi}{2}$ and hence $\frac{1}{\cosh z}$ is therefore analytic on this disk. By Taylor's theorem the Maclaurin series expansion must then exist in this disk. Now from Example 3 of section 59 we know that

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!}z^{2n} = 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \cdots \qquad (|z| < \infty).$$

This series has only even powers of z. Performing longdivision to extract the Maclaurin series of $\frac{1}{\cosh z}$ from this one will therefore also yield a series with only even powers, that is the coefficients (and hence also E_{2n+1}) of all the odd powers in the Maclaurin expansion of $\frac{1}{\cosh z}$ must be zero. We proceed to compute the first four non–zero terms in this expansion by means of long division.

$$\begin{array}{r|l}
 & 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \\
\hline
1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \frac{1}{6!}z^6 + \cdots & 1 \\
 & 1 + \frac{1}{2!}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots \\
\cline{2-2}
 & -\frac{1}{2}z^2 - \frac{1}{24}z^4 - \frac{1}{720}z^6 - \cdots \\
 & -\frac{1}{2}z^2 - \frac{1}{4}z^4 - \frac{1}{48}z^6 - \cdots \\
\cline{2-2}
 & \frac{5}{24}z^4 + \frac{7}{360}z^6 \cdots \\
 & \frac{5}{24}z^4 + \frac{5}{48}z^6 \cdots \\
\cline{2-2}
 & -\frac{61}{720}z^6 - \cdots \\
 & -\frac{61}{720}z^6 - \cdots \\
\cline{2-2}
 & \cdots
\end{array}$$

Clearly $\frac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \ \left(|z| < \frac{\pi}{2}\right)$. The first four non–zero coefficients are therefore $E_0 = 1,\ \frac{E_2}{2!} = -\frac{1}{2},\ \frac{E_4}{4!} = \frac{5}{24},\ \frac{E_6}{6!} = \frac{-61}{720}$. This implies that $E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61$.

CHAPTER 6

Residues and Poles

Study sections 68 – 70, and sections 72–77. Exercises in section 71 that pertain to earlier sections, should nevertheless be done (That is we will cover all material in chapter 6, except the theory and techniques of section 71.)

In this chapter we meet the final ingredient in our theory of integration (as promised in the overview of chapter 4), namely Cauchy's Residue Theorem. This theorem builds on the theory of Laurent series, and presents us with a powerful technique for computing integrals. The idea behind the theorem is easy enough to explain. Suppose we are given a function f which is analytic throughout a domain of the form $0 < |z-a| < r$, but not at a itself. Next suppose we want to integrate this function along a simply closed positively oriented contour C which lies completely inside this region and also encloses the troublesome point a. We saw from Laurent's theorem that f may be written as a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

in this region. To integrate f along C, we can now try to integrate term by term, and apply Cauchy's Integral Formulas to each of these termwise integrals to get

$$\begin{aligned} \int_C f(z)dz &= \sum_{n=0}^{\infty} a_n \int_C (z-a)^n dz + \sum_{n=1}^{\infty} b_n \int_C \frac{1}{(z-a)^n} dz \\ &= 2\pi i b_1 \end{aligned}$$

(Here we used the fact that $\int_C (z-a)^n dz = 0$ when $n \neq -1$, with $\int_C \frac{1}{(z-a)} dz = 2\pi i$.) So computing the integral, boils down to computing the coefficient b_1 of the $\frac{1}{(z-a)}$ term in the Laurent expansion of f. We call this coefficient b_1 the *residue* of f at a. Taking this argument to its logical conclusion presents us with Cauchy's residue theorem, which in its simplest form boils down to the statement that

$$\int_C f(z)dz = 2\pi i \{\text{sum of the residues inside } C\}.$$

It is very important to note that **ONLY** the residues of singularities inside C contribute to the value of the integral $\int_C f(z)dz$. Thus although f may have many other singularities besides those inside C, those other singularities make no contribution whatsoever to the value of $\int_C f(z)dz$. Now of course for us to be able to make good use of this theorem, we need to be skilled at computing these residues. For this reason a large part of chapter 6 is devoted to classifying singularities of complex functions, and developing techniques for computing the residues at these singularities.

Solutions to selected problems

Exercise 1, §71, p. 239
Find the residue at $z = 0$ of the function

$$(a)\ \frac{1}{z+z^2};\qquad (b)\ z\cos\left(\frac{1}{z}\right);\qquad (c)\ \frac{z-\sin z}{z};$$
$$(d)\ \frac{\cot z}{z^4};\qquad (e)\ \frac{\sinh z}{z^4\left(1-z^2\right)}.$$

Solution:

(a) The point $z = 0$ is the only singularity of $f(z) = \dfrac{1}{z+z^2}$ inside C where C is the positively oriented circle $|z| = \dfrac{1}{2}$. By (2) of section 69 the residue is

$$\begin{aligned}
\operatorname*{Res}_{z=0}(f) &= \frac{1}{2\pi i}\int_C \frac{1}{z+z^2}dz \\
&= \frac{1}{2\pi i}\int_C \frac{1/(1+z)}{z}dz \\
&= \left.\frac{1}{1+z}\right|_{z=0} \quad \text{(Cauchy Integration Formulae)} \\
&= 1.
\end{aligned}$$

(b) Since $\cos w = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n!} w^{2n}$ for all $|w| < \infty$, we may set $w = \dfrac{1}{z}$ to get

$$\begin{aligned}
z\cos\frac{1}{z} &= z\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{z^{2n}}\right) \\
&= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \frac{1}{6!}\frac{1}{z^5} + \cdots
\end{aligned}$$

Clearly

$$\operatorname*{Res}_{z=0}\left(z\cos\left(\frac{1}{z}\right)\right) = -\frac{1}{2}.$$

(c) Since $\sin z = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}z^{2n+1}$ (see section 59) it follows that

$$\begin{aligned}
\frac{1}{z}(z-\sin z) &= \frac{1}{z}\left(z - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right) \\
&= \frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \frac{1}{7!}z^6 - \ldots
\end{aligned}$$

Therefore the residue (coefficient of the $\dfrac{1}{z}$ term) is

$$\operatorname*{Res}_{z=0}\left(\frac{1}{z}(z-\sin z)\right) = 0.$$

(d) We use long division to compute the first few terms of the Laurent series of $\cot z = \dfrac{\cos z}{\sin z}$. Since

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \ldots\right)$$

and

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots\right),$$

it follows that

$$\begin{array}{r|l}
 & \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \ldots \\
\hline
z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots & 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \ldots \\
 & 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \ldots \\
\cline{2-2}
 & \quad -\frac{1}{3}z^2 + \frac{1}{30}z^4 - \ldots \\
 & \quad -\frac{1}{3}z^2 + \frac{1}{18}z^4 - \ldots \\
\cline{2-2}
 & \qquad -\frac{1}{45}z^4 + \ldots \\
 & \qquad -\frac{1}{45}z^4 + \ldots \\
\cline{2-2}
 & \qquad \ldots\ldots\ldots\ldots
\end{array}$$

Therefore

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \ldots$$

in which case

$$\frac{\cot z}{z^4} = \frac{1}{z^5} - \frac{1}{3}\frac{1}{z^3} - \frac{1}{45}\frac{1}{z}\ldots$$

But then

$$\operatorname*{Res}_{z=0}\left(\frac{\cot z}{z^4}\right) = -\frac{1}{45}.$$

(e) From section 59 we know that

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}.$$

In addition if $|z| < 1$ then $1/\left(1 - z^2\right) = \sum_{n=0}^{\infty} \left(z^2\right)^n$. Therefore if $0 < |z| < 1$, then

$$\begin{aligned}
\frac{\sinh z}{z^4\left(1 - z^2\right)} &= \frac{1}{z^4}\left(1 + z^2 + z^4 + \ldots\right)\left(z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots\right) \\
&= \frac{1}{z^3} + \left(1 + \frac{1}{3!}\right)\frac{1}{z} + \left(1 + \frac{1}{3!} + \frac{1}{5!}\right)z + \ldots
\end{aligned}$$

Consequently

$$\operatorname*{Res}_{z=0}\left(\frac{\sinh z}{z^4\left(1 - z^2\right)}\right) = 1 + \frac{1}{3!} = \frac{7}{6}.$$

Exercise 2, §71, p. 239

Use Cauchy's Residue Theorem (Section 70) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$(a)\ \frac{\exp(-z)}{z^2}; \qquad (b)\ \frac{\exp(-z)}{(z-1)^2}; \qquad (c)\ z^2 \exp\left(\frac{1}{z}\right); \qquad (d)\ \frac{z+1}{z^2 - 2z}.$$

Solution:
In each case let C be the positively oriented circle $|z| = 3$.

(a) The only singularity of $\frac{\exp(-z)}{z^2}$ is at $z = 0$. Therefore

$$\int_C \frac{\exp(-z)}{z^2} dz = 2\pi i \operatorname*{Res}_{z=0} \left(\frac{\exp(-z)}{z^2} \right).$$

Since $\frac{\exp(-z)}{z^2} = \frac{1}{z^2} \left(\sum_{n=0}^{\infty} \frac{1}{n!} (-z)^n \right) = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{1}{3!} z + \cdots$ it follows that $\operatorname*{Res}_{z=0} \left(\frac{\exp(-z)}{z^2} \right) = -1$, and hence that

$$\int_C \frac{\exp(-z)}{z^2} dz = -2\pi i.$$

(b) The only singularity of $\frac{\exp(-z)}{(z-1)^2}$ is at $z = 1$. To find the residue we compute the Laurent series at $z = 1$. Since

$$e^{-z} = \frac{1}{e} e^{-(z-1)} = \frac{1}{e} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(-(z-1) \right)^n \right)$$

it follows that

$$\begin{aligned} \frac{\exp(-z)}{(z-1)^2} &= \frac{1}{(z-1)^2} \cdot \frac{1}{e} \left[1 - (z-1) + \frac{1}{2!} (z-1)^2 - \dots \right] \\ &= \frac{1}{e} \frac{1}{(z-1)^2} - \frac{1}{e} \frac{1}{(z-1)} + \frac{1}{2e} - \dots \end{aligned}$$

and hence that

$$\operatorname*{Res}_{z=1} \left(\exp(-z) / (z-1)^2 \right) = -\frac{1}{e}.$$

Therefore

$$\int_C \frac{\exp(-z)}{(z-1)^2} dz = 2\pi i \left(-\frac{1}{e} \right).$$

(c) In Laurent form

$$\begin{aligned} z^2 \exp\left(\frac{1}{z} \right) &= z^2 \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z} \right)^n \right) \\ &= z^2 + z + \frac{1}{2!} + \frac{1}{3!} \frac{1}{z} + \frac{1}{4!} \frac{1}{z^2} + \dots \end{aligned}$$

Therefore

$$\operatorname*{Res}_{z=0} \left(z^2 \exp\left(\frac{1}{z} \right) \right) = \frac{1}{3!} = \frac{1}{6}.$$

Since in addition 0 is the only singularity of $z^2 \exp\left(\frac{1}{z} \right)$, it follows that

$$\int_C z^2 \exp\left(\frac{1}{z} \right) dz = 2\pi i \left(\frac{1}{6} \right) = \frac{\pi i}{3}.$$

(d) $\frac{z+1}{z^2 - 2z}$ has singularities at both 0 and 2 both of which lie inside the circle C. Therefore

$$\int_C \frac{z+1}{z^2 - 2z} dz = 2\pi i \left[\operatorname*{Res}_{z=0} (f) + \operatorname*{Res}_{z=2} (f) \right].$$

We compute the Laurent expansions at 0 and 2. At $z = 0$ we get

$$\begin{aligned}\frac{z+1}{z^2-2z} &= \left(-\frac{1}{2z}-\frac{1}{2}\right)\left(\frac{1}{1-\frac{z}{2}}\right)\\ &= \left(-\frac{1}{2z}-\frac{1}{2}\right)\left(1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right)\\ &= -\frac{1}{2z}-\frac{3}{4}-\frac{3}{8}z-\dots\end{aligned}$$

whenever $0 < |z| < 2$. Therefore

$$\operatorname*{Res}_{z=0}\left(\frac{z+1}{z^2-2z}\right) = -\frac{1}{2}.$$

At $z = 2$ we get

$$\begin{aligned}\frac{z+1}{z(z-2)} &= \frac{(z-2)+3}{2(z-2)}\frac{1}{1+\left(\frac{z-2}{2}\right)}\\ &= \left(\frac{1}{2}+\frac{3}{2(z-2)}\right)\left(1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\dots\right)\\ &= \frac{3}{2}\frac{1}{(z-2)}-\frac{1}{4}+\frac{1}{8}(z-2)-\dots\end{aligned}$$

whenever $0 < |z-2| < 2$, whence

$$\operatorname*{Res}_{z=2}\left(\frac{z+1}{z^2-2z}\right) = \frac{3}{2}.$$

Therefore

$$\int_C \frac{z+1}{z^2-2z}dz = 2\pi i\left[-\frac{1}{2}+\frac{3}{2}\right] = 2\pi i.$$

Exercise 4, §71, p. 239

Let C denote the circle $|z| = 1$, taken counterclockwise, and follow the steps below to show that

$$\int_C \exp\left(z+\frac{1}{z}\right)dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}.$$

(a) By using the Maclaurin series for e^z and referring to Theorem 1 in section 65, which justifies the term by term integration that is to be used, write the above integral as

$$\sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n\exp\left(\frac{1}{z}\right)dz.$$

(b) Apply the theorem in section 70 to evaluate the integrals appearing in part (a) to arrive at the desired result.

Solution:

Let C be the positively oriented circle $|z| = 1$. By example 1 of section 59

$$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n \qquad |z| < \infty.$$

Therefore on applying Theorem 1 of section 65 it follows that

$$\begin{aligned}\int_C e^{z+\frac{1}{z}}dz &= \int_C e^{\frac{1}{z}}e^z dz \\ &= \int_C e^{\frac{1}{z}}\sum_{n=0}^{\infty}\frac{1}{n!}z^n dz \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n e^{\frac{1}{z}}dz.\end{aligned}$$

Next notice that by example 1 of section 59

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{k!}\left(\frac{1}{z}\right)^k \qquad (0 < |z| < \infty).$$

Therefore

$$z^n e^{\frac{1}{z}} = z^n + z^{n-1} + \frac{1}{2!}z^{n-2} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!}\frac{1}{z} + \frac{1}{(n+2)!}\frac{1}{z^2} + \ldots$$

Clearly

$$\operatorname*{Res}_{z=0}\left(z^n e^{\frac{1}{z}}\right) = \frac{1}{(n+1)!}.$$

Therefore since $z = 0$ is the only singular point of $z^n e^{\frac{1}{z}}$, it follows from the theorem in section 70 that

$$\int_C z^n e^{\frac{1}{z}}dz = 2\pi i \operatorname*{Res}_{z=0}\left(z^n e^{\frac{1}{z}}\right) = \frac{2\pi i}{(n+1)!}.$$

Consequently

$$\int_C e^{z+\frac{1}{z}}dz = \sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n e^{\frac{1}{z}}dz = 2\pi i\sum_{n=0}^{\infty}\frac{1}{n!\,(n+1)!}.$$

Exercise 1, §72, p. 243

In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

$$(a)\ z\exp\left(\frac{1}{z}\right); \qquad (b)\ \frac{z^2}{1+z}; \qquad (c)\ \frac{\sin z}{z}; \qquad (d)\ \frac{\cos z}{z}; \qquad (e)\ \frac{1}{(2-z)^3}.$$

Solution:

(a) The only singularity of $z\exp\left(\frac{1}{z}\right)$ is $z = 0$. For $z = 0$ we have

$$z\,e^{\frac{1}{z}} = z\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}z^{1-n} \qquad |z| > 0.$$

Therefore the principal part of $z\,e^{\frac{1}{z}}$ is

$$\sum_{n=2}^{\infty}\frac{1}{n!}z^{1-n} = \sum_{k=1}^{\infty}\frac{1}{(k+1)!}z^{-k} \qquad |z| > 0$$

and the singularity is an essential singularity.

(b) $\dfrac{z^2}{1+z}$ has a singularity at $z=-1$. At this point the Laurent expansion is

$$\frac{z^2}{1+z} = \frac{1}{(z+1)}\left((z+1)-1\right)^2 = \frac{1}{(z+1)} - 2 + (z+1)$$

and the principal part of the function is $\dfrac{1}{z+1}$. Therefore $z=-1$ is a simple pole.

(c) $\dfrac{\sin z}{z}$ has a singular point at $z=0$. At this point

$$\frac{\sin z}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(-1)^n z^{2n+1}\right) = 1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 - \ldots$$

Therefore $z=0$ is a removable singular point and the principal part of $\dfrac{\sin z}{z}$ is 0.

