1.1 $x\dfrac{dy}{dx}=y+2\sqrt{xy}$ Homogeneous equation

$$\frac{dy}{dx}=\frac{y}{x}+\frac{2\sqrt{xy}}{x}$$

Put $y = vx$

then $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$$v+x\frac{dv}{dx}=v+\frac{2\sqrt{vx^2}}{x}$$

$$x\frac{dv}{dx}=2\sqrt{v}$$

$$\frac{1}{\sqrt{v}}dv=\frac{2}{x}dx$$

$$2\sqrt{v}=2\ell n|x|+c$$

$$\sqrt{\frac{y}{x}}=\ell n|Ax| \text{ where } \ell nA=\frac{C}{2}$$

(6)

1.1 Alternative solution $x\dfrac{dy}{dx}=y+2\sqrt{xy}$

$x\dfrac{dy}{dx}-y=2\sqrt{xy}$ Bernoulli equation

$$y^{-\frac{1}{2}}\frac{dy}{dx}-\frac{1}{x}y^{\frac{1}{2}}=2x^{-\frac{1}{2}} \qquad (1)$$

Put $z=y^{\frac{1}{2}}$

then $\dfrac{dz}{dx}=\dfrac{1}{2}y^{-\frac{1}{2}}\dfrac{dy}{dx}$

$$(1)\times\frac{1}{2}:\ y^{-\frac{1}{2}}\frac{dy}{dx}+\left(-\frac{1}{2x}\right)y^{\frac{1}{2}}=x^{-\frac{1}{2}}$$

$$\frac{dz}{dx}+\left(-\frac{1}{2x}\right)z=x^{-\frac{1}{2}}$$

Now $\displaystyle\int Pdx=-\frac{1}{2}\int\frac{1}{x}dx=-\frac{1}{2}\ell n\,x=\ell n\,x^{-\frac{1}{2}}$

and $e^{\int Pdx}=x^{-\frac{1}{2}}$ and $e^{-\int Pdx}=x^{\frac{1}{2}}$

$$z=x^{\frac{1}{2}}\int\left(x^{-\frac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right)dx$$

$$y^{\frac{1}{2}}=x^{\frac{1}{2}}\int x^{-1}dx$$

$$=x^{\frac{1}{2}}\left(\ell n\,x+C\right)$$

$$\sqrt{\frac{y}{x}}=\ell n\,x+C$$

1.2

$$\left[\sin y - 2xy + x^2\right]dx + \left[x\cos y - x^2\right]dy = 0$$

$$M \quad dx \quad + \quad N \quad dy = 0$$

$$\frac{\partial M}{\partial y} = \left(\cos y - 2x\right) = \frac{\partial N}{\partial x} = \left(\cos y - 2x\right)$$

Thus the equation is exact.

Now by direct integration

$$f(x,y) = \int\left[\sin y - 2xy + x^2\right]dx = x\sin y - x^2y + \frac{x^3}{3} + f(y) \qquad (1)$$

and

$$f(x,y) = \int\left[x\cos y - x^2\right]dy = x\sin y - x^2y + f(x) \qquad (2)$$

Comparing the answers (1) and (2) for $f(x,y)$

we find $f(y) = 0$

and $\quad f(x) = \frac{x^3}{3}$

Thus $\quad C = x\sin y - x^2y + \frac{x^3}{3}$

(6)

1.3

$$\frac{dy}{dx} - y = \frac{e^x}{x}, \text{ given that } y(e) = 0$$

[Given that $y(e) = 0$, means that when $x = e$ then $y = 0$. Recall functional notation]

Linear Equation

$$\int Pdx = \int (-1)dx = -x$$

Thus $y = e^x \int \left(e^{-x}\right)\left(\frac{e^x}{x}\right)dx$

$$= e^x \int \left(\frac{1}{x}\right)dx$$

$$= e^x\left(\ell n|x| + C\right)$$

Use given that $y(e) = 0$ to find $C$

$$0 = e^e\left(\ell n|e| + C\right)$$

$$0 = (1 + C)$$

$$C = -1$$

Thus $\mathrm{y} = e^x\left(\ell n|x| - 1\right)$

(8)

