POST TEST: MODULE 1 (LEARNING UNIT 1)

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QUESTION 1

1(a) Separable

$$\left(1+x^2\right)ydx+\left(1-y^2\right)xdy=0$$

(Divide each term by *xy*.)

$$\frac{\left(1+x^2\right)}{x}dx+\frac{\left(1-y^2\right)}{y}dy=0$$

(Variables are separated and we proceed to integrate.)

$$\int\frac{\left(1+x^2\right)}{x}dx+\int\frac{\left(1-y^2\right)}{y}dy=0$$

$$\int\frac{1}{x}dx+\int x\,dx+\int\frac{1}{y}dy-\int y\,dy=0$$

$$\ell n\,x+\frac{x^2}{2}+\ell n\,y-\frac{y^2}{2}=c$$

$\left(\text{Can combine terms to simplify the answer.}\right)$

$$\ell n\,xy+\frac{x^2-y^2}{2}=c$$

(8)

(b) Separable

$$\sin x\cos y\,dx=\sin y\cos x\,dy$$

$\left(\text{Divide by } \cos x\cos y\right)$

$$\frac{\sin x}{\cos x}dx=\frac{\sin y}{\cos y}dy$$

$$\tan x\,dx=\tan y\,dy$$

$\left(\text{Use trigonometric identities}\right)$

$$\int\tan x\,dx=\int\tan y\,dy$$

$$\ell n\sec x=\ell n\sec y+c$$

$$\ell n\sec x-\ell n\sec y=c$$

$$\ell n\frac{\sec x}{\sec y}=c$$

$\left(\text{Use the logarithm rules to simplify the answer}\right)$

QUESTION 2

$$\left(2xy-y^2+2x\right)dx+\left(x^2-2xy+2y\right)dy=0$$

$$Mdx+Ndy=0$$

Check: $\dfrac{\partial M}{\partial y}=2x-2y$ and $\dfrac{\partial N}{\partial x}=2x-2y$

Now $\dfrac{\partial f}{\partial x}=2xy-y^2+2x$ so that $f(x,y)=\int\left(2xy-y^2+2x\right)dx$

thus by direct integration $f(x,y)=x^2y-y^2x+x^2+f(y)$

Open Rubric

and $\dfrac{\partial f}{\partial y} = x^2 - 2xy + 2y$ so that $f(x, y) = \int \left(x^2 - 2xy + 2y \right) dx$

and by direct integration $f\left(x, y\right) = x^2 y - xy^2 + y^2 + f(x)$

Comparing the answers for $f\left(x, y\right)$ to find

$$f(x) = x^2 \text{ and } f(y) = y^2$$

Solution: $x^2 + y^2 + x^2 y - xy^2 = c$ (5)

QUESTION 3

$$x\frac{dy}{dx} - y = x^2 \qquad \left(\text{Can be written in linear form}\right)$$

$$\frac{dy}{dx} - \frac{y}{x} = x$$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right) y = x$$

$$\int P dx = -\int \tfrac{1}{x} dx = -\ell n x$$

$$\therefore e^{\int P dx} = e^{-\ell n\, x} = \frac{1}{x}$$

$$y = e^{-\int P dx} \int \left(e^{\int P dx} \right) Q\, dx$$

$$= x\left(\int \left(\frac{1}{x}\right)(x)\, dx \right)$$

$$= x\left(\int (1)\, dx \right)$$

$$= x\left(\int \left(x^0\right) dx \right)$$

$$= x(x + c)$$

$$= x^2 + cx$$

QUESTION 4

Method 1: Bernoulli

$$x^2 + y^2 = 2xy\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

$$= \frac{x}{2y} + \frac{y}{2x}$$

$$\frac{dy}{dx} - \frac{1}{2x} y = \frac{x}{2y}$$

$$y\frac{dy}{dx} - \frac{1}{2x} y^2 = \frac{x}{2}$$

Let $v = y^2$; then $\dfrac{dv}{dx} = 2y\dfrac{dy}{dx}$

$$\frac{1}{2}\ \frac{dv}{dx} - \frac{v}{2x} = \frac{x}{2}$$

$$\frac{dv}{dx} - \frac{v}{x} = x$$

$$\int P dx = -\int \frac{1}{x} dx = -\ell n \ \ x$$

$$e^{\int P dx} = e^{-\ell n\, x} = \frac{1}{x}$$

$$v = x\left(\int \frac{1}{x} \cdot x dx + c\right)$$

$$= x\left(\int dx + c\right)$$

$$= x(x + c)$$

$$y^2 = x^2 + xc \qquad (14)$$

Method 2: Homogeneous

$$x^2 + y^2 = 2xy\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x(vx)}$$

$$x\frac{dv}{dx} = \frac{x^2\left(1 + v^2\right)}{2x^2 v} - v$$

$$= \frac{\left(1 + v^2\right) - 2v^2}{2v}$$

$$= \frac{1 - v^2}{2v}$$

$$\frac{2v}{1-v^2}\,dv = \frac{1}{x}\,dx$$

$$-\int \frac{-2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\ell n\left|1-v^2\right| + \ell nC = \ell nx$$