(d) $\dfrac{\cos z}{z}$ has a singular point at $z=0$. At this point

$$\frac{\cos z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^{2n}}{(2n)!}z^{2n} = \frac{1}{z} - \frac{1}{2!}z + \frac{1}{4!}z^3 - \ldots$$

Therefore the principal part of $\dfrac{\cos z}{z}$ is $\dfrac{1}{z}$ and hence $z=0$ is a simple pole.

(e) $\dfrac{1}{(2-z)^3} = -\dfrac{1}{(z-2)^3}$ has a singular point at $z=2$. The principal part of the function at 2 is $-\dfrac{1}{(z-2)^3}$ and $z=2$ is therefore a pole of order 3.

Exercise 3, §72, p. 243

Suppose that a function f is analytic at z_0, and consider the quotient

$$g(z) = \frac{f(z)}{z-z_0}.$$

Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;
(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in section 57, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Solution:

Since f is analytic at z_0, Taylor's Theorem assures us that for some $R>0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < R$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0,1,2,3,\ldots$$

(a) If therefore $f(z_0) \neq 0$, then $a_0 \neq 0$ in which case the Laurent series expansion of $f(z)/(z-z_0)$ at z_0 is of the form

$$\frac{f(z)}{z-z_0} = \frac{a_0}{z-z_0} + a_1 + a_2(z-z_0) + \ldots$$

Thus $f(z)/(z-z_0)$ then has a simple pole at z_0 with residue $a_0 = f(z_0)$.

(b) If on the other hand $f(z_0) = 0$, then $a_0 = 0$ whence

$$f(z) = a_1(z-z_0) + a_2(z-z_0)^2 + \ldots \qquad |z-z_0| < R.$$

In this case

$$\frac{f(z)}{(z-z_0)} = a_1 + a_2(z-z_0) + a_3(z-z_0)^2 + \ldots \qquad 0 < |z-z_0| < R.$$

Clearly $f(z)/(z-z_0)$ then has a removable singularity at z_0.

Exercise 1, §74, p. 248

In each case, show that the singular points of the function are poles. Determine the order m of each pole, and find the corresponding residue B.

(a) $\dfrac{z^2+2}{z-1}$;

(b) $\left(\dfrac{z}{2z+1}\right)^3$;

(c) $\dfrac{\exp z}{z^2+\pi^2}$.

Solution:

(a) z^2+2 is non–zero at $z=1$ whereas $z-1$ has a zero of order 1 at $z=1$. Therefore $\left(z^2+2\right)/(z-1)$ has a simple pole at $z=1$. The residue is

$$\operatorname*{Res}_{z=1}\left(\frac{z^2+2}{z-1}\right) = \left.\frac{z^2+2}{\frac{d}{dz}(z-1)}\right|_{z=1} = 3.$$

(b) z^3 is nonzero at $z=-\dfrac{1}{2}$ whereas $2^3\left(z+\dfrac{1}{2}\right)^3$ has a zero of order 3 there. Therefore $z=-\dfrac{1}{2}$ is a pole of order 3 of $\left(\dfrac{z}{2z+1}\right)^3$. Since

$$\left(\frac{z}{2z+1}\right)^3 = \frac{\left(\frac{z}{2}\right)^3}{\left(z+\frac{1}{2}\right)^3}$$

with $\left(\dfrac{z}{2}\right)^3$ analytic at $z=-\dfrac{1}{2}$, the residue is given by

$$\begin{aligned}\operatorname*{Res}_{z=-\frac{1}{2}}\left(\frac{z}{2z+1}\right)^3 &= \left.\frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z}{2}\right)^3\right|_{z=-\frac{1}{2}} \\ &= \left.\frac{1}{2!}\,\frac{6z}{8}\right|_{z=-\frac{1}{2}} \\ &= -\frac{3}{16}.\end{aligned}$$

(c) $\exp(z)$ is never zero whereas $z^2+\pi^2=(z-i\pi)(z+i\pi)$ has simple zeros at $\pm i\pi$. Thus $\dfrac{\exp(z)}{(z^2+\pi^2)}$ has simple poles at $\pm i\pi$. The residues at these points are:

$$\begin{aligned}\operatorname*{Res}_{z=i\pi}\frac{\exp z}{z^2+\pi^2} &= \left.\frac{\exp z}{\frac{d}{dz}(z^2+\pi^2)}\right|_{z=i\pi}\\ &= \frac{\exp(i\pi)}{2(i\pi)}\\ &= \frac{i}{2\pi}\end{aligned}$$

and

$$\begin{aligned}\operatorname*{Res}_{z=-i\pi}\frac{\exp z}{z^2+\pi^2} &= \frac{\exp(-i\pi)}{2(-i\pi)}\\ &= -\frac{i}{2\pi}.\end{aligned}$$

Exercise 2 (a) & (b), §74, p. 248

Show that

(a) $\displaystyle\operatorname*{Res}_{z=-i\pi}\frac{z^{\frac14}}{z+1}=\frac{1+i}{\sqrt2}\quad(|z|>0,\ \ 0<\arg z<2\pi)\,;$

(b) $\displaystyle\operatorname*{Res}_{z=i}\frac{\operatorname{Log} z}{(z^2+1)^2}=\frac{\pi+2i}{8}.$

Solution:

(a) Here $z^{\frac14}$ is defined by

$$z^{\frac14}=\exp\left(\frac14\log z\right)$$

where

$$\log z=\ln|z|+i\arg(z)$$

for any z with $|z|>0,\ \ 0<\arg(z)<2\pi$. In particular this branch of $z^{\frac14}$ exists and is analytic at $z=-1$. With $\log z$ and $z^{\frac14}$ as above we have

$$\log(-1)=\ln 1+i\pi=i\pi$$

and hence that

$$\begin{aligned}(-1)^{\frac14} &= \exp\left(\frac14\log(-1)\right)\\ &= \exp\left(i\frac{\pi}{4}\right)\\ &= \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\\ &= \frac{1}{\sqrt2}+i\frac{1}{\sqrt2}.\end{aligned}$$

By contrast $z+1$ has a simple zero at $z=-1$ and hence $z^{\frac{1}{4}}/\left(z+1\right)$ must have a simple pole there. Thus

$$\operatorname*{Res}_{z=-1}\frac{z^{\frac{1}{4}}}{z+1}=\left.\frac{z^{\frac{1}{4}}}{\frac{d}{dz}\left(z+1\right)}\right|_{z=-1}=\left(-1\right)^{\frac{1}{4}}=\frac{1}{\sqrt{2}}+i\frac{i}{\sqrt{2}}.$$

(b) Recall that

$$\operatorname{Log}z=\ln|z|+i\operatorname{Arg}\left(z\right)$$

for any z with $|z|>0$ and $-\pi<\operatorname{Arg}(z)<\pi$. In particular $z=i$ lies in this region (note that $i=1e^{i\frac{\pi}{2}}$ in polar form) and hence $\operatorname{Log}z$ is analytic at $z=i$ with

$$\operatorname{Log}\left(i\right)=\ln 1+i\frac{\pi}{2}=i\frac{\pi}{2}.$$

By contrast $\left(z^2+1\right)^2=\left(z-i\right)^2\left(z+i\right)^2$ has a double zero at $z=i$ in which case $\operatorname{Log}z/\left(z^2+1\right)^2$ must then have a double pole at $z=i$. Therefore since

$$\frac{\operatorname{Log}z}{\left(z^2+1\right)^2}=\frac{\operatorname{Log}z/\left(z+i\right)^2}{\left(z-i\right)^2},$$

the theorem in section 73 ensures that

$$\begin{aligned}\operatorname*{Res}_{z=i}\left(\frac{\operatorname{Log}z}{\left(z^2+1\right)^2}\right) &= \left.\frac{1}{1!}\frac{d}{dz}\frac{\operatorname{Log}z}{\left(z+i\right)^2}\right|_{z=i}\\ &= \left.\frac{\left(z+i\right)^2\frac{1}{z}-2\left(z+i\right)\operatorname{Log}z}{\left(z+i\right)^4}\right|_{z=i}\\ &= \frac{\left(2i\right)^2\frac{1}{i}-2\left(2i\right)\left(i\frac{\pi}{2}\right)}{\left(2i\right)^4}\\ &= \frac{\pi+2i}{8}.\end{aligned}$$

Exercise 3, §74, p. 248
Find the value of the integral

$$\int_C\frac{dz}{z^3\left(z+4\right)},$$

taken counterclockwise around the circle

(a) $|z|=2$;

(b) $|z+2|=3$.

Solution:

$\dfrac{1}{z^3\left(z+4\right)}$ clearly has a simple pole at $z=-4$ and a pole of order 3 at $z=0$. By the theorem in section 73, the residue at $z=-4$ is given by

$$\operatorname*{Res}_{z=-4}\left(\frac{1}{z^3\left(z+4\right)}\right)=\left.\frac{1}{\left(z\right)^3}\right|_{z=-4}=-\frac{1}{64}$$

and at $z=0$ by

$$\begin{aligned}\operatorname*{Res}_{z=0}\left(\frac{1}{z^3(z+4)}\right) &= \frac{1}{2!}\frac{d^2}{dz^2}\left.\frac{1}{(z+4)}\right|_{z=0}\\ &= \left.\frac{1}{(z+4)^3}\right|_{z=0}\\ &= \frac{1}{64}.\end{aligned}$$

(a) Next observe that $|0|<2$ and $|-4|>2$. Of the two singular points only 0 is in the interior of $|z|=2$. Thus when integrating around this circle we get

$$\int_C \frac{dz}{z^3(z+1)} = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^3(z+1)}\right) = \frac{\pi i}{32}.$$

(b) Since $|0+2|<3$ and $|-4+2|<3$, both singular points lie inside $|z+2|=3$. Thus for this circle

$$\int_C \frac{dz}{z^3(z+1)} = 2\pi i \left[\operatorname*{Res}_{z=0}\left(\frac{1}{z^3(z+1)}\right) + \operatorname*{Res}_{z=-1}\left(\frac{1}{z^3(z+1)}\right)\right] = 0.$$

Exercise 5, §74, p. 248

Evaluate the integral

$$\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)}$$

where C is the circle $|z|=2$, described in the positive sense.

Solution:

$\dfrac{\cosh \pi z}{z(z^2+1)}$ has singular points where $z\left(z^2+1\right)=0$, i.e. where $z=0,\pm i$. Since $\cosh \pi z$ is non–zero at these points and each point is a simple zero of $z\left(z^2+1\right) = z(z-i)(z+i)$, these points are simple poles of $\dfrac{\cosh \pi z}{[z(z^2+1)]}$. Thus

$$\begin{aligned}\operatorname*{Res}_{z=i}\frac{\cosh \pi z}{z(z^2+1)} &= \left.\frac{\cosh \pi z}{\frac{d}{dz}\left(z(z^2+1)\right)}\right|_{z=i}\\ &= \left.\frac{\cosh \pi z}{3z^2+1}\right|_{z=i}\\ &= \frac{\cosh \pi i}{3(i)^2+1}\\ &= \frac{\cos \pi}{-2}\\ &= \frac{1}{2} \quad \text{(use (3) of section 35).}\end{aligned}$$

Similarly

$$\operatorname*{Res}_{z=-i}\frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh(-i\pi)}{3(-i)^2+1} = \frac{1}{2}$$

and

$$\operatorname*{Res}_{z=0}\frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh 0}{3(0)^2+1} = 1.$$

Since all these points are inside $|z| = 2$, we have

$$\begin{aligned}\int_C \frac{\cosh \pi z}{z\left(z^2+1\right)} dz &= 2\pi i \left[\frac{1}{2} + 1 + \frac{1}{2}\right] \\ &= 4\pi i.\end{aligned}$$

Exercise 2(b), §76, p. 255

Show that

(b) $\underset{z=\pi i}{\operatorname{Res}} \dfrac{\exp(zt)}{\sinh z} + \underset{z=-\pi i}{\operatorname{Res}} \dfrac{\exp(zt)}{\sinh z} = -2\cos \pi t.$

Solution:

Since

$$\left.\frac{d}{dz} \sinh z\right|_{z=\pm i\pi} = \cosh(\pm i\pi) = \cos(\pm\pi) = -1 \neq 0$$

(see (3) in section 35), $\sinh z$ clearly has simple zeros at $\pm i\pi$. Noting that $\exp(zt)$ is never zero, it follows that $\dfrac{\exp(zt)}{\sinh z}$ must have simple poles at $z = \pm i\pi$. Therefore

$$\begin{aligned}&\underset{z=\pi i}{\operatorname{Res}} \left(\frac{\exp(zt)}{\sinh z}\right) + \underset{z=-\pi i}{\operatorname{Res}} \left(\frac{\exp(zt)}{\sinh z}\right) \\ =\ & \frac{\exp(i\pi t)}{\cosh(i\pi)} + \frac{\exp(-i\pi t)}{\cosh(-i\pi)} \\ =\ & (-1)(\cos \pi t + i \sin \pi t) + (-1)(\cos(-\pi t) + i\sin(-\pi t)) \\ =\ & -2\cos \pi t.\end{aligned}$$

Exercise 3, §76, p. 255

Show that

(a) $\underset{z=z_n}{\operatorname{Res}} (z \sec z) = (-1)^{n+1} z_n$, where $z_n = \dfrac{\pi}{2} + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$;

(b) $\underset{z=z_n}{\operatorname{Res}} (\tanh z) = 1$, where $z_n = \left(\dfrac{\pi}{2} + n\pi\right) i \quad (n = 0, \pm 1, \pm 2, \ldots)$.

Solution:

(a) The zeros of $\cos z$ are at $z_n = \frac{\pi}{2} + n\pi (n \in \mathbb{Z})$. Observe that since

$$\left.\frac{d}{dz}(\cos z)\right|_{z=\frac{\pi}{2}+n\pi} = -\sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^{n+1} \neq 0$$

for each $n \in \mathbb{Z}$ (see section 34), $\cos z$ has simple zeros at $z_n = \dfrac{\pi}{2} + n\pi$. Thus $z \sec z = \dfrac{z}{\cos z}$ has simple poles at each of the z_n's. Therefore

$$\begin{aligned}\underset{z=z_n}{\operatorname{Res}} (z \sec z) &= \left.\frac{z}{\frac{d}{dz}\cos z}\right|_{z=z_n} \\ &= \frac{\left(\frac{\pi}{2} + n\pi\right)}{-\sin\left(\frac{\pi}{2} + n\pi\right)} \\ &= (-1)^{n+1}\left(\frac{\pi}{2} + n\pi\right) \quad n \in \mathbb{Z}.\end{aligned}$$

(b) $\tanh z = \dfrac{\sinh z}{\cosh z}$ has singular points where $\cosh z = 0$, i.e. where $z_n = i\,(2n+1)\,\dfrac{\pi}{2}$ $\;(n \in \mathbb{Z})$ (see section 26). At these points

$$\frac{d}{dz}\cosh z\Big|_{z=i(2n+1)\frac{\pi}{2}} = \sinh\left(i\,(2n+1)\,\frac{\pi}{2}\right) \neq 0.$$

Therefore $\cosh z$ has a simple zero and $\tanh z = \dfrac{\sinh z}{\cosh z}$ a simple pole at each $z_n = i\,(2n+1)\,\frac{\pi}{2}$ $\;(n \in \mathbb{Z})$. At each of these points the residue is

$$\operatorname*{Res}_{z=z_n}\,(\tanh z) = \frac{\sinh z}{\frac{d}{dz}\cosh z}\Bigg|_{z=z_n} = \frac{\sinh z_n}{\sinh z_n} = 1.$$

Exercise 4, §76, p. 255

Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\int_C \tan z\,dz$;

(b) $\displaystyle\int_C \frac{dz}{\sinh 2z}$.