**[20]**

**Maximum: [20]**

## QUESTION 1

1.1 $\left(D^2 - 36\right)y = \cosh 3x$

$$m^2 - 36 = 0$$
$$(m-6)(m+6) = 0$$
$$m = 6 \text{ or } m = -6$$
$$y_{CF} = Ae^{-6x} + Be^{6x}$$

$$y_{PI} = \frac{1}{D^2 - 36}\{\cosh 3x\}$$
$$= \frac{1}{9-36}\{\cosh 3x\}$$
$$= -\frac{\cosh 3x}{27}$$

or put $\cosh 3x = \frac{e^{3x} + e^{-3x}}{2}$

or $y_{PI} = \frac{1}{D^2 - 36}\left\{\frac{e^{3x}}{2} + \frac{e^{-3x}}{2}\right\}$
$$= \frac{1}{2}\left(\frac{e^{3x}}{(-27)} + \frac{e^{-3x}}{(-27)}\right)$$
$$= \frac{1}{-27}\left(\frac{e^{3x} + e^{-3x}}{2}\right)$$

(5)

$$y_{gen} = Ae^{-6x} + Be^{6x} - \frac{\cosh 3x}{27}$$

1.2 $\left(D^2 + 2D + 4\right)y = e^{2x}\sin 2x$

$$\left(m^2 + 2m + 4\right) = 0$$
$$m = \frac{-2 \pm \sqrt{4-16}}{2}$$
$$= -1 \pm \sqrt{3}j$$
$$y_{CF} = e^{-x}\left(A\sin\sqrt{3}x + B\cos\sqrt{3}x\right)$$

$$y_{PI} = \frac{1}{D^2 + 2D + 4}\left\{e^{2x}\sin 2x\right\}$$
$$= e^{2x}\frac{1}{(D+2)^2 + 2(D+2) + 4}\{\sin 2x\}$$
$$= e^{2x}\frac{1}{D^2 + 6D + 12}\{\sin 2x\}$$
$$= e^{2x}\frac{1}{-4 + 6D + 12}\{\sin 2x\}$$
$$= e^{2x}\frac{1}{6D + 8}\{\sin 2x\}$$



Open Rubric

$$= \left(\frac{e^{2x}}{2}\right)\frac{1}{(3D+4)}\frac{(3D-4)}{(3D-4)}\{\sin 2x\}$$

$$= \left(\frac{e^{2x}}{2}\right)\frac{(3D-4)}{9(-4)-16}\{\sin 2x\}$$

$$= \left(\frac{e^{2x}}{-104}\right)(3D-4)\{\sin 2x\}$$

$$= \left(\frac{e^{2x}}{-104}\right)(6\cos 2x - 4\sin 2x)$$

(8)

$$= \left(\frac{e^{2x}}{-52}\right)(3\cos 2x - 2\sin 2x)$$

$$y_{gen} = e^{-x}\left(A\sin\sqrt{3}x + B\cos\sqrt{3}x\right) - \left(\frac{e^{2x}}{52}\right)(3\cos 2x - 2\sin 2x)$$

**[13]**

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**QUESTION 2**

Solve for **only** $x$ in the following set of simultaneous differential equations by using **D-operator** methods:

$$(D+1)x - Dy = -1 \qquad (1)$$

$$(2D-1)x - \left(D - \tfrac{1}{2}\right)y = 1 \qquad (2)$$

(8)

**Answer:** Using elimination

$$\left(D-\tfrac{1}{2}\right)(D+1)x - D\left(D-\tfrac{1}{2}\right)y = \left(D-\tfrac{1}{2}\right)(-1) \qquad (3) = \left(D-\tfrac{1}{2}\right)\times(1)$$

$$D(2D-1)x - D\left(D-\tfrac{1}{2}\right)y = D(1) \qquad (4) = (D)\times(2)$$

$(3)-(4)$:

$$\left(D-\tfrac{1}{2}\right)(D+1)x - D(2D-1)x = \left(0+\tfrac{1}{2}\right) - (0)$$

$$\left(D^2 + \tfrac{1}{2}D - \tfrac{1}{2} - 2D^2 + D\right)x = \tfrac{1}{2}$$

$$\left(-D^2 + \tfrac{3}{2}D - \tfrac{1}{2}\right)x = \tfrac{1}{2}$$

$$-m^2 + \tfrac{3}{2}m - \tfrac{1}{2} = 0$$

$$2m^2 - 3m + 1 = 0$$

$$(2m-1)(m-1) = 0$$

$$m = \tfrac{1}{2} \text{ or } m = 1$$

$$x_{CF} = Ae^{\frac{1}{2}t} + Be^{t}$$

$$x_{PI} = \frac{1}{-D^2 + \frac{3}{2}D - \frac{1}{2}}\left\{\tfrac{1}{2}e^{0t}\right\}$$