$$\ell nC = \ell nx + \ell n\left|1-\left(\tfrac{y}{x}\right)^2\right|$$

$$= \ell n\, x\left(1-\frac{y^2}{x^2}\right)$$

$$C = x - \frac{y^2}{x}$$

$$y^2 = x^2 + Cx$$

QUESTION 5

Bernoulli

$$\frac{dy}{dx} + y = e^x y^4$$

$$y^{-4}\frac{dy}{dx} + y^{-3} = e^x$$

Let z $= y^{-3}$

then $\dfrac{dz}{dx} = -3y^{-4}\ \dfrac{dy}{dx}$

thus $-\dfrac{1}{3}\dfrac{dz}{dx} + z = e^x$

$$\frac{dz}{dx} - 3z = -3e^x$$

$$\int P dx = -\int 3\,dx = -3x$$

and $e^{\int Pdx} = e^{-3x}$

$$\therefore z = -3e^{3x}\left(\int e^{-3x}\cdot -3e^x dx + c\right)$$

$$= -3e^{3x}\left(\int e^{-2x} dx + c\right)$$

$$= -3e^{3x}\left(-\frac{1}{2}e^{-2x} + c\right)$$

$$y^{-3} = \frac{3}{2}e^x + ce^{3x}$$

(11)

QUESTION 6

$$-2x\ \frac{dy}{dx}+y=x(x+1)y^3$$

$$-2\ \frac{dy}{dx}+y\cdot\frac{1}{x}=(x+1)y^3$$

$$-2y^{-3}\frac{dy}{dx}+y^{-2}\cdot\frac{1}{x}=x+1$$

Let $z=y^{-2}$

Then $\dfrac{dz}{dx}=-2y^{-3}\dfrac{dy}{dx}$

$$\therefore\ \frac{dz}{dx}+z\cdot\frac{1}{x}=x+1$$

Now $e^{\int Pdx}=e^{\int\frac{1}{x}dx}$

$$=e^{\ell n\,x}$$

$$=x$$

$$\therefore\ z=\frac{1}{x}\left(\int x\left(x+1\right)dx+c\right)$$

$$=\frac{1}{x}\left(\frac{x^3}{3}+\frac{x^2}{2}+c\right)$$

$$y^{-2}=\frac{x^2}{3}+\frac{x}{2}+\frac{c}{x}$$

$$y^2=\frac{6x}{2x^3+3x^2+6c}$$ (10)

QUESTION 7

$$\frac{dy}{dx}+2y\cdot\frac{1}{x}=3x^2y^{\frac{4}{3}}$$

$$y^{-\frac{4}{3}}\ \frac{dy}{dx}+2y^{-\frac{1}{3}}\cdot\frac{1}{x}=3x^2$$

Let $z=y^{-\frac{1}{3}}$

Then $\dfrac{dz}{dx}=-\dfrac{1}{3}y^{-\frac{4}{3}}\ \dfrac{dy}{dx}$

$$\therefore \frac{dz}{dx} - z\cdot\frac{2}{3x} = -x^2$$

$$e^{\int P dx} = e^{\int -\frac{2}{3}\cdot\frac{1}{x}dx}$$

$$= e^{-\frac{2}{3}\ell n\, x}$$

$$= x^{-\frac{2}{3}}$$

$$\therefore z = x^{\frac{2}{3}}\left(-\int x^{-\frac{2}{3}}.\ x^2\ \ dx + c\right)$$

$$= x^{\frac{2}{3}}\left(-\frac{3}{7}x^{\frac{7}{3}} + c\right)$$

$$y^{-\frac{1}{3}} = -\frac{3}{7}x^3 + cx^{\frac{2}{3}} \qquad (9)$$

QUESTION 8

$$\frac{dy}{dx} + y\cos\ \ x = y^n \sin 2x$$

$$y^{-n}\frac{dy}{dx} + y^{1-n}.\ \cos x = 2\sin x\cos x$$

$$(1-n)y^{-n}\ \frac{dy}{dx} + (1-n)y^{1-n}.\ \cos x = 2(1-n)\sin x\cos x$$

$$Let\ \ z = y^{1-n}$$

$$Then\ \ \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

$$\therefore \frac{dz}{dx} + z(1-n)\cos x = 2(1-n)\ \ \sin\ \ x\ \ \cos\ \ x$$

$$e^{\int P dx} = e^{\int (1-n)\cos x\, dx}$$

$$= e^{(1-n)\sin x}$$

$$z.e^{(1-n)\sin x} = 2\int\ e^{(1-n)\sin x}.(1-n)\sin x\cos x dx + c$$

Put $u\ =\ \sin x$ then $du = \cos x$ and $dv = e^{(1-n)\sin x}.(1-n)\cos x dx$

$$v = \int\ e^{(1-n)\sin x}.(1-n)\cos x dx = e^{(1-n)\sin x}$$

$$= 2\sin x.e^{(1-n)\sin x} - 2\int\ e^{(1-n)\sin x}.\cos x dx + c$$

$$= 2\sin x.e^{(1-n)\sin x} - \frac{2}{(1-n)}\cdot e^{(1-n)\sin x} + c$$

$$z = 2\sin x - \frac{2}{(1-n)} + c.e^{(n-1)\sin x}$$