Solution:

(a) $\tan z = \dfrac{\sin z}{\cos z}$ has singular points where $\cos z = 0$, i.e. where $z_n = (2n+1)\,\dfrac{\pi}{2}$ $\;(n \in \mathbb{Z})$. Now the inequality $\left|(2n+1)\,\dfrac{\pi}{2}\right| < 2$ only holds if $n = 0, -1$ and hence only $\pm\dfrac{\pi}{2}$ lie inside the circle $|z| = 2$. Since

$$\frac{d}{dz}\cos z\Big|_{z=\pm\frac{\pi}{2}} = -\sin\left(\pm\frac{\pi}{2}\right) = \mp 1,$$

the points $z = \pm\dfrac{\pi}{2}$ are simple zeros of $\cos z$ and hence simple poles of $\tan z = \dfrac{\sin z}{\cos z}$. Thus

$$\operatorname*{Res}_{z=\frac{\pi}{2}}\tan z = \frac{\sin z}{\frac{d}{dz}\cos z}\Bigg|_{z=\frac{\pi}{2}} = \frac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1$$

and similarly

$$\operatorname*{Res}_{z=-\frac{\pi}{2}}\tan z = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1.$$

Hence

$$\int_C \tan z dz \;=\; 2\pi i\left[\operatorname*{Res}_{z=\frac{\pi}{2}}\tan z + \operatorname*{Res}_{z=-\frac{\pi}{2}}\tan z\right] = -4\pi i.$$

(b) $\dfrac{1}{\sinh 2z}$ has singular points where $\sinh 2z = 0$; i.e. where $z_n = i\,n\,\frac{\pi}{2}$ $(n \in \mathbb{Z})$. Since

$$\begin{aligned}
\frac{d}{dz}\sinh 2z\Big|_{z=i\,n\frac{\pi}{2}} &= 2\cosh\left(i\,n\,\pi\right)\\
&= 2\cos\left(n\pi\right) \quad ((3) \text{ of section 35})\\
&= 2\left(-1\right)^n \qquad n \in \mathbb{Z}
\end{aligned}$$

it is clear that these points are simple zeros of $\sinh z$ and simple poles of $\dfrac{1}{\sinh z}$. At each $z_n = i\,n\,\dfrac{\pi}{2}$

$$\operatorname*{Res}_{z=i\,n\,\frac{\pi}{2}} \frac{1}{\sinh 2z} = \frac{1}{2\cosh(i\,n\,\pi)} = \frac{(-1)^n}{2}.$$

Of all these singular points only the ones corresponding to $n = -1, 0, 1$ lie inside the circle $|z| = 2$. Consequently

$$\begin{aligned} \int_C \frac{dz}{\sinh 2z} &= 2\pi i \left[\sum_{n=-1}^{1} \operatorname*{Res}_{z=z_n} \left(\frac{1}{\sinh 2z} \right) \right] \\ &= 2\pi i \left[-\frac{1}{2} + \frac{1}{2} - \frac{1}{2} \right] \\ &= -\pi i. \end{aligned}$$

Exercise 5, §76, p. 255

Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 8, section 43), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Solution:

$\dfrac{1}{z^2 \sin z}$ has singular points where $z = 0$ and where $\sin z = 0$, i.e. where $z_n = n\pi \ (n \in \mathbb{Z})$. Since

$$\left.\frac{d}{dz}\sin z\right|_{z_n = n\pi} = \cos(n\pi) \neq 0,$$

$\sin z$ has simple zeros at each z_n with z^2 having a zero of order 2 at $z = 0$. Thus $z^2 \sin z$ has a zero of order 3 at $z = 0$ and simple zeros at the other z_n's, whence $\dfrac{1}{z^2 \sin z}$ of course has a triple pole at 0 and simple poles at the other z_n's. Recall that $\sin z = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} z^{2n+1}$. Hence

$$\frac{1}{z^2 \sin z} = \frac{1}{z^3\left(1 - \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \dots\right)}$$

and so by the theorem in section 73

$$\begin{aligned}
\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} &= \frac{1}{2!} \frac{d^2}{dz^2} \left. \frac{1}{\left(1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - \ldots\right)} \right|_{z=0} \\
&= \frac{1}{2} \frac{d}{dz} \left. \frac{-\left(-\frac{2}{3!} z + \frac{4}{5!} z^3 - \ldots\right)}{\left(1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - \ldots\right)^2} \right|_{z=0} \\
&= \frac{1}{2\left(1 - \frac{1}{3!} z^2 + \ldots\right)^4} \left[\left(\frac{2}{3!} - \frac{12}{5!} z^2 + \ldots\right) \left(1 - \frac{1}{3!} z^2 + \ldots\right)^2 \right. \\
&\quad \left. \left. + 2\left(1 - \frac{1}{3!} z^2 + \ldots\right) \left(-\frac{2}{3!} z + \frac{4}{5!} z^3 - \ldots\right)^2 \right] \right|_{z=0} \\
&= \frac{1}{6}.
\end{aligned}$$

For each $z_n = n\pi \quad (n = \pm 1, \pm 2, \ldots)$ we have

$$\begin{aligned}
\operatorname*{Res}_{z=n\pi} \frac{1}{z^2 \sin z} &= \left. \frac{1/z^2}{\frac{d}{dz}(\sin z)} \right|_{z=n\pi} \\
&= \left. \frac{1/z^2}{\cos z} \right|_{z=n\pi} \\
&= \frac{(-1)^n}{(n\pi)^2}.
\end{aligned}$$

Since $z_n = n\pi$ is inside the square C_N whenever $-N \leq n \leq N$, it follows that

$$\begin{aligned}
&\int_{C_N} \frac{dz}{z^2 \sin z} \\
&= 2\pi i \left[\sum_{n=-N}^{N} \operatorname*{Res}_{z=z_n} \left(\frac{1}{z^2 \sin z} \right) \right] \\
&= 2\pi i \left[\operatorname*{Res}_{z=0} \left(\frac{1}{z^2 \sin z} \right) + \sum_{n=1}^{N} \left(\operatorname*{Res}_{z=z_n} \left(\frac{1}{z^2 \sin z} \right) + \operatorname*{Res}_{z=z_{-n}} \left(\frac{1}{z^2 \sin z} \right) \right) \right] \\
&= 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{(n\pi)^2} \right].
\end{aligned}$$

However in exercise 8 of section 43 we saw that

$$\lim_{N \to \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0.$$

Hence as $N \to \infty$, it follows that

$$0 = 2\pi i \left[\frac{1}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\pi)^2} \right].$$

This simplifies to

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Exercise 6, §76, p. 256
Show that

$$\int_C \frac{dz}{\left(z^2-1\right)^2+3} = \frac{\pi}{2\sqrt{2}}$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2,\ y = 0$ and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q\left(z\right) = \left(z^2-1\right)^2+3$ are the square roots of the numbers $1 \pm \sqrt{3i}$, show that the reciprocal $\dfrac{1}{q\left(z\right)}$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \text{ and } -\overline{z}_0 = \frac{-\sqrt{3}+i}{\sqrt{2}}.$$

Then apply Theorem 2 in section 76.

Solution:

$\dfrac{1}{\left(z^2-1\right)^2+3}$ has singular points where $\left(z^2-1\right)^2 = -3$, i.e. where $z^2 = 1 \pm i\sqrt{3}$. Now in polar form

$$1 + i\sqrt{3} = 2e^{\frac{i\pi}{3}} \text{ and } 1 - i\sqrt{3} = 2e^{-\frac{i\pi}{3}}.$$

The square roots of these numbers are precisely the singular points of $\dfrac{1}{\left(z^2-1\right)+3}$ and correspond to

$$\begin{aligned}
\sqrt{2}e^{\frac{i\pi}{6}} &= \sqrt{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = \sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}} \\
\sqrt{2}e^{\frac{i7\pi}{6}} &= \sqrt{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = -\sqrt{\frac{3}{2}}-i\frac{1}{\sqrt{2}} \\
\sqrt{2}e^{-\frac{i\pi}{6}} &= \sqrt{2}\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right) = \sqrt{\frac{3}{2}}-i\frac{1}{\sqrt{2}} \\
\sqrt{2}e^{\frac{i5\pi}{6}} &= \sqrt{2}\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = -\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}.
\end{aligned}$$

Of these points only

$$z_0 = \sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}} \text{ and } -\overline{z}_0 = -\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}$$

lie inside the given rectangle. All these points are simple poles and so

$$\begin{aligned}
\operatorname*{Res}_{z=z_0} \frac{1}{(z^2-1)^2+3} &= \left.\frac{1}{\frac{d}{dz}\left[(z^2-1)^2+3\right]}\right|_{z=z_0} \\
&= \left.\frac{1}{4(z^2-1)z}\right|_{z=z_0} \\
&= \frac{1}{4\left(\left(\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}\right)^2-1\right)\left(\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}\right)} \\
&= -\frac{\sqrt{3}}{24\sqrt{2}}-i\frac{1}{8\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname*{Res}_{z=-\overline{z}_0} \frac{1}{(z^2-1)^2+3} &= \left.\frac{1}{4(z^2-1)z}\right|_{z=-\overline{z}_0} \\
&= \frac{1}{4\left(\left(-\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}\right)^2-1\right)\left(-\sqrt{\frac{3}{2}}+i\frac{1}{\sqrt{2}}\right)} \\
&= \frac{\sqrt{3}}{24\sqrt{2}}-i\frac{1}{8\sqrt{2}}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\int_C \frac{dz}{(z^2-1)^2+3} &= 2\pi i\left[\operatorname*{Res}_{z=z_0}\frac{1}{(z^2-1)^2+3}+\operatorname*{Res}_{z=-\overline{z}_0}\frac{1}{(z^2-1)^2+3}\right] \\
&= 2\pi i\left(-\frac{i}{4\sqrt{2}}\right) \\
&= \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

Exercise 7, §76, p. 256

Consider the function

$$f(z)=\frac{1}{[q(z)]^2},$$

where q is analytic at z_0, $q(z_0)=0$, and $q'(z_0)\neq 0$. Show that z_0 is a pole of order $m=2$ of the function f, with residue

$$B_0=\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Suggestion: Note that z_0 is a zero of order $m=1$ of the function q, so that equation (3), section 76, holds. Then write

$$f(z)=\frac{\phi(z)}{(z-z_0)^2} \quad \text{where } \phi(z)=\frac{1}{[g(z)]^2}.$$

The desired form of the residue $B_0=\phi'(z_0)$ can be obtained by showing that $q'(z_0)=g(z_0)$ and $q''(z_0)=2g'(z_0)$.

Solution:

Since by hypothesis q has a simple zero at z_0, it follows from (3) of section 76 that

$$(*) \qquad q(z) = (z - z_0)\, g(z)$$

where g is non–zero and analytic at z_0. Hence

$$f(z) = \frac{1}{q(z)^2} = \frac{1/g(z)^2}{(z-z_0)^2}$$

where $\dfrac{1}{g(z)^2}$ is analytic and nonzero at $z = z_0$. Thus by the theorem in section 73

$$\begin{aligned} \operatorname*{Res}_{z=z_0} f(z) &= \frac{1}{1!}\frac{d}{dz}\left.\frac{1}{[g(z)]^2}\right|_{z=z_0} \\ &= -\frac{2g'(z_0)}{[g(z_0)]^3}. \end{aligned}$$

However by $(*)$ above

$$q'(z) = g(z) + (z - z_0)\, g'(z)$$

and

$$q''(z) = 2g'(z) + (z - z_0)\, g''(z).$$

Consequently $q'(z_0) = g(z_0)$ and $q''(z_0) = 2g'(z_0)$. Therefore

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{-2g'(z_0)}{[g(z_0)]^3} = -\frac{q''(z_0)}{[q'(z_0)]^3}.$$

Exercise 8, §76, p. 256

Use the result in Exercise 7 to find the residue at $z = 0$ of the function

(a) $f(z) = \csc^2 z$;

(b) $f(z) = \dfrac{1}{(z+z^2)^2}$.

Solution:

(a) Since $\left.\dfrac{d}{dz}\sin z\right|_{z=0} = \cos 0 = 1$, $\sin z$ has a simple zero at $z = 0$. We may therefore directly apply the results of Exercise 7 above to the function

$$f(z) = \csc^2 z = \frac{1}{\sin^2 z}$$

to see that

$$\operatorname*{Res}_{z=0} \csc^2 z = -\left.\frac{\left(\frac{d^2}{dz^2}\sin z\right)}{\left[\frac{d}{dz}\sin z\right]^3}\right|_{z=0} = \left.\frac{\sin z}{\cos^3 z}\right|_{z=0} = 0.$$

(b) Observe that $z + z^2$ has a zero at 0 whereas $\left.\dfrac{d}{dz}\left(z+z^2\right)\right|_{z=0} = 1$. As in part (a), we may therefore directly apply the results of Exercise 7 above to

$$f(z) = \frac{1}{(z+z^2)^2},$$

to conclude that

$$\operatorname*{Res}_{z=0} \frac{1}{\left(z+z^2\right)^2} = \left. \frac{-\left(\frac{d^2}{dz^2}\left(z+z^2\right)\right)}{\left[\frac{d}{dz}\left(z+z^2\right)\right]^3} \right|_{z=0} = - \left. \frac{2}{\left[1+2z\right]^3} \right|_{z=0} = -2.$$

CHAPTER 7

Applications of Residues

Study only sections 78–83 and 85–87. Note that some of the exercises at the end of section 84 refer to section 83. Although the material of section 84 does not form part of the course, these particular exercises should be attempted. Thus in section 84 attempt at least exercises 1, 3, 4, and 6(a), using the techniques of section 83.

The importance of Cauchy's Residue Theorem (discussed in chapter 6) stretches far beyond complex analysis itself. Residue theory has many important applications including the computation of real integrals, and finding the roots of polynomials. Using a few clever tricks many classes of *real* integrals can be written in a form where we can use residue theory to compute the integral. A number of these classes play a very important role in physics, and are typically extremely difficult to compute by other means. The largest part of chapter 7 is spent describing precisely how one goes about using residue theory to compute these real integrals. A second important consequence of residue theory, is Rouché's theorem. By the fundamental theorem of algebra we know that a polynomial of degree n, has n roots. Unfortunately for polynomials of degree 4 or higher, there is no simple formula for computing the roots. So knowing that the roots exist is very nice, but where are they? Using Rouché's theorem, we can approximate complicated polynomials by simpler ones, and in this way at least get some idea of where to search for the roots of such complicated polynomials.

Solutions to selected problems

Exercise 1, §79, p. 267

Use residues to evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

Solution:

The function $f(z) = \dfrac{1}{1+z^2}$ has singularities at $\pm i$. We integrate f around the contour $[-R, R] \cup C_R$ $(R > 0)$ where

$$\begin{aligned} C_R &: \quad z(t) = R\,e^{it} \qquad 0 \le t \le \pi \\ [-R, R] &: \quad z(t) = t \qquad\quad -R \le t \le R. \end{aligned}$$

For $R > 1$ the singular point $z = i$ is in the interior of $[-R, R] \cup C_R$. So by Cauchy's residue theorem

$$\int_{-R}^{R} f(t)\,dt + \int_{C_R} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=i} f(z)\right].$$

Now by Theorem 2 in section 76, $z = i$ is a simple pole of f with

$$\operatorname*{Res}_{z=i} \frac{1}{1+z^2} = \left.\frac{1}{2z}\right|_{z=i} = \frac{1}{2i},$$

and hence

$$\int_{-R}^{R} f(t)\,dt = \pi - \int_{C_R} f(z)\,dz.$$

For $z = R\,e^{it}$ we have $\left|z^2+1\right| \ge R^2 - 1$ whence $\left|\dfrac{1}{1+z^2}\right| \le \dfrac{1}{R^2-1}$. Then

$$\left|\int_{C_R} \frac{1}{1+z^2} dz\right| \le \left(\frac{1}{R^2-1}\right) \pi R$$

where πR is the length of C_R. From this it is clear that $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$. Consequently

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{1+t^2} dt &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+t^2} dt \\ &= \pi - \lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^2} dz \\ &= \pi. \end{aligned}$$

Since the integrand is even it follows that

$$\int_0^\infty \frac{1}{1+t^2} dt = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+t^2} = \frac{\pi}{2}.$$

(Note: We didn't actually need residues to compute this integral. Since $\dfrac{d}{dt} \arctan t = \dfrac{1}{1+t^2}$, it follows that

$$\begin{aligned} \int_0^\infty \frac{1}{1+t^2} dt &= \lim_{b\to\infty} \int_0^b \frac{1}{1+t^2} dt \\ &= \lim_{b\to\infty} \arctan b - \arctan 0 \\ &= \frac{\pi}{2}.) \end{aligned}$$

Exercise 2, §79, p. 267

Use residues to evaluate the integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$.

Solution:

Consider $f(z) = \frac{1}{(z^2+1)^2}$. By the theorem in section 73 f has poles of order 2 at $z = \pm i$. We integrate f over the same contour used in exercise (1) above. As before $z = i$ is in the interior of $[-R, R] \cup C_R$ when $R > 1$, in which case Cauchy's residue theorem tells us that

$$\int_{-R}^{R} f(t)\,dt + \int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z).$$

Now since

$$\frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2}\left(\frac{1}{(z+i)^2}\right),$$

it follows from the theorem in section 73 that

$$\begin{aligned}\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2} &= \left.\frac{d}{dz}\frac{1}{(z+i)^2}\right|_{z=i} \\ &= \left.\frac{-2}{(z+i)^3}\right|_{z=i} \\ &= -\frac{i}{4}\end{aligned}$$

and hence that

$$\begin{aligned}\int_{-R}^{R} \frac{1}{(t^2+1)^2}dt &= 2\pi i\left(-\frac{i}{4}\right) - \int_{C_R} \frac{1}{(z^2+1)^2}dz \\ &= \frac{\pi}{2} - \int_{C_R} \frac{1}{(z^2+1)^2}dz.\end{aligned}$$

As before if $z = Re^{it}$ ($|z| = R$), then

$$\left|\frac{1}{(z^2+1)^2}\right| \le \frac{1}{\left(|z|^2-1\right)^2} = \frac{1}{(R^2-1)^2}.$$

Since the length of C_R is πR, this means that

$$\left|\int_{C_R} \frac{1}{(z^2+1)^2}dz\right| \le \frac{\pi R}{(R^2-1)^2} \to 0$$

as $R \to \infty$. But then

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+1)^2}dt = \frac{\pi}{2} - \lim_{R\to\infty}\int_{C_R} \frac{1}{(z^2+1)^2}dz = \frac{\pi}{2}.$$

Since the integrand is an even function this yields

$$\int_0^\infty \frac{1}{(t^2+1)^2}dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(t^2+1)^2}dt = \frac{\pi}{4}.$$

Exercise 3, §79, p. 267

Use residues to evaluate the improper integral $\int_0^\infty \frac{dx}{x^4+1}$.