$$= \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)}$$

$$= -1$$

$$x_{GEN}(t) = Ae^{\frac{1}{2}t} + Be^{t} - 1$$

**Answer:** Using Determinants and Cramer's rule

$$\begin{vmatrix} (D+1) & (-D) \\ (2D-1) & -\left(D-\tfrac{1}{2}\right) \end{vmatrix} x = \begin{vmatrix} -1 & (-D) \\ 1 & -\left(D-\tfrac{1}{2}\right) \end{vmatrix}$$

$$\left[-(D+1)\left(D-\tfrac{1}{2}\right)+D(2D-1)\right]x = \left(D-\tfrac{1}{2}\right)(1)-(-D)(1)$$

$$\left[-D^2-\tfrac{1}{2}D+\tfrac{1}{2}+2D^2-D\right]x = 0-\tfrac{1}{2}+0$$

$$\left(D^2+\tfrac{1}{2}D-\tfrac{1}{2}-2D^2+D\right)x = \tfrac{1}{2}$$

$$\left(-D^2+\tfrac{3}{2}D-\tfrac{1}{2}\right)x = \tfrac{1}{2}$$

$$-m^2+\tfrac{3}{2}m-\tfrac{1}{2}=0$$

$$2m^2-3m+1=0$$

$$(2m-1)(m-1)=0$$

$$m=\tfrac{1}{2} \text{ or } m=1$$

$$x_{CF} = Ae^{\frac{1}{2}t}+Be^t$$

$$x_{PI} = \frac{1}{-D^2+\frac{3}{2}D-\frac{1}{2}}\left\{\tfrac{1}{2}e^{0t}\right\} = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -1$$

$$x_{GEN}(t) = Ae^{\frac{1}{2}t}+Be^t-1$$

**[8]**

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**QUESTION 3**

3.1.1 $L\{2t\sin 2t\} = 2\left(\dfrac{4s}{(s^2+4)^2}\right) = \dfrac{8s}{(s^2+4)^2}$ . (1)

3.1.2 $L\{3H(t-2)-\delta(t-4)\} = \dfrac{3e^{-2s}}{s}-e^{-4s}$ (2)

3.2 Use partial fractions to find the inverse Laplace transform of $\dfrac{5s+2}{s^2+3s+2}$ (5)

$$\frac{5s+2}{(s+1)(s+2)} = \frac{A}{s+1}+\frac{B}{s+2}$$

$$5s+2 = A(s+2)+(s+1)$$

Put $s=-1$: $-3=A$

Put $s=-2$: $-8=-B$

$8=B$

$$L^{-1}\left\{\frac{5s+2}{s^2+3s+2}\right\} = L^{-1}\left\{\frac{-3}{s+1}\right\}+L^{-1}\left\{\frac{8}{s+2}\right\} = -3e^{-t}+8e^{-2t}$$

**[8]**

## QUESTION 4

Determine the unique solution of the following differential equation by using **Laplace transforms:**

$$y''(t)+2y'(t)+10y(t)=\left(25t^2+16t+2\right)e^{3t}\text{, if } y(0)=0 \text{ and } y'(0)=0.$$

**Answer:**

$$y''(t) \quad +2y'(t) \quad +10\,y(t) \quad = 25t^2e^{3t}+16te^{3t} \quad +2e^{3t}$$

$$\left(s^2Y(s)-sy(0)-y'(0)\right) \quad +2\left(sY(s)-y(0)\right) \quad +10\left(Y(s)\right)=\frac{2(25)}{(s-3)^3}+\frac{16}{(s-3)^2}+\frac{2}{(s-3)}$$

given $y(0)=0$ and $y'(0)=0$

$$\left(s^2+2s+10\right)Y(s)=\frac{50+16(s-3)+2(s-3)^2}{(s-3)^3}$$

$$Y(s)=\frac{2s^2+4s+20}{(s-3)^3}$$

$$=\frac{2\left(s^2+2s+10\right)}{(s-3)^3\left(s^2+2s+10\right)}$$

$$=\frac{2}{(s-3)^3}$$

$$y(t)=t^2e^{3t}$$

**[9]**

## QUESTION 5

The motion of a mass on a spring is described by the differential equation $\frac{d^2x}{dt^2}+100x=36\cos 8t$. If $x=0$ and $\frac{dx}{dt}=0$, at $t=0$ find the steady state solution for *x(t)* and discuss the motion**.**

