QUESTION 1

1.1 $\left(D^2-36\right)y=\cosh 3x$

$$m^2-36=0$$
$$(m-6)(m+6)=0$$
$$m=6 \text{ or } m=-6$$
$$y_{CF}=Ae^{-6x}+Be^{6x}$$

$$y_{PI}=\frac{1}{D^2-36}\{\cosh 3x\}$$
$$=\frac{1}{9-36}\{\cosh 3x\}$$
$$=-\frac{\cosh 3x}{27}$$

or put $\cosh 3x=\dfrac{e^{3x}+e^{-3x}}{2}$

$$\text{or } y_{PI}=\frac{1}{D^2-36}\left\{\frac{e^{3x}}{2}+\frac{e^{-3x}}{2}\right\}$$
$$=\frac{1}{2}\left(\frac{e^{3x}}{(-27)}+\frac{e^{-3x}}{(-27)}\right)$$
$$=\frac{1}{-27}\left(\frac{e^{3x}+e^{-3x}}{2}\right)$$

(5)

$$y_{gen}=Ae^{-6x}+Be^{6x}-\frac{\cosh 3x}{27}$$

1.2 $\left(D^2+2D+4\right)y=e^{2x}\sin 2x$

$$\left(m^2+2m+4\right)=0$$
$$m=\frac{-2\pm\sqrt{4-16}}{2}$$
$$=-1\pm\sqrt{3}j$$
$$y_{CF}=e^{-x}\left(A\sin\sqrt{3}x+B\cos\sqrt{3}x\right)$$

$$y_{PI}=\frac{1}{D^2+2D+4}\left\{e^{2x}\sin 2x\right\}$$
$$=e^{2x}\frac{1}{(D+2)^2+2(D+2)+4}\{\sin 2x\}$$
$$=e^{2x}\frac{1}{D^2+6D+12}\{\sin 2x\}$$
$$=e^{2x}\frac{1}{-4+6D+12}\{\sin 2x\}$$
$$=e^{2x}\frac{1}{6D+8}\{\sin 2x\}$$

$$=\left(\frac{e^{2x}}{2}\right)\frac{1}{(3D+4)}\frac{(3D-4)}{(3D-4)}\{\sin 2x\}$$

$$=\left(\frac{e^{2x}}{2}\right)\frac{(3D-4)}{9(-4)-16}\{\sin 2x\}$$

$$=\left(\frac{e^{2x}}{-104}\right)(3D-4)\{\sin 2x\}$$

$$=\left(\frac{e^{2x}}{-104}\right)(6\cos 2x-4\sin 2x)$$

(8)

$$=\left(\frac{e^{2x}}{-52}\right)(3\cos 2x-2\sin 2x)$$

$$y_{gen}=e^{-x}\left(A\sin\sqrt{3}x+B\cos\sqrt{3}x\right)-\left(\frac{e^{2x}}{52}\right)(3\cos 2x-2\sin 2x)$$

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QUESTION 2

Solve for **only** x in the following set of simultaneous differential equations by using **D-operator** methods:

$$(D+1)x-Dy=-1 \qquad (1)$$

$$(2D-1)x-\left(D-\tfrac{1}{2}\right)y=1 \qquad (2)$$

(8)

Answer: Using elimination

$$\left(D-\tfrac{1}{2}\right)(D+1)x-D\left(D-\tfrac{1}{2}\right)y=\left(D-\tfrac{1}{2}\right)(-1) \qquad (3)=\left(D-\tfrac{1}{2}\right)\times(1)$$

$$D(2D-1)x-D\left(D-\tfrac{1}{2}\right)y=D(1) \qquad (4)=(D)\times(2)$$

$(3)-(4)$:

$$\left(D-\tfrac{1}{2}\right)(D+1)x-D(2D-1)x=\left(0+\tfrac{1}{2}\right)-(0)$$

$$\left(D^2+\tfrac{1}{2}D-\tfrac{1}{2}-2D^2+D\right)x=\tfrac{1}{2}$$

$$\left(-D^2+\tfrac{3}{2}D-\tfrac{1}{2}\right)x=\tfrac{1}{2}$$

$$-m^2+\tfrac{3}{2}m-\tfrac{1}{2}=0$$

$$2m^2-3m+1=0$$

$$(2m-1)(m-1)=0$$

$$m=\tfrac{1}{2}\text{ or }m=1$$

$$x_{CF}=Ae^{\frac{1}{2}t}+Be^{t}$$

$$x_{PI}=\frac{1}{-D^2+\frac{3}{2}D-\frac{1}{2}}\left\{\tfrac{1}{2}e^{0t}\right\}$$

$$=\frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)}$$

$$=-1$$

$$x_{GEN}(t)=Ae^{\frac{1}{2}t}+Be^{t}-1$$

Answer: Using Determinants and Cramer's rule

$$\begin{vmatrix} (D+1) & (-D) \\ (2D-1) & -\left(D-\frac{1}{2}\right) \end{vmatrix} x = \begin{vmatrix} -1 & (-D) \\ 1 & -\left(D-\frac{1}{2}\right) \end{vmatrix}$$

$$\left[-(D+1)\left(D-\tfrac{1}{2}\right)+D(2D-1)\right]x = \left(D-\tfrac{1}{2}\right)(1)-(-D)(1)$$