Solution:

Consider $f(z) = \dfrac{1}{1+z^4}$. Such an f has singular points where $z^4 = -1 = e^{i\pi}$, i.e. where

$$z_0 = e^{\frac{i\pi}{4}},\ z_1 = e^{\frac{i3\pi}{4}},\ z_2 = e^{\frac{i5\pi}{4}},\ z_3 = e^{\frac{i7\pi}{4}}.$$

(See section 9.) Of these only z_0 and z_1 are in the upper half–plane. Integrate f over the same contour as in exercise (1) above. Then for $R > 1$ it follows from Cauchy's Residue Theorem that

$$\int_{-R}^{R} \frac{1}{1+t^4}dt + \int_{C_R} \frac{1}{1+z^4}dz = 2\pi i \left[\operatorname*{Res}_{z=z_0} \frac{1}{1+z^4} + \operatorname*{Res}_{z=z_1} \frac{1}{1+z^4}\right].$$

By the Theorem in section 76 we have

$$\operatorname*{Res}_{z=z_0} \frac{1}{1+z^4} = \frac{1}{4z_0^3} = \frac{z_0}{4z_0^4} = \frac{e^{\frac{i\pi}{4}}}{-4} = \frac{-1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}.$$

Similarly

$$\operatorname*{Res}_{z=z_1} \frac{1}{1+z^4} = \frac{z_1}{4z_1^4} = \frac{e^{\frac{i3\pi}{4}}}{-4} = \frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}.$$

Moreover for $|z| = R$

$$\left|\frac{1}{1+z^4}\right| \le \frac{1}{|z|^4 - 1} = \frac{1}{R^4-1}.$$

Therefore

$$\left|\int_{C_R} \frac{1}{1+z^4}dz\right| \le \left(\frac{1}{R^4-1}\right)\pi R \to 0$$

as $R \to \infty$. Consequently

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{1+t^4}dt \\ = \ & 2\pi i \left[\left(-\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}\right) + \left(\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}\right)\right] - \lim_{R\to\infty} \int_{C_R} \frac{1}{1+z^4}dz \\ = \ & \frac{\pi}{\sqrt{2}}, \end{aligned}$$

whence

$$\int_0^{\infty} \frac{1}{1+t^4}dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+t^4}dt = \frac{\pi}{2\sqrt{2}}$$

since $\dfrac{1}{1+t^4}$ is an even function.

Exercise 5, §79, p. 267

Use residues to evaluate the improper integral $\displaystyle\int_0^{\infty} \frac{x^2dx}{(x^2+9)(x^2+4)^2}$.

Solution:

We integrate $z^2/\left[\left(z^2+9\right)\left(z^2+4\right)^2\right]$ around the positively oriented contour $C_R \cup [-R, R]$ where

$$\begin{aligned} C_R &: \ z(t) = R\,e^{it} \qquad 0 \le t \le \pi \\ [-R, R] &: \ z(t) = t \qquad -R \le t \le R. \end{aligned}$$

Now $f(z) = z^2/\left[\left(z^2+9\right)\left(z^2+4\right)^2\right]$ has singularities where $\left(z^2+9\right)\left(z^2+4\right)^2 = 0$, i.e. where $z = \pm 3i$, $z = \pm 2i$. Of these only $3i$ and $2i$ will lie inside $C_R \cup [-R, R]$ for R large enough. For R large it then follows from the residue theorem that

$$(*) \quad \int_{C_R} f(z)\,dz + \int_{[-R,R]} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=3i}(f(z)) + \operatorname*{Res}_{z=2i}(f(z))\right].$$

Since $\left(z^2+9\right)\left(z^2+4\right)^2 = (z-3i)(z+3i)(z-2i)^2(z+2i)^2$ it follows from §73 that $f(z)$ has a simple pole at $z = 3i$ and a double pole at $z = 2i$. Therefore by the Theorem in §76

$$\begin{aligned}
\operatorname*{Res}_{z=3i}(f) &= \lim_{z\to 3i}(z-3i)\left[\frac{z^2}{(z-3i)(z+3i)\left(z^2+4\right)^2}\right] \\
&= \lim_{z\to 3i}\frac{z^2}{(z+3i)\left(z^2+4\right)^2} \\
&= \frac{-9}{6i(-9+4)^2} \\
&= \frac{3i}{50}
\end{aligned}$$

and by the Theorem in §73

$$\begin{aligned}
\operatorname*{Res}_{z=2i}(f) &= \frac{1}{1!}\frac{d}{dz}\left(\frac{z^2}{\left(z^2+9\right)(z+2i)^2}\right)\Bigg|_{z=2i} \\
&= \frac{\left[\left(z^2+9\right)(z+2i)^2\right]2z - \left[2z(z+2i)^2 + 2\left(z^2+9\right)(z+2i)\right]z^2}{\left(z^2+9\right)^2(z+2i)^4}\Bigg|_{z=2i} \\
&= -\frac{13i}{200}
\end{aligned}$$

(Note that $f(z) = (z-2i)^{-2}g(z)$ where $g(z) = z^2/\left[\left(z^2+9\right)(z+2i)^2\right]$.) Now for $z = Re^{it}$ we have

$$\begin{aligned}
\left|\left(z^2+9\right)\left(z^2+4^2\right)\right| &= \left|z^2+9\right|\left|z^2+4\right|^2 \\
&\geq \left(|z|^2-9\right)\left(|z|^2-4\right)^2 \\
&= \left(R^2-9\right)\left(R^2-4\right)^2,
\end{aligned}$$

and so with $z(t) = Re^{it}$ we have that

$$\begin{aligned}
\left|\int_{C_R} f(z)\,dz\right| &\leq \int_0^\pi \left|\frac{z(t)^2}{(z(t)^2+9)(z(t)^2+4)^2}\right|\left|iRe^{it}\right|dt \\
&\leq \int_0^\pi \frac{R^3}{\left(R^2-9\right)\left(R^2-4\right)^2}dt = \frac{R^3\pi}{\left(R^2-9\right)\left(R^2-4\right)^2} \to 0
\end{aligned}$$

as $R \to \infty$. Applying all this to $(*)$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx &= \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz + \int_{-R}^{R} \frac{x^2}{(x^2+9)(x^2+4)^2} dx \\ &= 2\pi i \left[\frac{3i}{50} - \frac{13i}{200}\right] \\ &= \frac{\pi}{100}.\end{aligned}$$

Since the integrand is an even function it then follows that

$$\begin{aligned}\int_{0}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)} dx \\ &= \frac{\pi}{200}.\end{aligned}$$

Exercise 7, §79, p. 267
Use residues to find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x dx}{(x^2+1)(x^2+2x+2)}.$$

Solution:
Let $f(z) = \dfrac{z}{(z^2+1)(z^2+2z+2)}$. This function has singular points where $z = \pm i$ and where $z = \frac{1}{2}\left(-2 \pm \sqrt{4-8}\right) = -1 \pm i$. Of these only $z = i$ and $z = -1+i$ lie in the upper half–plane. By Theorem 2 in section 76

$$\begin{aligned}\operatorname*{Res}_{z=i} \frac{z}{(z^2+1)(z^2+2z+2)} &= \left.\frac{\frac{z}{(z^2+2z+2)}}{\frac{d}{dz}(z^2+1)}\right|_{z=i} \\ &= \frac{\frac{i}{(i^2+2i+2)}}{2i} \\ &= \frac{1}{2(1+2i)} \\ &= \frac{1}{10} - i\frac{2}{10}.\end{aligned}$$

Similarly

$$\begin{aligned}\operatorname*{Res}_{z=-1+i} \frac{z}{(z^2+1)(z^2+2z+2)} &= \left.\frac{\frac{z}{(z^2+1)}}{2z+2}\right|_{z=-1+i} \\ &= \frac{\frac{(-1+i)}{(1-2i)}}{2i} \\ &= -\frac{1}{10} + \frac{3i}{10}.\end{aligned}$$

Now let C_R be as in exercise (1) above. Then for $R > 0$ large enough it will follow from Cauchy's Residue theorem that

$$(*)\qquad \begin{aligned} \int_{-R}^{R} \frac{xdx}{(x^2+1)(x^2+2x+2)} &+ \int_{C_R} \frac{1}{(z^2+1)(z^2+2z+2)}dz \\ &= 2\pi i \left[\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-1+i} f(z) \right] \\ &= 2\pi i \left(\frac{i}{10} \right) \\ &= -\frac{\pi}{5}. \end{aligned}$$

Next note that for z on C_R (i.e. $|z| = R$)

$$\begin{aligned} \left|z^2+1\right| &\geq |z|^2 - 1 = R^2 - 1 \\ \left|z^2+2z+2\right| &\geq \left|z^2+2z\right| - 2 \geq |z|^2 - 2|z| - 2 = R^2 - 2R - 2 \end{aligned}$$

and hence

$$\left| \frac{z}{(z^2+1)(z^2+2z+2)} \right| \leq \frac{R}{(R^2-1)(R^2-2R-2)}$$

for such z. Therefore

$$\left| \int_{C_R} \frac{zdz}{(z^2+1)(z^2+2z+2)} \right| \leq \frac{\pi R^2}{(R^2-1)(R^2-2R-2)} \to 0$$

as $R \to \infty$. (Recall that the length of C_R is πR.) If in $(*)$ above we let $R \to \infty$, we then get

$$\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}.$$

Exercise 8, §79, p. 267

Use residues and the contour shown in Fig. 95 in the textbook, where $R > 1$, to establish the integration formula

$$\int_0^{\infty} \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

Solution:

To compute $\int\limits_0^{\infty} \frac{1}{1+x^3}dx$ we follow the hint and integrate around the perimeter of the sector corresponding to the join of the interval $[0, R]$, the curve $C_R : R\,e^{it}$ $\left(0 \leq t \leq \frac{2\pi}{3}\right)$, and the line segment $L_R : (R-t)\,e^{\frac{i2\pi}{3}}$ $(0 \leq t \leq R)$from $R\,e^{\frac{i\pi 2}{3}}$ to 0.

$\dfrac{1}{1+z^3}$ has singularities at the 3rd roots of -1, i.e. at $z = e^{\left(\frac{i(2k+1)\pi}{3}\right)}$, $k = 0, 1, 2$. Of these only the root corresponding to $k = 0$ (the one with argument $\dfrac{\pi}{3}$) will lie inside the given curve for R big enough. Hence

$$(1)\qquad \int\limits_0^R \frac{1}{1+x^3}dx + \int\limits_{C_R} \frac{1}{1+z^3}dz + \int\limits_{L_R} \frac{1}{1+z^3}dz = 2\pi i \operatorname*{Res}_{z=e^{\frac{i\pi}{3}}} \left(\frac{1}{1+z^3} \right).$$

Now by Theorem 2 in §76, $\frac{1}{1+z^3}$ has a simple pole at $e^{\frac{i\pi}{3}}$ with

$$\operatorname*{Res}_{z=e^{\frac{i\pi}{3}}} \left(\frac{1}{1+z^3} \right) = \frac{1}{3z^2}\Big|_{z=e^{\frac{i\pi}{3}}} = \frac{z}{3z^3}\Big|_{z=e^{\frac{i\pi}{3}}} = -\frac{1}{3}e^{\frac{i\pi}{3}}$$

(Here we used the fact that $e^{\frac{i\pi}{3}}$ is a 3^{rd} root of -1.)

Moreover

$$\begin{aligned}\left|\int\limits_{C_R}\frac{1}{1+z^3}dz\right| &\leq \int\limits_0^{\frac{2\pi}{3}}\left|\frac{1}{1+R^3e^{i3t}}\right|Rdt\\ &\leq \int\limits_0^{\frac{2\pi}{3}}\frac{R}{R^3-1}dt\\ &= \frac{2R\pi}{3\left(R^3-1\right)}\to 0 \text{ as } R\to\infty\end{aligned}$$

and

$$\begin{aligned}\int\limits_{L_R}\frac{1}{1+z^3}dz &= \int\limits_0^R\frac{1}{1+[(R-t)\,e^{\frac{i2\pi}{3}}]^3}\left(-e^{\frac{i2\pi}{3}}\right)dt\\ &= -e^{\frac{i2\pi}{3}}\int\limits_0^R\frac{1}{1+(R-t)^3}dt.\end{aligned}$$

Setting $x = R - t$ yields

$$\int\limits_{L_R}\frac{1}{1+z^3}dz = -e^{\frac{i2\pi}{3}}\int\limits_0^R\frac{1}{1+x^3}dx.$$

Thus (1) reduces to

$$\left(1-e^{\frac{i2\pi}{3}}\right)\int\limits_0^R\frac{1}{1+x^3}dx+\int\limits_{C_R}\frac{1}{1+z^3}dz=\frac{(-2\pi ie^{\frac{i\pi}{3}})}{3}.$$

Letting $R\to\infty$ it follows that

$$\left(1-e^{\frac{i2\pi}{3}}\right)\int\limits_0^\infty\frac{1}{1+x^3}dx=\frac{-2\pi ie^{\frac{i\pi}{3}}}{3}$$

or equivalently that

$$\begin{aligned}\int\limits_0^\infty\frac{1}{1+x^3}dx &= \frac{\pi}{3}\left(\frac{2ie^{\frac{i\pi}{3}}}{e^{\frac{i2\pi}{3}}-1}\right)\times\frac{e^{\frac{-i\pi}{3}}}{e^{\frac{-i\pi}{3}}}\\ &= \frac{\pi}{3}\left(\frac{2i}{e^{\frac{i\pi}{3}}-e^{\frac{-i\pi}{3}}}\right)\\ &= \frac{\pi}{3}\frac{1}{\sin\frac{\pi}{3}}\\ &= \frac{\pi}{3}\text{cosec}\frac{\pi}{3}.\end{aligned}$$

Exercise 9, §79, p. 268

Let m and n be integers, where $0\leq m<n$. Follow the steps below to derive the integration formula

$$\int_0^\infty\frac{x^{2m}}{x^{2n}+1}dx=\frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\,\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

and that there are none on that axis.

(b) With the aid of Theorem 2 in Sec. 76, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)\,.$$

where c_k are the zeros found in part (a) and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the identity (see Exercise 9, Sec. 8)

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \neq 1)$$

to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2n}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part (b) to complete the derivation of the integration formula.

Solution:

(a) The zeros of the polynomial $z^{2n}+1$ are at all z for which $z^{2n} = -1 = e^{i\pi}$. Thus by the formula in section 9 the roots are

$$z_k = \exp\left(i\left(\frac{\pi}{2n} + \frac{2k\pi}{2n}\right)\right) = \exp\left(i\frac{(2k+1)\,\pi}{2n}\right)$$

where $0 \le k \le 2n-1$. For such a root to be on the real axis $\frac{(2k+1)}{2n}\pi$ would have to be an integer multiple of π. But since $2n$ is even and $2k+1$ odd, this is not possible since $\frac{2k+1}{2n}$ can never be an integer. The roots that are in the upper half–plane are those for which

$$0 \le \frac{2k+1}{2n}\pi \le \pi,$$

i.e. those corresponding to all k with

$$0 \le \frac{2k+1}{2n} \le 1.$$

We conclude that

$$z_k = \exp\left(i\frac{(2k+1)\,\pi}{2n}\right) \qquad 0 \le k \le n-1$$

are all the roots that lie in the upper half–plane.

(b) Let $z_k = \exp\left(i\frac{(2k+1)\pi}{2n}\right)$ $(0 \leq k \leq n-1)$ be as in (a). By Theorem 2 in section 76 each of these is a simple pole of $\frac{z^{2m}}{(z^{2n}+1)}$ with

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} \frac{z^{2m}}{z^{2n}+1} &= \left.\frac{z^{2m}}{2nz^{2n-1}}\right|_{z=z_k} \\
&= \frac{1}{2n}(z_k)^{2(m-n)+1} \\
&= \frac{1}{2n}\left[e^{i(2k+1)\pi/2n}\right]^{2(m-n)+1} \\
&= \frac{1}{2n}e^{-i(2k+1)\pi}e^{i(2k+1)\alpha} \\
&= -\frac{1}{2n}e^{i(2k+1)\alpha}
\end{aligned}$$

where $\alpha = \frac{2m+1}{2n}\pi$. (Since $2k+1$ is odd, $e^{-i(2k+1)\pi} = -1$ by (3) of section 6.) Given that

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad z \neq 1$$

it therefore follows that

$$\begin{aligned}
2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{z^{2n}+1} &= 2\pi i \sum_{k=0}^{n-1} \frac{-1}{2n} e^{i(2k+1)\alpha} \\
&= -\frac{\pi i}{n} e^{i\alpha} \sum_{k=0}^{n-1} \left(e^{i2\alpha}\right)^k \\
&= -\frac{\pi i}{n} e^{i\alpha} \frac{1-e^{i2n\alpha}}{1-e^{i2\alpha}} \\
&= \frac{\pi i}{n} \frac{1-e^{i(2m+1)\pi}}{\left(e^{i\alpha}-e^{-i\alpha}\right)} \\
&= \frac{\pi i}{n} \frac{1-(-1)}{2i\sin\alpha} \\
&= \frac{\pi}{n\sin\alpha}
\end{aligned}$$

(recalling that $\alpha = \frac{(2m+1)\pi}{2n}$ and that $\frac{1}{2i}\left(e^{i\alpha}-e^{-i\alpha}\right) = \sin\alpha$).