**Answer:** Using D-operator methods

$$\frac{d^2x}{dt^2}+100x=36\cos 8t$$

$$\left(D^2+100\right)x=36\cos 8t$$

$$m^2+100=0$$

$$m=\pm 10j$$

$$x_{CF}=A\cos 10t+B\sin 10t$$

$$x_{PI} = \frac{1}{D^2+100}\{36\cos 8t\}$$
$$= \frac{36}{-(8)^2+100}\cos 8t$$
$$= \cos 8t$$
$$x_{GEN} = A\cos 10t + B\sin 10t + \cos 8t \qquad (1)$$

Given if t = 0 then $x = 0$

$$0 = A\cos 0 + B\sin 0 + \cos 0$$
$$A = -1$$

$(1)$ becomes $x_{GEN} = -\cos 10t + B\sin 10t + \cos 8t \qquad (2)$

$$x'_{GEN} = 10\sin 10t + 10B\cos 10t - 8\sin 8t$$

Given if t = 0 then $x' = 0$

$$0 = 10B$$
$$B = 0$$

$(2)$ becomes $x_{GEN} = -\cos 10t + \cos 8t$

therefore the stationary solution $(t \to \infty)$ is a oscillating motion

**Answer:** Using Laplace Transforms

$x''(t) + 100x(t) = 36\cos 8t$ given $x = 0$ and $\frac{dx}{dt} = 0$, at $t = 0$

$$(s^2+100)X(s) = 36\left(\frac{s}{s^2+64}\right)$$
$$X(s) = \frac{36s}{(s^2+64)(s^2+100)}$$

Using partial fractions: $\frac{36s}{(s^2+64)(s^2+100)} = \frac{As+B}{s^2+64} + \frac{Cs+D}{s^2+100}$

$$36s = (As+B)(s^2+100) + (Cs+D)(s^2+64)$$

$\vdots$

We find $B = D = 0,\ A = 1$ and $C = -1$

$$X(s) = \frac{s}{s^2+(8)^2} - \frac{s}{s^2+(10)^2}$$
$$x(t) = \cos 8t - \cos 10t$$

therefore the stationary solution $(t \to \infty\ )$ is a oscillating motion

**[12]**

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**Maximum: [50]**

## QUESTION 1

If $A=\begin{bmatrix}3 & 1\\ -1 & 5\end{bmatrix}$, find an eigenvalue and an eigenvector of $A$.

**Answer:**

For eigenvalues put $|A-\lambda I|=0$

$$\begin{vmatrix}3-\lambda & 1\\ -1 & 5-\lambda\end{vmatrix}=0$$

$$(3-\lambda)(5-\lambda)+1=0$$

$$15-8\lambda+\lambda^2+1=0$$

$$\lambda^2-8\lambda+16=0$$

$$(\lambda-4)^2=0$$

$$\lambda=4\,(\text{twice})$$

For an eigenvalue corresponmding to $\lambda=4$

$$\begin{bmatrix}(3-4) & 1\\ -1 & (5-4)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$$