$$\left[-D^2-\tfrac{1}{2}D+\tfrac{1}{2}+2D^2-D\right]x = 0-\tfrac{1}{2}+0$$

$$\left(D^2+\tfrac{1}{2}D-\tfrac{1}{2}-2D^2+D\right)x = \tfrac{1}{2}$$

$$\left(-D^2+\tfrac{3}{2}D-\tfrac{1}{2}\right)x = \tfrac{1}{2}$$

$$-m^2+\tfrac{3}{2}m-\tfrac{1}{2}=0$$

$$2m^2-3m+1=0$$

$$(2m-1)(m-1)=0$$

$$m=\tfrac{1}{2} \text{ or } m=1$$

$$x_{CF} = Ae^{\frac{1}{2}t}+Be^t$$

$$x_{PI} = \frac{1}{-D^2+\frac{3}{2}D-\frac{1}{2}}\left\{\tfrac{1}{2}e^{0t}\right\} = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -1$$

$$x_{GEN}(t) = Ae^{\frac{1}{2}t}+Be^t-1$$

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QUESTION 3

3.1.1 $\quad L\{2t\sin 2t\} = 2\left(\dfrac{4s}{(s^2+4)^2}\right) = \dfrac{8s}{(s^2+4)^2}$. (1)

3.1.2 $\quad L\{3H(t-2)-\delta(t-4)\} = \dfrac{3e^{-2s}}{s}-e^{-4s}$ (2)

3.2 Use partial fractions to find the inverse Laplace transform of $\dfrac{5s+2}{s^2+3s+2}$ (5)

$$\frac{5s+2}{(s+1)(s+2)} = \frac{A}{s+1}+\frac{B}{s+2}$$

$$5s+2 = A(s+2)+(s+1)$$

Put $s=-1$: $\quad -3=A$

Put $s=-2$: $\quad -8=-B$

$$8=B$$

$$L^{-1}\left\{\frac{5s+2}{s^2+3s+2}\right\} = L^{-1}\left\{\frac{-3}{s+1}\right\}+L^{-1}\left\{\frac{8}{s+2}\right\}$$

$$= -3e^{-t}+8e^{-2t}$$

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QUESTION 4

Determine the unique solution of the following differential equation by using **Laplace transforms:**

$$y''(t)+2y'(t)+10y(t)=\left(25t^2+16t+2\right)e^{3t}\text{, if } y(0)=0 \text{ and } y'(0)=0.$$

Answer:

$$y''(t) \quad +2y'(t) \quad +10\,y(t) \quad = 25t^2e^{3t}+16te^{3t} \quad +2e^{3t}$$

$$\left(s^2Y(s)-sy(0)-y'(0)\right) \quad +2\left(sY(s)-y(0)\right) \quad +10\left(Y(s)\right)=\frac{2(25)}{(s-3)^3}+\frac{16}{(s-3)^2}+\frac{2}{(s-3)}$$

given $y(0)=0$ and $y'(0)=0$

$$\left(s^2+2s+10\right)Y(s)=\frac{50+16(s-3)+2(s-3)^2}{(s-3)^3}$$

$$Y(s)=\frac{2s^2+4s+20}{(s-3)^3}$$

$$=\frac{2\left(s^2+2s+10\right)}{(s-3)^3\left(s^2+2s+10\right)}$$

$$=\frac{2}{(s-3)^3}$$

$$y(t)=t^2e^{3t}$$

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QUESTION 5

The motion of a mass on a spring is described by the differential equation $\frac{d^2x}{dt^2}+100x=36\cos 8t$. If $x=0$ and $\frac{dx}{dt}=0$, at $t=0$ find the steady state solution for *x(t)* and discuss the motion**.**

Answer: Using D-operator methods

$$\frac{d^2x}{dt^2}+100x=36\cos 8t$$

$$\left(D^2+100\right)x=36\cos 8t$$

$$m^2+100=0$$

$$m=\pm 10j$$

$$x_{CF}=A\cos 10t+B\sin 10t$$

$$x_{PI} = \frac{1}{D^2+100}\{36\cos 8t\}$$
$$= \frac{36}{-(8)^2+100}\cos 8t$$
$$= \cos 8t$$
$$x_{GEN} = A\cos 10t + B\sin 10t + \cos 8t \qquad (1)$$

Given if t = 0 then $x = 0$

$$0 = A\cos 0 + B\sin 0 + \cos 0$$
$$A = -1$$

(1) becomes $x_{GEN} = -\cos 10t + B\sin 10t + \cos 8t \qquad (2)$

$$x'_{GEN} = 10\sin 10t + 10B\cos 10t - 8\sin 8t$$

Given if t = 0 then $x' = 0$

$$0 = 10B$$
$$B = 0$$

(2) becomes $x_{GEN} = -\cos 10t + \cos 8t$

therefore the stationary solution $(t \to \infty)$ is a oscillating motion

Answer: Using Laplace Transforms

$x''(t) + 100x(t) = 36\cos 8t$ given $x = 0$ and $\dfrac{dx}{dt} = 0$, at $t = 0$

$$(s^2+100)X(s) = 36\left(\frac{s}{s^2+64}\right)$$
$$X(s) = \frac{36s}{(s^2+64)(s^2+100)}$$

Using partial fractions: $\dfrac{36s}{(s^2+64)(s^2+100)} = \dfrac{As+B}{s^2+64} + \dfrac{Cs+D}{s^2+100}$

$$36s = (As+B)(s^2+100) + (Cs+D)(s^2+64)$$

⋮

We find $B = D = 0,\ A = 1$ and $C = -1$

$$X(s) = \frac{s}{s^2+(8)^2} - \frac{s}{s^2+(10)^2}$$
$$x(t) = \cos 8t - \cos 10t$$

therefore the stationary solution $(t \to \infty\ )$ is a oscillating motion

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Maximum: [50]