(c) We now integrate $f(z) = \frac{z^{2m}}{z^{2n}+1}$ over the curve $[-R, R] \cup C_R$ where C_R is as in exercise 1 above. For R large enough $(R > 1)$ the curve $[-R, R] \cup C_R$ will contain all singular points of $f(z)$ in the upper half–plane. Therefore with z_k as in (a), we have by (a) and (b) that

$$\begin{aligned}
&\int_{-R}^{R} \frac{x^{2m}}{x^{2n}+1}dx + \int_{C_R} \frac{z^{2m}}{z^{2n}+1}dz \\
&= 2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=z_k} \frac{z^{2m}}{z^{2n}+1} \\
&= \frac{\pi}{n\sin\alpha} \\
&= \frac{\pi}{n}\operatorname{cosec}\left(\frac{(2m+1)\pi}{2n}\right)
\end{aligned}$$

(where $\alpha = (2m+1)\pi/2n$). Now for z on C_R (i.e. $|z| = R$) we have that $\left|z^{2n}+1\right| \geq |z|^{2n} - 1 = R^{2n} - 1$ and hence that

$$\left|\frac{z^{2m}}{z^{2n}+1}\right| \leq \frac{R^{2m}}{R^{2n}-1}.$$

Therefore

$$\begin{aligned}
\left|\int_{C_R} \frac{z^{2m}}{z^{2n}+1} dz\right| &\leq \left(\frac{R^{2m}}{R^{2n}-1}\right)\pi R \qquad \frac{\div R^{2n}}{\div R^{2n}} \\
&= \frac{R^{-[2(n-m)-1]}\pi}{1-R^{-2n}} \\
&\to \frac{0}{1} = 0
\end{aligned}$$

as $R \to \infty$. (To see that this is the case, recall that $2(n-m) - 1 > 0$ since $n > m$.) Letting $R \to \infty$ it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{z^{2m}}{z^{2n}+1} dz &= \frac{\pi}{n}\operatorname{cosec}\left(\frac{(2m+1)\pi}{2n}\right) - \lim_{R\to\infty}\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz \\
&= \frac{\pi}{n}\operatorname{cosec}\left(\frac{(2m+1)\pi}{2n}\right).
\end{aligned}$$

Since the integrand is even this means that

$$\begin{aligned}
\int_{0}^{\infty} \frac{z^{2m}}{z^{2n}+1} dz &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{z^{2m}}{z^{2n}+1}dz \\
&= \frac{\pi}{2n}\operatorname{cosec}\left[\frac{(2m+1)\pi}{2n}\right].
\end{aligned}$$

Exercise 1, §81, p. 275

Use residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} \qquad (a > b > 0).$$

Solution:

We integrate

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$$

around the contour $C_R \cup [-R, R]$ where as before C_R is the circle sector

$$C_R : z(t) = R\,e^{it} \quad 0 \leq t \leq \pi.$$

The function $f(z)$ has singularities where $\left(z^2+a^2\right)\left(z^2+b^2\right) = 0$, i.e. where $z = \pm ai,\ z = \pm bi$. Of these only ai and bi lie inside $C_R \cup [-R, R]$ for large R and so by the residue theorem

$$\int_{-R}^{R} f(\dot{x})\,dx + \int_{C_R} f(z)\,dz = 2\pi i\left[\operatorname*{Res}_{z=ai}(f) + \operatorname*{Res}_{z=bi}(f)\right]$$

for large R. The function f has simple poles at ai and bi and therefore by Theorem 2 in §76

$$\begin{aligned}\operatorname*{Res}_{z=ai}(f) &= \lim_{z\to ai}\left[\frac{e^{iz}}{(z+ai)(z^2+b^2)}\right]\\ &= \frac{e^{-a}}{2ai(b^2-a^2)} = \frac{e^{-a}i}{2a(a^2-b^2)}\\ \operatorname*{Res}_{z=bi}(f) &= \lim_{z\to bi}\left[\frac{e^{iz}}{(z^2+a^2)(z+bi)}\right]\\ &= -\frac{e^{-b}i}{2b(a^2-b^2)}.\end{aligned}$$

Moreover for $z = R\,e^{it} = R\cos t + iR\sin t$

$$\begin{aligned}\left|e^{iz}\right| &= e^{-R\sin t} \le e^0 = 1 \text{ for } t\in[0,\pi],\\ \left|\left(z^2+a^2\right)\left(z^2+b^2\right)\right| &\ge \left(|z|^2-a^2\right)\left(|z|^2-b^2\right) = \left(R^2-a^2\right)\left(R^2-b^2\right).\end{aligned}$$

Therefore

$$\begin{aligned}\left|\int_{C_R}\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}dz\right| &\le \int_0^\pi \frac{\left|e^{iz(t)}\right|}{(R^2-a^2)(R^2-b^2)}\left|iR\,e^{it}\right|dt\\ &\le \frac{R\pi}{(R^2-a^2)(R^2-b^2)} \to 0\end{aligned}$$

as $R\to\infty$. Applying all of this to the first equality, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}dx &= \lim_{R\to\infty}\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx\\ &= 2\pi i\left[\frac{e^{-a}i}{2a(a^2-b^2)} - \frac{e^{-b}i}{2b(a^2-b^2)}\right]\\ &= \frac{\pi}{(a^2-b^2)}\left(\frac{1}{be^b}-\frac{1}{ae^a}\right).\end{aligned}$$

Finally recall that $\Re\left(e^{ix}\right) = \cos x$. If therefore we compare the real parts of the previous equality, we get that

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{(a^2-b^2)}\left(\frac{1}{be^b}-\frac{1}{ae^a}\right).$$

Exercise 3, §81, p. 275

Use residues to evaluate the improper integral

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)}dx \qquad (a>0,\ b>0).$$

Solution:

We integrate $g(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$ around the contour $C_R\cup[-R;R]$ where C_R is parametrised by $z(t) = R\,e^{it}$ $(0\le t\le\pi)$. The function g has singularities at $\pm bi$. Of these only bi will lie inside $C_R\cup[-R,R]$ for large $R>0$. Therefore by the residue theorem

$$\int_{-R}^{R}\frac{e^{iax}}{(x^2+b^2)^2}dx + \int_{C_R}\frac{e^{iaz}}{(z^2+b^2)^2}dz = 2\pi i\left(\operatorname*{Res}_{z=bi} g(z)\right).$$

Since

$$g(z) = \frac{1}{(z-bi)^2}\left[\frac{e^{iaz}}{(z+bi)^2}\right],$$

$z = bi$ is a double pole. By the theorem in section 73

$$\begin{aligned}\operatorname*{Res}_{z=bi} g(z) &= \frac{d}{dz}\left(\frac{e^{iaz}}{(z+bi)^2}\right)\Bigg|_{z=bi} \\ &= \frac{(z+bi)^2\, iae^{iaz} - 2(z+bi)\,e^{iaz}}{(z+bi)^4}\Bigg|_{z=bi} \\ &= \frac{(ab+1)\,e^{-ab}}{i4b^3}.\end{aligned}$$

Therefore by equating real parts

$$\int_{-R}^{R} \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi(ab+1)\,e^{-ab}}{2b^3} - \Re\int_{C_R}\frac{e^{iaz}}{(z^2+b^2)^2}dz.$$

Next observe that since $\left|e^{iaz}\right| = e^{-ay} \leq 1$ whenever $y \geq 0$, and $\left|\left(z^2+b^2\right)^2\right| \geq \left(|z|^2 - b^2\right)^2$, it follows that

$$|g(z)| \leq \frac{1}{(R^2-b^2)^2}$$

whenever $z \in C_R$. Consequently

$$\left|\Re\int_{C_R} g(z)\,dz\right| \leq \left|\int_{C_R} g(z)\,dz\right| \leq \frac{\pi R}{(R^2-b^2)^2} \to 0$$

as $R \to \infty$. If therefore we let $R \to \infty$, then surely

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)}dx = \frac{\pi(ab+1)}{2b^3e^{ab}}.$$

Since $\frac{\cos a(-x)}{\left((-x)^2+b^2\right)^2} = \frac{\cos ax}{(x^2+b^2)^2}$ (the integrand is even), this means that

$$\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi(ab+1)}{4b^3e^{ab}}.$$

Exercise 5, §81, p. 276

Use residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}dx \qquad (a > 0).$$

Solution:

We integrate $g(z) = \frac{ze^{iaz}}{(z^4+4)}$ over the contour $[-R, R] \cup C_R$ where C_R is the semicircle $C_R : z(t) = R\,e^{it}\,(0 \leq t \leq \pi)$.

The function g has singular points where $z^4 = -4 = e^{i\pi}$, i.e. at

$$z_k = \sqrt{2}e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)} \qquad 0 \leq k \leq 3.$$

Of these z_0 and z_1 are in the upper half–plane. Therefore for R big enough $\left(R > \sqrt{2}\right)$, Cauchy's residue theorem ensures that

$$\int_{-R}^{R} \frac{xe^{iax}}{x^4+4}dx + \int_{C_R}\frac{ze^{iaz}}{z^4+4}dz = 2\pi i\left[\operatorname*{Res}_{z=z_0} g(z) + \operatorname*{Res}_{z=z_1} g(z)\right].$$

On comparing imaginary parts we see that

$$\int_{-R}^{R}\frac{x\sin ax}{x^4+4}dx=\Im\left(2\pi i\left[\operatorname*{Res}_{z=z_0}g(z)+\operatorname*{Res}_{z=z_1}g(z)\right]\right)-\Im\left(\int_{C_R}\frac{ze^{iaz}}{z^4+4}dz\right).$$

Now if $R>\sqrt{2}$, $f(z)=\frac{z}{z^4+4}$ is analytic outside and on the circle $|z|=R$. Now for $|z|=R$, $\left|z^4+4\right|\geq|z|^4-4=R^4-4$ whence

$$\left|\frac{z}{z^4+4}\right|\leq\frac{R}{R^4-4}\to 0\quad\text{as}\quad R\to\infty.$$

Therefore by Jordan's lemma (the Theorem in section 81)

$$\int_{C_R}\frac{ze^{iaz}}{z^4+4}dz\to 0\quad\text{as}\quad R\to\infty,$$

i.e.

$$\Im\left(\int_{C_R}\frac{ze^{iaz}}{z^4+4}dz\right)\to 0\quad\text{as}\quad R\to\infty.$$

Now by Theorem 2 in section 76

$$\operatorname*{Res}_{z=z_0}\frac{ze^{iaz}}{z^4+4}=\frac{z_0e^{iaz_0}}{4z_0^3}=\frac{z_0^2e^{iaz_0}}{4z_0^4}=\frac{2e^{i\frac{\pi}{2}}e^{ia(1+i)}}{-16}=\frac{-i}{8}e^{-a}e^{ia}.$$

Similarly

$$\operatorname*{Res}_{z=z_1}\frac{ze^{iaz}}{z^4+1}=\frac{z_1^2e^{iaz_1}}{4z_1^4}=\frac{i}{8}e^{-a}e^{-ia}.$$

Therefore

$$\operatorname*{Res}_{z=z_0}\frac{ze^{iaz}}{z^4+4}+\operatorname*{Res}_{z=z_1}\frac{ze^{iaz}}{z^4+4}=-\frac{i}{8}e^{-a}\left[e^{ia}-e^{-ia}\right]=\frac{1}{4}e^{-a}\sin a.$$

(Recall that $2i\sin a=e^{ia}-e^{-ia}$.) Then

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+1}dx &= \Im\left[2\pi i\left(\frac{1}{4}e^{-a}\sin a\right)-\lim_{R\to\infty}\int_{C_R}\frac{ze^{iaz}}{z^4+1}dz\right]\\ &= \frac{\pi}{2}e^{-a}\sin a.\end{aligned}$$

Exercise 7, §81, p. 276

Use residues to evaluate the improper integral

$$\int_{-\infty}^{\infty}\frac{x\sin x dx}{(x^2+1)(x^2+4)}.$$

Solution:

We integrate $g(z)=\frac{ze^{iz}}{(z^2+1)(z^2+4)}$ over the same contour as in exercise (5) above. The function g has singular points where $z=\pm i$, $z=\pm 2i$. Of these $z=i$, $z=2i$ are in the upper half–plane. So for $R>2$, Cauchy's residue theorem ensures that

$$\begin{aligned}&\int_{-R}^{R}\frac{xe^{ix}}{(x^2+1)(x^2+4)}dx+\int_{C_R}\frac{ze^{iz}}{(z^2+1)(z^2+4)}dz\\ =\ &2\pi i\left[\operatorname*{Res}_{z=i}\frac{ze^{iz}}{(z^2+1)(z^2+4)}+\operatorname*{Res}_{z=2i}\frac{ze^{iz}}{(z^2+1)(z^2+4)}\right].\end{aligned}$$

Now take imaginary parts to get

$$(*)\qquad\int_{-R}^{R}\frac{x\sin x}{(x^2+1)(x^2+4)}dx=\Im\left(2\pi i\left[\operatorname*{Res}_{z=i}g(z)+\operatorname*{Res}_{z=2i}g(z)\right]-\int_{C_R}g(z)\,dz\right).$$

Next notice that $f(z) = \frac{z}{(z^2+1)(z^2+4)}$ is analytic outside the circle $|z| = 2$. In addition if $R > 2$ and $|z| = R$, then

$$\left|\frac{z}{(z^2+1)(z^2+4)}\right| \leq \frac{R}{(R^2-1)(R^2-4)} \to 0$$

as $R \to \infty$ (since $\left|z^2+4\right| \geq |z|^2 - 4 = R^2 - 4$ and $\left|z^2+1\right| \geq R^2 - 1$). So by the Theorem in section 81

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz = 0.$$

In addition Theorem 2 in section 76 reveals that

$$\begin{aligned}
\operatorname*{Res}_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \left.\frac{\frac{ze^{iz}}{(z^2+4)}}{\frac{d}{dz}(z^2+1)}\right|_{z=i} = \frac{\frac{ie^{-1}}{(-1+4)}}{2i} = \frac{e^{-1}}{6}, \\
\operatorname*{Res}_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} &= \left.\frac{\frac{ze^{iz}}{(z^2+1)}}{\frac{d}{dz}(z^2+4)}\right|_{z=2i} = \frac{\frac{2ie^{-2}}{(-4+1)}}{4i} = \frac{-e^{-2}}{6}.
\end{aligned}$$

If therefore we let $R \to \infty$ in $(*)$ above, it follows that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} dx = \Im\left[2\pi i \left(\frac{e^{-1}}{6} - \frac{e^{-2}}{6}\right)\right] = \frac{\pi}{3} e^{-1}\left(1 - e^{-1}\right).$$

Exercise 10, §81, p. 276

Use residues to find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5} dx.$$

Solution:

Consider the integral of the function $g(z) = \frac{(z+1)e^{iz}}{(z^2+4z+5)}$ on $[-R, R] \cup C_R$ where as before C_R is the upper half of the circle $|z| = R$. This function has singular points where $z^2 + 4z + 5 = 0$, i.e. where $z = \frac{1}{2}\left(-4 \pm \sqrt{16-20}\right) = -2 \pm i$. Since $z = -2+i$ is in the upper half–plane, it follows from Cauchy's residue theorem that

$$\int_{-R}^{R} \frac{(x+1)e^{ix}}{x^2+4x+5} dx + \int_{C_R} \frac{(z+1)e^{iz}}{z^2+4z+5} dz = 2\pi i \left(\operatorname*{Res}_{z=-2+i} \frac{(z+1)e^{iz}}{z^2+4z+5}\right),$$

for R big enough. Taking real parts we get that

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5} dx = \Re\left[2\pi i \left(\operatorname*{Res}_{z=-2+i} g(z)\right) - \int_{C_R} g(z)\, dz\right].$$

By Theorem 2 in section 76

$$\begin{aligned}
\operatorname*{Res}_{z=-2+i} \frac{(z+1)e^{iz}}{z^2+4z+5} &= \left.\frac{(z+1)e^{iz}}{2z+4}\right|_{z=-2+i} \\
&= \frac{1+i}{2} e^{-1-2i} \\
&= \frac{e^{-1}}{2}\left((\cos 2 + \sin 2) + i(\cos 2 - \sin 2)\right).
\end{aligned}$$

Moreover since $f(z) = \frac{(z+1)}{(z^2+4z+5)}$ is analytic outside the circle $|z| = |-2 \pm i| = \sqrt{5}$, and since

$$\left|\frac{(z+1)}{z^2+4z+5}\right| \leq \frac{R+1}{R^2-4R-5}$$

whenever $|z| = R$ with the right hand side tending to 0 as $R \to \infty$, it follows from the Theorem in section 81 that

$$\lim_{R\to\infty} \int_{C_R} \frac{(z+1)\, e^{iz}}{z^2+4z+5} dz = 0.$$

Therefore on letting $R \to \infty$, it follows from the above that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5} dx &= \Re \left[2\pi i \frac{e^{-1}}{2} \left((\cos 2 + \sin 2) + i (\cos 2 - \sin 2) \right) \right] \\ &= \pi e^{-1} (\sin 2 - \cos 2) . \end{aligned}$$

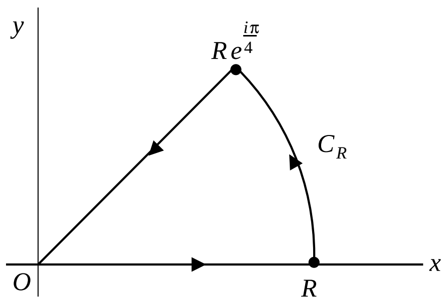
Exercise 12, §81, p. 276

Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^\infty \cos\left(x^2\right) dx = \int_0^\infty \sin\left(x^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(a) By integrating the function $\exp\left(iz^2\right)$ around the positively oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{4}$ (Fig. 99) and appealing to the Cauchy–Goursat theorem, show that

$$\int_0^R \cos\left(x^2\right) dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \Re \int_{C_R} e^{iz^2} dz$$



and

$$\int_0^R \sin\left(x^2\right) dx = \frac{1}{\sqrt{2}} \int_0^R e^{-r^2} dr - \Im \int_{C_R} e^{iz^2} dz,$$

where C_R is the arc $z = R\, e^{i\theta}$ $\left(0 \le \theta \le \frac{\pi}{4}\right)$.