$$-x_1+x_2=0$$

$$x_1=x_2$$

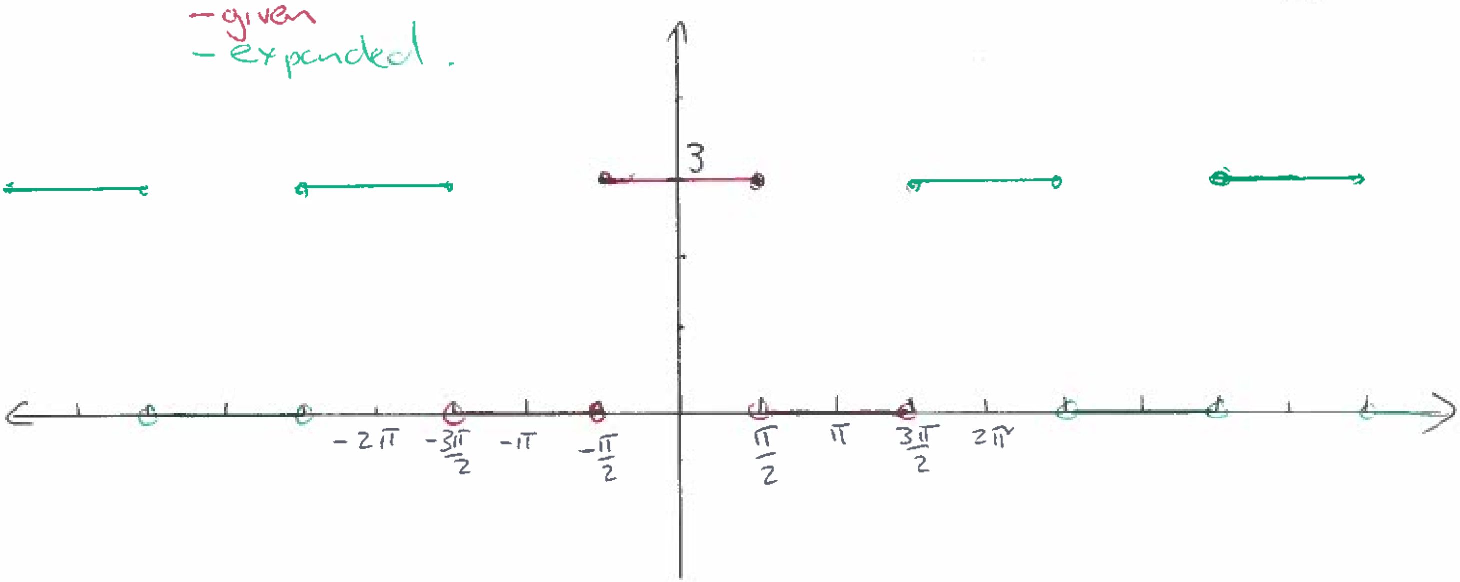
Choose $x_1=1$

Thus an eigenvalue is $\begin{bmatrix}1\\ 1\end{bmatrix}$ **[8]**

## QUESTION 2

Given the function defined by $f(t)=\begin{cases}0 & -\pi<t<-\frac{\pi}{2}\\ 3 & -\frac{\pi}{2}\le t\le\frac{\pi}{2}\\ 0 & \frac{\pi}{2}<t<\pi\end{cases}$, with period $2\pi$:

2.1 Sketch the function. (3)



2.2 even (1)

2.3 Find the Fourier series for *f(t)*. (8)

For even functions $b_n = 0$.

Period = $2\pi$, thus L = $\pi$

$$a_0 = \tfrac{1}{L}\int_{-L}^{L} f(t)\,dt$$
$$= \tfrac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\,dt + \int_{\frac{\pi}{2}}^{\pi} 0\,dt\right]$$
$$= \tfrac{1}{\pi}\left[3t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \tfrac{3}{\pi}\left[\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right]$$
$$= 3$$

or because it is an even function $a_0 = \tfrac{2}{\pi}\int_0^{\frac{\pi}{2}} 3\,dt$

$$a_n = \tfrac{1}{L}\int_{-L}^{L} f(t)\cos\left(\tfrac{n\pi t}{L}\right)dt$$
$$= \tfrac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} 0\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\cos(nt)\,dt + \int_{\frac{\pi}{2}}^{\pi} 0\,dt\right]$$
$$= \tfrac{3}{n\pi}\left[\sin(nt)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \tfrac{3}{n\pi}\left[\sin\left(\tfrac{n\pi}{2}\right) - \sin\left(-\tfrac{n\pi}{2}\right)\right]$$
$$= \tfrac{3}{n\pi}\left[\sin\left(\tfrac{n\pi}{2}\right) + \sin\left(\tfrac{n\pi}{2}\right)\right] \qquad \text{Remember } \sin(-x) = -\sin(x)$$
$$= \tfrac{6}{n\pi}\left[\sin\left(\tfrac{n\pi}{2}\right)\right]$$
$$= \begin{cases} 0 & \text{n even} \\ \tfrac{6}{n\pi}\left[\sin\left(\tfrac{n\pi}{2}\right)\right] & \text{odd} \end{cases}$$

$$f(t) = \frac{3}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sin\left(\tfrac{n\pi}{2}\right)\right)\cos(nt)$$

$$= \frac{3}{2} + \frac{6}{\pi}(\cos t) - \frac{2}{\pi}(\cos 3t) + \ldots\ldots\ldots$$

**[12]**