(b) Show that the value of the integral along the arc C_R in part (a) tends to zero as R tends to infinity by obtaining the inequality

$$\left| \int_{C_R} e^{iz^2} dz \right| \le \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \sin\theta} d\theta$$

and then referring to the form (3), Sec. 81, of Jordan's inequality.

(c) Use the results in parts (a) and (b), together with the known integration formula

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

Solution:

(a) The boundary of the sector consists of $[0, R]$, C_R where $C_R : z(t) = R e^{it}$ and the line segment L_R from $R e^{\frac{i\pi}{4}} = \left(\frac{R}{\sqrt{2}} + i\frac{R}{\sqrt{2}}\right)$ to 0 which we may parametrise by $L_R : z(t) = -\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) t$, $-R \leq t \leq 0$. Now since e^{iz^2} is analytic on all of $\mathbb{C}$, it follows from the Cauchy-Goursat theorem that

$$\begin{aligned} 0 &= \int_{C_R} e^{iz^2} dz + \int_{L_R} e^{iz^2} dz + \int_{[0,R]} e^{iz^2} dz \\ &= \int_{C_R} e^{iz^2} dz + \int_{-R}^{0} e^{i\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^2 t^2} \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) dt + \int_0^R e^{it^2} dt \\ &= \int_{C_R} e^{iz^2} dz - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \int_0^R e^{-s^2} ds + \int_0^R e^{it^2} dt. \end{aligned}$$

(In the second integral we set $s = -t$.) Since

$$\int_0^R e^{it^2} dt = \int_0^R \cos t^2 dt + i \int_0^R \sin t^2 dt$$

we may therefore compare real and imaginary parts to get

$$\int_0^R \cos t^2 dt = \Re \int_0^R e^{it^2} dt = \frac{1}{\sqrt{2}} \int_0^R e^{-s^2} ds - \Re \int_{C_R} e^{iz^2} dz$$

and similarly

$$\int_0^R \sin t^2 dt = \frac{1}{\sqrt{2}} \int_0^R e^{-s^2} ds - \Im \int_{C_R} e^{iz^2} dz.$$

(b) Note that if $z(t) = R e^{it}$, then

$$iz^2(t) = iR^2 e^{i2t} = i\left(R^2 \cos 2t + iR^2 \sin 2t\right) = -R^2 \sin 2t + iR^2 \cos 2t.$$

Hence for such $z(t)$'s we have by (7) of section 29 that

$$\left| e^{iz^2(t)} \right| = e^{\Re\left(iz^2(t)\right)} = e^{-R^2 \sin 2t}$$

Therefore

$$\left| \int e^{iz^2} dz \right| = \left| \int_0^{\frac{\pi}{4}} e^{iz^2(t)} iR e^{it} dt \right| \leq \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2t} R dt.$$

Now make the substitution $s = 2t$ in the last integral to get

$$\left| \int_{C_R} e^{iz^2} dz \right| \leq \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \sin s} ds.$$

(c) Finally apply (3) of section 81 to see that

$$\left| \int_{C_R} e^{iz^2} dz \right| \leq \frac{R}{2} \left(\frac{\pi}{2R^2}\right) = \frac{\pi}{4R} \to 0$$

as $R \to \infty$. But then $\lim_{R\to\infty} \int_{C_R} e^{iz^2} dz = 0$, and hence by the theorem in section 55

$$\lim_{R\to\infty} \Re\left(\int_{C_R} e^{iz^2} dz\right) = 0 = \lim_{R\to\infty} \Im\left(\int_{C_R} e^{iz^2} dz\right).$$

Moreover

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

If therefore we let $R \to \infty$ in (a) we surely have $\int_0^\infty \cos t^2 dt = \frac{1}{\sqrt{2}} \int_0^\infty e^{-s^2} ds = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ and similarly $\int_0^\infty \sin t^2 dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Exercise 1, §84, p. 286
Derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a) \qquad (a \geq 0,\ b \geq 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Solution:
Consider the function $g(z) = \frac{\left(e^{iaz} - e^{ibz}\right)}{z^2}$. We integrate this function across the contour $[-R, -\rho] \cup C_\rho \cup [\rho, R] \cup C_R$ where

$$\begin{aligned} C_R &: \ z(t) = R\, e^{it} \qquad 0 \leq t \leq \pi \\ C_\rho &: \ z(t) = \rho e^{-it} \qquad -\pi \leq t \leq 0 \end{aligned}$$

(see figure 101 in section 82). The only singular point of the integrand is at $z = 0$ and so by the residue theorem

$$\int_{-R}^{-\rho} \frac{e^{iax} - e^{ibx}}{x^2} dx + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2} dz + \int_\rho^R \frac{e^{iax} - e^{ibx}}{x^2} dx + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2} dz = 0.$$

It follows that

$$\begin{aligned} & -\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2} dz - \int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2} dz \\ = & \int_\rho^R \frac{e^{iax} - e^{ibx}}{x^2} dx + \int_{-R}^{-\rho} \frac{e^{iax} - e^{ibx}}{x^2} dx \\ & \text{(set } x = -s \text{ in the 2}^{\text{nd}} \text{ integral)} \\ = & \int_\rho^R \frac{e^{iax} - e^{ibx}}{x^2} dx + \int_\rho^R \frac{e^{-ias} - e^{ibs}}{s^2} ds \\ = & \ 2\int_\rho^R \frac{\cos(ax) - \cos(bx)}{x^2} dx. \\ & \text{(Set } x = s \text{ and recall that } \cos ax = \frac{1}{2}\left(e^{iax} + e^{-iax}\right), \text{ etc.)} \end{aligned}$$

Notice that

$$\begin{aligned} \frac{1}{z^2}\left(e^{iaz} - e^{ibz}\right) &= \frac{1}{z^2}\left(\sum_{n=0}^\infty \frac{i^n}{n!}(az)^n - \sum_{n=0}^\infty \frac{i^n}{n!}(bz)^n\right) \\ &= \frac{1}{z^2}\left(\sum_{n=1}^\infty \frac{(i)^n}{n!}(a^n - b^n) z^n\right) \\ &= \frac{i(a-b)}{z} - \frac{\left(a^2 - b^2\right)}{2!} - \frac{i\left(a^3 - b^3\right)}{3!} z + \cdots . \end{aligned}$$

Therefore $\frac{1}{z^2}\left(e^{iaz} - e^{ibz}\right)$ has a simple pole at 0 with residue $i(a-b)$. Therefore by the theorem in section 82

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2} dz = (-i(a-b))\pi i = \pi(a-b).$$

Now for $|z| = R$, $\left|\frac{1}{z^2}\right| = \frac{1}{R^2} \to 0$ as $R \to \infty$. Therefore since $\frac{1}{z^2}$ is analytic whenever $|z| > 0$, it follows from the theorem in section 81 that both

$$\int_{C_R} \frac{e^{iaz}}{z^2} dz \quad \text{and} \quad \int_{C_R} \frac{e^{ibz}}{z^2} dz$$

tend to 0 as $R \to \infty$. If therefore we let $R \to \infty$ and $\rho \to 0$ in the above formula, it follows that

$$2\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = -\pi (a - b)$$

i.e.

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} (b - a).$$

Now notice that $2\sin^2 x = 1 - \cos 2x = \cos 0 - \cos 2x$. Therefore by what we've just shown (with $a = 0$ and $b = 2$) we get that

$$2\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2} (2 - 0).$$

Hence

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Exercise 3, §84, p. 286
Use the function

$$f(z) = \frac{z^{\frac{1}{3}} \log z}{z^2 + 1} = \frac{e^{\left(\frac{1}{3}\right)\log z} \log z}{z^2 + 1} \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to derive this pair of integration formulas:

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2 + 1} dx = \frac{\pi^2}{6}, \qquad \int_0^\infty \frac{\sqrt[3]{x}}{x^2 + 1} dx = \frac{\pi}{\sqrt{3}}.$$

Solution:

We integrate the function $g(z) = \frac{e^{\frac{1}{3}\log z} \log z}{(z^2+1)}$ over the contour $[-R, -\rho] \cup C_\rho \cup [\rho, R] \cup C_R$ where C_ρ and C_R are as in exercise 1 above and where

$$\log(z) = \ln|z| + i\arg(z) \qquad -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}.$$

Now for z in the interval $[-R, -\rho]$ we get $\log z = \ln|z| + i\pi$ whereas $\log z = \ln|z|$ for z in $[\rho, R]$. Moreover for $R > 1$ and $0 < \rho < 1$ the singular point $z = i$ of the function lies inside the given contour. By the residue theorem we then have

$$
\begin{aligned}
& 2\pi i \operatorname*{Res}_{z=i} g(z) \\
= & \int_{-R}^{-\rho} g(x)\,dx + \int_{C_\rho} g(z)\,dz + \int_{\rho}^{R} g(x)\,dx + \int_{C_R} g(z)\,dz \\
= & \int_{-R}^{-\rho} \frac{\exp\left(\frac{1}{3}(\ln|x| + i\pi)\right)(\ln|x| + i\pi)}{x^2+1}dx + \int_{C_\rho} g(z)\,dz \\
& + \int_{\rho}^{R} \frac{\exp\left(\frac{1}{3}\ln|x|\right)\ln|x|}{x^2+1}dx + \int_{C_R} g(z)\,dz \\
= & e^{\frac{i\pi}{3}} \int_{-R}^{-\rho} \frac{(|x|)^{\frac{1}{3}}(\ln|x| + i\pi)}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_{\rho}^{R} \frac{x^{\frac{1}{3}}\ln x}{x^2+1}dx + \int_{C_R} g(z)\,dz \\
& (\text{set } x = -s \text{ in the 1}^{\text{st}} \text{ integral}) \\
= & \left(1 + e^{\frac{i\pi}{3}}\right) \int_{\rho}^{R} \frac{x^{\frac{1}{3}}\ln x}{x^2+1}dx + i\pi e^{\frac{i\pi}{3}} \int_{\rho}^{R} \frac{x^{\frac{1}{3}}}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_{C_R} g(z)\,dz.
\end{aligned}
$$

By Theorem 2 in section 76

$$
\begin{aligned}
\operatorname*{Res}_{z=i} g(z) &= \left.\frac{\exp\left(\frac{1}{3}\log z\right)\log z}{2z}\right|_{z=i} \\
&= \frac{\exp\left(\frac{1}{3}\left(\ln 1 + i\frac{\pi}{2}\right)\right)\left(\ln 1 + i\frac{\pi}{2}\right)}{2i} \\
&= \frac{\pi}{4}e^{\frac{i\pi}{6}} \\
&= \frac{\pi}{4}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right).
\end{aligned}
$$

Now for $|z| = R$

$$
\left|\frac{z^{\frac{1}{3}}\log(z)}{z^2+1}\right| \leq \frac{|z|^{\frac{1}{3}}\,|\ln|z| + i\arg(z)|}{|z|^2 - 1} \leq \frac{R^{\frac{1}{3}}\left(\ln R + \frac{3\pi}{2}\right)}{R^2-1}.
$$

Therefore

$$
\left|\int_{C_R} g(x)\,dz\right| \leq \left[\frac{R^{\frac{1}{3}}\left(|\ln R| + \frac{3\pi}{2}\right)}{R^2-1}\right]\pi R \to 0 \quad \text{as} \quad R \to \infty,
$$

that is

$$
\lim_{R\to\infty} \int_{C_R} g(z)\,dz = 0.
$$

Similarly

$$
\left|\int_{C_\rho} g(x)\,dz\right| \leq \left[\frac{\rho^{\frac{1}{3}}\left(|\ln \rho| + \frac{3\pi}{2}\right)}{\rho^2-1}\right]\pi\rho.
$$

Since $\rho\ln\rho \to 0$ as $\rho \to 0$ (we can show this by L'Hospital's theorem), this means that $\left|\int_{C_\rho} g(z)\,dz\right| \to 0$ as $\rho \to 0$ and hence that

$$
\lim_{\rho\to 0} \int_{C_\rho} g(z)\,dz = 0.
$$

If therefore we let $R \to \infty$ and $\rho \to 0$ in the above integration formula it follows that

$$-\frac{\pi^2}{4} + i\frac{\pi^2\sqrt{3}}{4} = \left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)\int_0^\infty \frac{x^{\frac{1}{3}}\ln x}{x^2+1}dx + \left(-\frac{\pi\sqrt{3}}{2} + i\frac{\pi}{2}\right)\int_0^\infty \frac{x^{\frac{1}{3}}}{x^2+1}dx.$$

Now let I and J respectively denote the first and second integral above. Comparing real and imaginary parts reveals that

$$-\frac{\pi^2}{4} = \frac{3}{2}I - \frac{\sqrt{3}\pi}{2}J \text{ and } \frac{\sqrt{3}\pi^2}{4} = \frac{\sqrt{3}}{2}I + \frac{\pi}{2}J.$$

From these equations we may now solve for I and J to get

$$\int_0^\infty \frac{x^{\frac{1}{3}}\ln x}{x^2+1}dx = I = \frac{\pi^2}{6}$$

and

$$\int_0^\infty \frac{x^{\frac{1}{3}}}{x^2+1}dx = J = \frac{\pi}{\sqrt{3}}.$$

Exercise 4, §84, p. 286

Use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0,\ -\frac{\pi}{2} < \arg < \frac{3\pi}{2}\right)$$

to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx = \frac{\pi^3}{8}, \qquad \int_0^\infty \frac{\ln x}{x^2+1}dx = 0.$$

Suggestion: The integration formula obtained in Exercise 1, Sec. 79, is needed here.

Solution:

We integrate the function $g(z) = \frac{(\log z)^2}{(z^2+1)}$ over the same contour as in exercise 3 above where as before

$$\log(z) = \ln|z| + i\arg(z)$$

and

$$-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}.$$

The singular point $z = i$ is the only singular point inside $[-R, -\rho] \cup C_\rho \cup [\rho, R] \cup C_R$. In addition $\log z = \ln|z| + i\pi$ on $[-R, -\rho]$ and $\log z = \ln|z|$ on $[\rho, R]$. Therefore by the residue theorem

$$\begin{aligned}
&2\pi i \operatorname*{Res}_{z=i} g(z) \\
=\ & \int_{-R}^{-\rho} \frac{(\ln|x| + i\pi)^2}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_\rho^R \frac{(\ln x)^2}{x^2+1}dx + \int_{C_R} g(x)\,dz \\
& \text{(Set } s = -x \text{ in the 1}^{\text{st}} \text{ integral)} \\
=\ & 2\int_\rho^R \frac{(\ln x)^2}{x^2+1}dx - \pi^2\int_\rho^R \frac{1}{x^2+1}dx + 2\pi i\int_\rho^R \frac{\ln x}{(x^2+1)}dx \\
& + \int_{C_\rho} g(z)\,dz + \int_{C_R} g(z)\,dz.
\end{aligned}$$

Now by Theorem 2 in section 76 we have that

$$\operatorname*{Res}_{z=i} \frac{(\log z)^2}{z^2+1} = \left.\frac{(\log z)^2}{2z}\right|_{z=i} = \frac{\left(\ln 1 + i\frac{\pi}{2}\right)^2}{2i} = \frac{\pi^2}{8}i.$$

For $|z| = R$, $|\log z| \leq |\ln|z|| + |\arg(z)| \leq \ln R + \frac{3\pi}{2}$. Therefore

$$\left|\frac{(\log z)^2}{z^2+1}\right| \leq \frac{\left(\ln R + \frac{3\pi}{2}\right)^2}{(R^2-1)}$$

and hence

$$\left|\int_{C_R} \frac{(\log z)^2}{z^2+1} dz\right| \leq \frac{\left(\ln R + \frac{3\pi}{2}\right)^2}{(R^2-1)} \pi R \to 0$$

as $R \to \infty$. (To see this note that we can use L'Hospital's theorem to show that both $\frac{\ln R}{R}$ and $\frac{(\ln R)^2}{R}$ tend to 0 as $R \to \infty$.) Similarly

$$\left|\int_{C_\rho} \frac{(\log z)^2}{z^2+1} dz\right| \leq \frac{\left(\ln \rho + \frac{3\pi}{2}\right)^2}{(\rho^2-1)} \pi \rho \to 0 \quad \text{as} \quad \rho \to 0.$$

(Again this can be seen by using L'Hospital's theorem to show that both $\rho \ln \rho$ and $\rho (\ln \rho)^2$ tend to 0 as $\rho \to 0$.) Thus

$$\lim_{R\to\infty} \int_{C_R} g(z)\, dz = 0 = \lim_{\rho\to 0} \int_{C_\rho} g(z)\, dz.$$

Finally recall that in exercise (1) of section 79 we showed that

$$\int_0^\infty \frac{1}{x^2+1} dx = \frac{\pi}{2}.$$

If therefore we let $R \to \infty$ and $\rho \to 0$ in the first integration formula we obtained, we get

$$\frac{\pi^3}{4} = 2\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx + 2\pi i \int_0^\infty \frac{\ln x}{1+x^2} dx.$$

Comparing real and imaginary parts now reveals that

$$\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{1+x^2} dx = 0.$$

Exercise 6(a), §84, p. 287

Show that

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating an appropriate branch of the multiple–valued function

$$f(z) = \frac{z^{-\frac{1}{2}}}{z^2+1} = \frac{e^{\left(-\frac{1}{2}\right)\log z}}{z^2+1}$$

over the indented path in Fig. 101, Sec. 82.

Solution:

Consider the function $g(z) = \frac{z^{-\frac{1}{2}}}{(z^2+1)} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{(z^2+1)}$ where as before

$$\log z = \ln|z| + i\arg(z)\,;\ -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}.$$

We again integrate this function over $[-R,-\rho] \cup C_\rho \cup [\rho, R] \cup C_R$ where C_ρ and C_R are as before. As in exercise (1) above it follows that $\log(z) = \ln|z| + i\pi$ on $[-R,-\rho]$ and $\log(z) = \ln|z|$ on $[\rho, R]$. In addition g has a singular point at $z = i$ which will lie inside the contour if $0 < \rho < 1 < R$. It will then follow from the residue theorem that

$$
\begin{aligned}
&2\pi i \operatorname*{Res}_{z=i} g(z) \\
= &\int_{-R}^{-\rho} \frac{\exp\left(-\frac{1}{2}\log x\right)}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_\rho^R \frac{\exp\left(-\frac{1}{2}\log x\right)}{x^2+1}dx + \int_{C_R} g(z)\,dz \\
= &\int_{-R}^{-\rho} \frac{\exp\left(-\frac{1}{2}\ln|x| - i\frac{\pi}{2}\right)}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_\rho^R \frac{\exp\left(-\frac{1}{2}\ln|x|\right)}{x^2+1}dx \\
&+ \int_{C_R} g(z)\,dz \\
= &\int_{-R}^{-\rho} \frac{-i|x|^{-\frac{1}{2}}}{x^2+1}dx + \int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_{C_R} g(z)\,dz \\
= &(1-i)\int_\rho^R \frac{x^{-\frac{1}{2}}}{x^2+1}dx + \int_{C_\rho} g(z)\,dz + \int_{C_R} g(z)\,dz.
\end{aligned}
$$

(Note that $\int_{-R}^{-\rho} \frac{|x|^{-\frac{1}{2}}}{x^2+1}dx = \int_\rho^R \frac{s^{-\frac{1}{2}}}{s^2+1}ds$. To see this set $s = -x$.)

By Theorem 2 in section 76

$$
\operatorname*{Res}_{z=i} g(z) = \left.\frac{\exp\left(-\frac{1}{2}\log z\right)}{2z}\right|_{z=i} = \frac{\exp\left(-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right)}{2i} = \frac{1}{2\sqrt{2}i}(1-i).
$$

Now for $|z| = R$

$$
\left|\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1}\right| = \left|\frac{z^{-\frac{1}{2}}}{z^2+1}\right| \le \frac{R^{-\frac{1}{2}}}{R^2-1}.
$$

(To see this note that

$$
\left|\exp\left(-\frac{1}{2}\log z\right)\right| = \exp\left(\Re\left(-\frac{1}{2}\log(z)\right)\right) = \exp\left(-\frac{1}{2}\ln|z|\right) = |z|^{-\frac{1}{2}}.)
$$

But then

$$
\left|\int_{C_R} g(z)\,dz\right| \le \frac{\pi R}{R^{\frac{1}{2}}(R^2-1)} \to 0 \quad \text{as} \quad R \to \infty.
$$

Similarly

$$
\left|\int_{C_\rho} g(z)\,dz\right| \le \frac{\pi \rho}{\rho^{\frac{1}{2}}(\rho^2-1)} \to 0 \quad \text{as} \quad \rho \to 0.
$$

Consequently

$$
\lim_{R\to\infty} \int_{C_R} g(z)\,dz = 0 = \lim_{\rho\to 0} \int_{C_\rho} g(z)\,dz.
$$

Therefore on letting $R \to \infty$ and $\rho \to 0$ in the above integration formula, we get that

$$
2\pi i\left[\frac{1}{2\sqrt{2}i}(1-i)\right] = (1-i)\int_0^\infty \frac{x^{-\frac{1}{2}}}{x^2+1}dx,
$$

i.e. that

$$
\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.
$$

Note As an alternative we could have set $s = \sqrt{x}$ (i.e. $s^2 = x$). Then $\frac{ds}{dx} = \frac{1}{2\sqrt{x}}$ and hence

$$
\int_0^\infty \frac{1}{x^2+1}\frac{1}{\sqrt{x}}dx = 2\int_0^\infty \frac{1}{s^4+1}ds.
$$

The integral on the right hand side can now be solved by means of the techniques of section 79.

Exercise 1, §85, p. 290
Use residues to evaluate the definite integral

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}.$$

Solution:
By (1), (2) and (3) of section 85 the given integral becomes

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{1}{5+4\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}\frac{1}{iz}dz = \int_C \frac{dz}{2z^2+5iz-2}$$

where C is the positively oriented circle $|z|=1$. The integrand of the complex integral has singular points where $2z^2+5iz-2=0$, that is where $z = \frac{1}{4}\left(-5i \pm \sqrt{(5i)^2+16}\right) = \frac{1}{4}(-5i \pm 3i) = -2i,\ -\frac{1}{2}i$. Of these points only $z=-\frac{1}{2}i$ lies inside C. Therefore by the residue theorem

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \left(\operatorname*{Res}_{z=-\frac{1}{2}i} \frac{1}{2z^2+5iz-2}\right).$$

Applying Theorem 2 in section 76 we conclude that

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \left(\frac{1}{4z+5i}\right)\Bigg|_{z=-\frac{1}{2}i} = \frac{2\pi}{3}.$$

Exercise 3, §85, p. 290
Use residues to evaluate the definite integral

$$\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5-4\cos 2\theta}.$$

Solution:
By the remark at the end of section 85 we have that

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \qquad \sin n\theta = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

for each $n \in \mathbb{N}$ where $z = e^{i\theta}$. Therefore

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}d\theta &= \int_{|z|=1} \frac{\left[\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)\right]^2}{5-4\left[\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\right]}\frac{1}{zi}dz \\ &= \frac{i}{4}\int_{|z|=1} \frac{\left(z^6+1\right)^2}{z^5\left(2z^4-5z^2+2\right)}dz.\end{aligned}$$

The integrand $g(z) = \dfrac{\left(z^6+1\right)^2}{z^5\left(2z^4-5z^2+2\right)}$ has singularities where $z=0$ and where $2z^4-5z^2+2=0$. The roots of the last equation are at $z^2 = \frac{1}{4}\left(5 \pm \sqrt{25-16}\right) = \left\{2, \frac{1}{2}\right\}$ (i.e. $z = \pm\sqrt{2}$ and $z = \pm\frac{1}{\sqrt{2}}$). Thus the integrand has singularities at $0, \pm\sqrt{2}$ and $\pm\frac{1}{\sqrt{2}}$. Of all these singularities only $0, \frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ lie inside the circle

$|z| = 1$. The integrand has simple poles at $\pm\frac{1}{\sqrt{2}}$ and a pole of order 5 at 0. We may therefore use Theorem 2 in §76 to find the residues at $\pm\frac{1}{\sqrt{2}}$. By this result

$$\begin{aligned}\operatorname*{Res}_{z=\frac{1}{\sqrt{2}}} g(z) &= \left[\frac{\left(z^6+1\right)^2/z^5}{\frac{d}{dz}\left(2z^4-5z^2+2\right)}\right]_{z=\frac{1}{\sqrt{2}}} \\ &= \left[\frac{\left(z^6+1\right)^2/z^5}{8z^3-10z}\right]_{z=\frac{1}{\sqrt{2}}} \\ &= -\frac{27}{16}\end{aligned}$$

and similarly

$$\operatorname*{Res}_{z=-\frac{1}{\sqrt{2}}} g(z) = \left[\frac{\left(z^6+1\right)^2/z^5}{8z^3-10z}\right]_{z=-\frac{1}{\sqrt{2}}} = -\frac{27}{16}.$$

To find the residue at 0 we have by the theorem in §73 that

$$\operatorname*{Res}_{z=0} g(z) = -\frac{1}{4!}\frac{d^4}{dz^4}\left[\frac{\left(z^6+1\right)^2}{\left(2z^4-5z^2+2\right)}\right]_{z=0}.$$

However to avoid the pain of differentiating 4 times we rather look at the Laurent series of

$$\frac{\left(z^6+1\right)^2}{z^5\left(2z^4-5z^2+2\right)} = \left(z^7+2z+\frac{1}{z^5}\right)\frac{1}{\left(2z^2-1\right)\left(z^2-2\right)}.$$

Now by partial fractions

$$\frac{1}{\left(2z^2-1\right)\left(z^2-2\right)} = \frac{1}{3}\left[\frac{1}{\left(z^2-2\right)} - \frac{2}{\left(2z^2-1\right)}\right].$$

Therefore for $|z|$ small enough

$$\begin{aligned}\frac{1}{\left(2z^2-1\right)\left(z^2-z\right)} &= \frac{1}{3}\left[-\frac{1}{2}\frac{1}{\left(1-\frac{1}{2}z^2\right)} + \frac{2}{\left(1-2z^2\right)}\right] \\ &= -\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{2^k}z^{2k} + \frac{2}{3}\sum_{k=0}^{\infty}2^k z^{2k} \\ &= \sum_{k=0}^{\infty}\frac{1}{3}\left(2^{k+1}-\frac{1}{2^{k+1}}\right)z^{2k}\end{aligned}$$

whence

$$\frac{\left(z^6+1\right)^2}{z^5\left(2z^4-5z^2+2\right)} = \left(z^7+2z+\frac{1}{z^5}\right)\left(\sum_{k=0}^{\infty}\frac{1}{3}\left(2^{k+1}-\frac{1}{2^{k+1}}\right)z^{2k}\right).$$

In the resulting expansion the $\frac{1}{z}$ term will be

$$\frac{1}{z^5}\left(\frac{1}{3}\left(8-\frac{1}{8}\right)z^4\right) = \frac{63}{24}\frac{1}{z}.$$

Therefore

$$\operatorname*{Res}_{z=0} g(z) = \frac{63}{24}.$$

By means of the residue theorem we now have that

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta &= \frac{i}{4} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{z^5\left(2z^4 - 5z^2 + 2\right)} dz \\
&= \frac{i}{4} 2\pi i \left[\operatorname*{Res}_{z=\frac{1}{\sqrt{2}}} g(z) + \operatorname*{Res}_{z=-\frac{1}{\sqrt{2}}} g(z) + \operatorname*{Res}_{z=0} g(z) \right] \\
&= -\frac{\pi}{2} \left[-\frac{27}{16} - \frac{27}{16} + \frac{63}{24} \right] \\
&= \frac{3\pi}{8}.
\end{aligned}$$

Alternative: In the above integral the difficulty posed by computing the residue at a pole of order 5 can be avoided altogether if we use partial fractions. Observe that

$$\frac{1}{z^5\left(2z^4 - 5z^2 + 2\right)} = \frac{1}{z^5\left(z^2 - 2\right) 2\left(z - \frac{1}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}}\right)}.$$

Hence we can write

$$\frac{1}{z^5\left(2z^4 - 5z^2 + 2\right)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z^4} + \frac{E}{z^5} + \frac{1}{2}\left[\frac{F}{\left(z - \frac{1}{\sqrt{2}}\right)} + \frac{G}{\left(z + \frac{1}{\sqrt{2}}\right)} \right] + \frac{Hz + I}{\left(z^2 - 2\right)}$$

and solve for the constants to get

$$B = D = I = 0, \quad A = \frac{21}{8}, \quad C = \frac{5}{4}, \quad E = \frac{1}{2}, \quad F = G = -\frac{8}{3}, \quad H = \frac{1}{24}.$$

On multiplying both sides by $\left(z^6+1\right)^2$ and integrating over $|z| = 1$ Cauchy's integration formulae then ensure that

$$\begin{aligned}
&\int_{|z|=1} \frac{\left(z^6+1\right)^2}{z^5\left(2z^4 - 6z^2 + z\right)} dz \\
= \;& \frac{21}{8} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{z} dz + \frac{5}{4} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{z^3} dz + \frac{1}{2} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{z^5} dz \\
& -\frac{4}{3} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{\left(z - \frac{1}{\sqrt{2}}\right)} dz - \frac{4}{3} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{\left(z + \frac{1}{\sqrt{2}}\right)} dz + \frac{1}{24} \int_{|z|=1} \frac{\left(z^6+1\right)^2}{\left(z^2 - 2\right)} dz \\
= \;& \frac{21}{8} (2\pi i \times 1) + \frac{5}{4} \left(\frac{2\pi i}{2!} \frac{d^2}{dz^2} \left(z^6+1\right)^2 \bigg|_{z=0} \right) + \frac{1}{2} \left(\frac{2\pi i}{4!} \frac{d^4}{dz^4} \left(z^6+1\right)^2 \bigg|_{z=0} \right) \\
& -\frac{4}{3} \left(2\pi i \left(\left(\frac{1}{\sqrt{2}} \right)^6 + 1 \right)^2 \right) - \frac{4}{3} \left(2\pi i \left(\left(-\frac{1}{\sqrt{2}} \right)^6 + 1 \right)^2 \right) + 0 \\
= \;& \frac{21\pi i}{4} + 0 + 0 - \frac{81\pi i}{24} - \frac{81\pi i}{24} + 0 \\
= \;& -\frac{3\pi i}{2}.
\end{aligned}$$

(The last integral is zero since there the integrand is analytic inside and on $|z| = 1$.) Therefore

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta = \frac{i}{4} \int_{|z|=1} \frac{\left(z^6 + 1\right)^2}{z^5 \left(2z^4 - 5z^2 + z\right)} dz = \frac{i}{4} \left(-\frac{3\pi i}{2}\right) = \frac{3\pi}{8}.$$

Exercise 5, §85, p. 291
Use residues to evaluate the definite integral

$$\int_0^{\infty} \frac{\cos 2\theta d\theta}{1 - 2a\cos\theta + a^2} \qquad (-1 < a < 1).$$

Solution:
By (1), (2) and (3) of section 85 we see that

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = \int_C \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{1 - 2a\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right) + a^2} \frac{1}{iz} dz = \int_C \frac{i\left(z^4 + 1\right)}{2z^2\left(az^2 - \left(a^2 + 1\right)z + a\right)} dz$$

where C is the positively oriented circle $|z| = 1$. The denominator of the integrand has zeros where $z = 0$ and where $z = \frac{1}{2a}\left(a^2 + 1 \pm \sqrt{\left(a^2 + 1\right)^2 - 4a^2}\right) = \frac{1}{2a}\left(\left(a^2 + 1\right) \pm \left(1 - a^2\right)\right) = \frac{1}{a}, a$. None of these zeros are also zeros of $z^4 + 1$ and hence all are singular points of the integrand. Of these points only 0 and a are inside C. Therefore the residue theorem tells us that

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = 2\pi i \left[\operatorname*{Res}_{z=0} g(z) + \operatorname*{Res}_{z=a} g(z)\right]$$

where $g(z) = \frac{i\left(z^4+1\right)}{\left[2z^2\left(az^2 - \left(a^2+1\right)z + a\right)\right]}$. By the theorem in section 73

$$\operatorname*{Res}_{z=0} g(z) = \frac{d}{dz} \left.\frac{i\left(z^4 + 1\right)}{2\left(az^2 - \left(a^2 + 1\right)z + a\right)}\right|_{z=0} = \frac{1}{2} \left.\frac{i4z^3\left(az^2 - \left(a^2 + 1\right)z + a\right) - i\left(z^4 + 1\right)\left(2az - \left(a^2 + 1\right)\right)}{\left(az^2 - \left(a^2 + 1\right)z + a\right)^2}\right|_{z=0} = \frac{i}{2}\left(\frac{a^2 + 1}{a^2}\right).$$

From Theorem 2 in section 76 it follows that

$$\operatorname*{Res}_{z=a} g(z) = \left.\frac{i\frac{\left(z^4+1\right)}{2z^2}}{\frac{d}{dz}\left(az^2 - \left(a^2 + 1\right)z + a\right)}\right|_{z=a} = \frac{i\left(a^4 + 1\right)}{2a^2\left(a^2 - 1\right)}.$$

Thus

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = 2\pi i \left[\frac{i}{2}\left(\frac{a^2 + 1}{a^2}\right) + \frac{i}{2} \frac{\left(a^4 + 1\right)}{a^2\left(a^2 - 1\right)}\right] = -\pi \frac{\left(a^4 - 1\right) + \left(a^4 + 1\right)}{a^2\left(a^2 - 1\right)} = \frac{2a^2\pi}{\left(1 - a^2\right)}.$$

Finally notice that

$$\frac{\cos\left(2\left(-\theta\right)\right)}{1-2a\cos\left(-\theta\right)+a^2}=\frac{\cos 2\theta}{1-2a\cos\theta+a^2},$$

i.e. that $\frac{\cos 2\theta}{(1-2a\cos\theta+a^2)}$ is an even function . Therefore

$$\int_0^{\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos 2\theta}{1-2a\cos\theta+a^2}d\theta=\frac{a^2\pi}{(1-a^2)}.$$

Exercise 7, §85, p. 291

Use residues to evaluate the definite integral

$$\int_0^{\pi}\sin^{2n}\theta d\theta \qquad (n=1,2,\ldots).$$

Solution:

Again by (1), (2) and (3) of section 85

$$\int_{-\pi}^{\pi}\sin^{2n}\theta d\theta=\int_C\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^{2n}\frac{1}{iz}dz=\frac{i\left(-1\right)^{n+1}}{2^{2n}}\int_c\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}dz$$

where C is the positively oriented circle $|z|=1$ and where $n=1,2,\ldots$. The function $\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}$ has a singular point at $z=0$, and hence by the residue theorem

$$\int_C\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}dz=2\pi i\quad\operatorname*{Res}_{z=0}\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}.$$

By the binomial theorem

$$\begin{aligned}\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}} &= \frac{1}{z^{2n+1}}\sum_{k=0}^{2n}\binom{2n}{k}\left(-1\right)^{2n-k}z^{2k}\\ &= \sum_{k=0}^{2n}\binom{2n}{k}\left(-1\right)^{2n-k}z^{2k-(2n+1)}\end{aligned}$$

for all $0<|z|$. The coefficient of the $\frac{1}{z}$ term (corresponding to $k=n$) is $\binom{2n}{n}\left(-1\right)^n$. That is

$$\operatorname*{Res}_{z=0}\frac{\left(z^2-1\right)^{2n}}{z^{2n+1}}=\binom{2n}{n}\left(-1\right)^n=\frac{(2n)!}{\left(n!\right)^2}\left(-1\right)^n.$$

Therefore

$$\int_{-\pi}^{\pi}\sin^{2n}\theta d\theta=\frac{i\left(-1\right)^{n+1}}{2^{2n}}\left(2\pi i\frac{(2n)!}{\left(n!\right)^2}\left(-1\right)^n\right)=2\pi\frac{(2n)!}{2^{2n}\left(n!\right)^2}.$$

Since $\sin^{2n}\theta$ is an even function we conclude that

$$\int_0^{\pi}\sin^{2n}\theta d\theta=\pi\frac{(2n)!}{2^{2n}\left(n!\right)^2}.$$

Exercise 1, §87, p. 296

Let C denote the unit circle $|z|=1$, described in the positive sense. Use the theorem in Sec. 86 to determine the value of $\Delta_C\arg f\left(z\right)$ when

(a) $f\left(z\right)=z^2$; (b) $f\left(z\right)=\left(z^3+2\right)/z$; (c) $f\left(z\right)=\left(2z-1\right)^7/z^3$.

Solution:

(a) $f(z) = z^2$ has a double zero at $z = 0$ and no poles inside $|z| = 1$. Hence here

$$\Delta_C \arg\left(z^2\right) = 2\pi(2 - 0) = 4\pi.$$

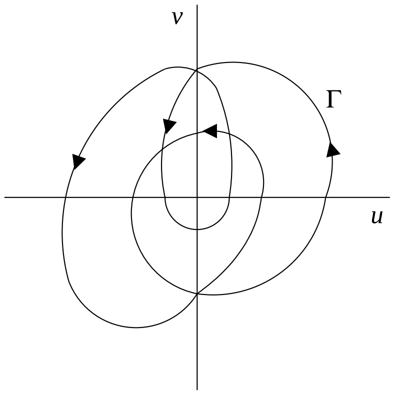
(b) $f(z) = \left(z^3 + 2\right)/z$ has a simple pole at $z = 0$ and zeros at the roots of $z^3 = -2$. However all these roots lie outside $|z| = 1$. Thus

$$\Delta_C \arg\left(\frac{z^3 + 2}{z}\right) = 2\pi(0 - 1) = -2\pi.$$

(c) $f(z) = \frac{(2z-1)^7}{z^3}$ has a triple pole at $z = 0$ and a zero of order 7 at $z = \frac{1}{2}$. Hence $\Delta_C \arg\left(\frac{(2z-1)^7}{z^3}\right) = 2\pi(7 - 3) = 8\pi$.

Exercise 2, §87, p. 296

Let f be a function which is analytic inside and on a simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown in Fig. 107. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 86, determine the number of zeros, counting multiplicities, of f interior to C.



Solution:

Since f is analytic inside and on C, it clearly has no poles inside or on C. So by the theorem in section 86

$$Z = \frac{1}{2\pi}\Delta_C \arg f(z)$$

where Z is the number of zeros (counting multiplicities) of f inside C. Now from the sketch it is clear that as z traverses once around the contour C, $f(z)$ will effectively circle the origin 3 times. That is

$$\Delta_C \arg f(z) = 3 \times 2\pi = 6\pi.$$

Therefore counting multiplicities f has $\frac{1}{2\pi}(6\pi) = 3$ zeros inside C.

Exercise 3, §87, p. 296

Using the notation in Sec. 86, suppose that Γ does not enclose the origin $w = 0$ and that there is a ray from that point which does not intersect Γ. By observing that the absolute value of $\Delta_C \arg f(z)$ must be less than 2π when a point z makes one cycle around C and recalling that $\Delta_C \arg f(z)$ is an integral multiple of 2π, point out why the winding number of Γ with respect to the origin $w = 0$ must be zero.

Solution:

Let f and Γ be as stated and suppose that the ray $w = re^{i\alpha}$ $(0 \leq r < \infty)$ does not intersect Γ. Since 0 is by assumption outside Γ and also on the ray, the entire ray must then be outside Γ (or else it would have crossed Γ). Now let $f(z)$ traverse around Γ starting from say $f(z_0)$ where $\arg f(z_0) = \varphi_0$ and $\alpha - 2\pi < \varphi_0 < \alpha$. Although $\arg f(z)$ may increase or decrease as $f(z)$ traverses Γ, the fact that $f(z)$ can never be on the ray $w = re^{i\alpha}$ $(0 \leq r < \infty)$ ensures that $\arg f(z)$ can never cross either $\theta = \alpha - 2\pi$ or $\theta = \alpha$. (To see this note that a point $w \neq 0$ lies on the ray if and only if $\arg w = \alpha + 2k\pi$ $(k \in \mathbb{Z})$.) Clearly then $|\arg f(z) - \arg f(z_0)| < 2\pi$ for each $f(z)$ on Γ, whence

$$|\Delta_C \arg f(z)| < 2\pi.$$

Since $\Delta_C \arg f(z)$ must be an integer multiple of 2π, this inequality can only hold if in fact

$$\Delta_C \arg f(z) = 0.$$

The claim follows.

Exercise 6, §87, p. 297

Determine the number of zeros, counting multiplicities, of the polynomial

$$\text{(a) } z^6 - 5z^4 + z^3 - 2z; \qquad \text{(b) } 2z^4 - 2x^3 + 2z^2 - 2z + 9$$

inside the circle $|z| = 1$.

Solution:

(a) Let $f(z) = z^6 - 5z^4$ and $g(z) = z^3 - 2z$. Now $f(z) = z^4\left(z^2 - 5\right)$ has 6 zeros counting multiplicities, but only 4 are inside the circle $|z| = 1$. On the circle $|z| = 1$ we have $|f(z)| \geq 5|z|^4 - |z|^6 = 4$ and $|g(z)| \leq |z|^3 + 2|z| = 3$. So by Rouché's theorem $f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$ also has 4 zeros inside $|z| = 1$.

(b) Let $f(z) = 9$ for all z and $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$. Clearly f has no zeros anywhere. In addition if $|z| = 1$ then

$$|g(z)| \leq 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8 < 9 = |f(z)|.$$

Thus by Rouché's theorem $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ also has no zeros inside $|z| = 1$.

Exercise 8, §87, p. 297

Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \leq |z| < 2$.

Solution:

First let $f(z) = -6z^2$ and $g(z) = 2z^5 + z + 1$. Here f has 2 zeros inside $|z| = 1$. Now for $|z| = 1$ we have

$$|g(z)| \leq 2|z|^5 + |z| + 1 = 4 \quad \text{and} \quad |f(z)| = 6|z|^2 = 6.$$

Hence by Rouché's theorem $f(z) + g(z) = 2z^5 - 6z^2 + z + 1$ also has 2 zeros inside $|z| = 1$. Next let $\tilde{f}(z) = 2z^5$ and $\tilde{g}(z) = -6z^2 + z + 1$. The polynomial $f(z) = 2z^5$ has 5 zeros inside $|z| = 2$ (all at $z = 0$.) For $|z| = 2$ we have

$$|\tilde{g}(z)| \leq 6|z|^2 + |z| + 1 = 27 \text{ and } \left|\tilde{f}(z)\right| = 2|z|^5 = 64.$$

So by Rouché's theorem $\tilde{f}(z)+\tilde{g}(z)=2z^5-6z^2+z+1$ also has 5 zeros inside $|z|=2$. So in the region inside $|z|=2$ but not inside $|z|=1$ (i.e. the region $1\leq|z|<2$), the polynomial $2z^5-6z^2+z+1$ must have exactly $5-2=3$ roots.

Exercise 10, §87, p. 298

Let the functions f and g be as in the statement of Rouché's theorem in Sec. 87, and let the orientation of the contour C there be positive. Then define the function

$$\Phi(t)=\frac{1}{2\pi i}\int_C\frac{f'(z)+tg'(z)}{f(z)+tg(z)}dz \qquad (0\leq t\leq 1),$$

and follow the steps below to give another proof of that theorem.

(a) Point out why the denominator in the integrand of the integral defining $\Phi(t)$ is never zero on $\mathbb{C}$. This ensures the existence of the integral.

(b) Let t and t_0 be any two points in the interval $0\leq t\leq 1$, and show that

$$|\Phi(t)-\Phi(t_0)|=\frac{|t-t_0|}{2\pi}\left|\int_C\frac{fg'-f'g}{(f+tg)(f+t_0g)}dz\right|.$$

Then, after pointing out why

$$\left|\frac{fg'-f'g}{(f+tg)(f+t_0g)}\right|\leq\frac{|fg'-f'g|}{(|f|-|g|)^2}$$

at points on C, show that there is a positive constant A, which is independent of t and t_0, such that

$$|\Phi(t)-\Phi(t_0)|\leq A|t-t_0|.$$

Conclude from this inequality that $\Phi(t)$ is continuous on the interval $0\leq t\leq 1$.

(c) By referring to equation (8), Sec. 86, state why the value of the function Φ is, for each value of t in the interval $0\leq t\leq 1$, an integer representing the number of zeros of $f(z)+tg(z)$ inside C. Then conclude from the fact that Φ is continuous, as shown in part (b), that $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, inside C.

Solution:

(a) By assumption $|g(x)|<|f(z)|$ for all z on the contour C. Thus for any $0\leq t\leq 1$ this implies that

$$|f(z)+tg(z)|\geq|f(z)|-t|g(z)|\geq|f(z)|-|g(z)|>0$$

whenever z is a point on C.

(b) Now let t, $t_0\in[0,1]$. Then

$$\begin{aligned}&\Phi(t)-\Phi(t_0)\\ &=\frac{1}{2\pi i}\int_C\left(\frac{f'(z)+tg'(z)}{f(z)+tg(z)}-\frac{f'(z)+t_0g'(z)}{f(z)+t_0g(z)}\right)dz\\ &=\frac{1}{2\pi i}\int_C\frac{(f'(z)+tg'(z))(f(z)+t_0g(z))-(f'(z)+t_0g'(z))(f(z)+tg(z))}{(f(z)+tg(z))(f(z)+t_0g(z))}dz\\ &=\frac{(t-t_0)}{2\pi i}\int_C\frac{g'(z)f(z)-f'(z)g(z)}{(f(z)+tg(z))(f(z)+t_0g(z))}dz. \qquad (*)\end{aligned}$$

Now as we saw in part (a)

$$|f(z)+sg(z)|\geq|f(z)|-|g(z)|$$

for all $z \in C$ and all $0 \leq s \leq 1$. In particular for t and t_0 it then follows that

$$|(f(z) + tg(z))(f(z) + t_0 g(z))| \geq (|f(z)| - |g(z)|)^2$$

for all z on C, whence

$$(**) \qquad \left| \frac{fg' - f'g}{(f + tg)(f + t_0 g)} \right| \leq \frac{|fg' - f'g|}{(|f| - |g|)^2}$$

for all z on C. Since f, g are analytic on C, each of f, g, f' and g' are then continuous on C. (This follows from for example Theorem 1 of section 52 and the discussion at the end of section 19.) Since $\frac{|fg' - f'g|}{(|f| - |g|)^2}$ is then defined and continuous on C, it follows from (6) of section 18 that we can find a constant $M > 0$ so that

$$\frac{|fg' - f'g|}{(|f| - |g|)^2} \leq M \qquad \text{for all } z \text{ on } C.$$

Comparing this with $(**)$ above, it then follows from (1) of section 43 that

$$\left| \int_C \frac{f(z) g'(z) - f'(z) g(z)}{(f(z) + tg(z))(f(z) + t_0 g(z))} dz \right| \leq ML$$

where L is the length of C and M is clearly independent of t and t_0. By $(*)$ we then have

$$|\Phi(t) - \Phi(t_0)| \leq \frac{ML}{2\pi} |t - t_0| .$$

Now let $\varepsilon < 0$ be given and set $\delta = \frac{2\pi\varepsilon}{ML}$. Then

$$|t - t_0| < \delta \Rightarrow |\Phi(t) - \Phi(t_0)| < \frac{ML}{2\pi}\delta = \varepsilon.$$

Thus by definition $t \to \Phi(t)$ is continuous on $[0, 1]$.

(c) For each fixed t, $f(z) + tg(z)$ is analytic inside and on C (i.e. it has no poles inside or on C), and hence it then follows from (8) of section 86 that

$$\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz$$

is an integer representing the number of zeros of $f(z) + tg(z)$ inside C. But then $\Phi(t)$ must be constant on $[0, 1]$ since if Φ was not constant the fact that all its values are integers means that it would then have to admit of a "jump discontinuity" somewhere on $[0, 1]$. But there can be no such jump since Φ is continuous. Therefore $t \to \Phi(t)$ is constant on $[0, 1]$ as claimed. In particular $\Phi(0) = \Phi(1)$, or in other words f and $f + g$ have the same number of zeros inside C (counting multiplicities of course).