Tutorial letter 202/2/2018

Statistical Inference III

STA3702

Semester 2

Department of Statistics

SOLUTIONS TO ASSIGNMENT 02

Question 1 **[30]**

(a) **(5)**

$$\begin{aligned} f(x|\theta) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\} \\ &= \exp\left\{\theta x - \frac{x^2}{2} - \ln\sqrt{2\pi} - \frac{\theta^2}{2}\right\} \\ &= \exp\{p(\theta)K(x) + S(x) + q(\theta)\} \end{aligned}$$

where $p(\theta) = \theta$, $K(x) = x$, $S(x) = -\frac{x^2}{2} - \ln\sqrt{2\pi}$ and $q(\theta) = -\frac{\theta^2}{2}$.

(b) **(3)**

$-\sum_{i=1}^{n} X_i$ is the complete sufficient statistic for θ - a property of the members of the exponential family of distributions.

(c) **(8)**

$$p'(\theta) = 1,\ \ p''(\theta) = 0,\ \ q'(\theta) = -\theta,\ \ q''(\theta) = -1.$$

(2)

$$E\left[\sum_{i=1}^{n} X_i\right] = -n\frac{q'(\theta)}{p'(\theta)} = -n\frac{-\theta}{(1)} = n\theta.$$

(2)

$$\begin{aligned} Var\left[\sum_{i=1}^{n} X_i\right] &= \frac{n}{[p'(\theta)]^3}\left\{p''(\theta)q'(\theta) - q''(\theta)p'(\theta)\right\} \\ &= \frac{n}{[1]^3}\{0 \times (-\theta) - (-1) \times (1)\} = n. \end{aligned}$$

(4)

(d) **(9)**

The likelihood function is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2\right\}$$

and the log likelihood function is

$$l(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2.$$

(4)

Now,

$$l'(\theta) = \sum_{i=1}^{n}(x_i - \theta) \text{ and } l''(\theta) = -n.$$

(2)

The Fisher information about θ in the sample is

$$I(\theta) = -E\left[l''(\theta)\right] = n.$$

(2)

Hence the required Cramer-Rao lower bound is $\dfrac{1}{I(\theta)} = \dfrac{1}{n}$. (1)

(e) **(5)**

From part (c) above, $E\left[\dfrac{1}{n}\sum_{i=1}^{n} X_i\right] = \theta$. Therefore, $\dfrac{1}{n}\sum_{i=1}^{n} X_i$ is a minimum variance unbiased estimator of θ because it is an unbiased estimator of θ and a function of a complete sufficient statistic for θ. (3+2=5)

Question 2 [25]

(a) **(5)**

The probability density function of Y is

$$\begin{aligned} g(y|\theta) &= \frac{\partial G(y|\theta)}{\partial y} = n[F(y|\theta)]^{n-1} f(y|\theta) \\ &= \begin{cases} n\exp\{(n-1)(y-\theta)\} \times \exp\{(y-\theta)\} = n\exp\{n(y-\theta)\} & \text{if } -\infty < y \leq \theta, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) **(6)**

The moment generating function of Y is

$$\begin{aligned} M_y(t) &= \int_{\theta}^{\infty} n\exp\{ty\}\exp\{n(y-\theta)\}dy \\ &= \int_{\theta}^{\infty} \frac{n}{n+t}(n+t)\exp\{(n+t)(y-\theta)+t\theta\}dy \\ &= \frac{n}{n+t}\exp\{t\theta\}\int_{\theta}^{\infty}(n+t)\exp\{(n+t)(y-\theta)\}dy = \frac{n}{n+t}\exp\{t\theta\} \end{aligned}$$

(4)

$$E[Y] = \left.\frac{\partial \ln M_y(t)}{\partial t}\right|_{t=0} = -\frac{1}{n} + \theta.$$

(2)

(c) **(5)**

From part (b) above, $E\left[Y + \dfrac{1}{n}\right] = \theta$. Therefore, $Y + \dfrac{1}{n}$ is a minimum variance unbiased estimator of θ because it is an unbiased estimator of θ and a function of a complete sufficient statistic for θ. (3+2=5)

(d) **(5)**

$$0.95 = P(c < Y < \theta) = 1 - \exp\{n(c-\theta)\} \Longrightarrow n(c-\theta) = \ln(1-0.95) \Longrightarrow c = \theta - \frac{2.9957}{n}.$$

(e) **(4)**

From part (d)

$$\theta - \frac{2.9957}{n} < Y < \theta \Longrightarrow -\frac{2.9957}{n} < Y - \theta < 0 \Longrightarrow 0 < \theta - Y < \frac{2.9957}{n} \Longrightarrow Y < \theta < Y + \frac{2.9957}{n}$$

is the 95% confidence interval for θ.

Question 3 **[15]**

(a) **(13)**

The likelihood function of θ is: $L(\theta|\mathbf{x}) = \dfrac{e^{-10\theta}\theta^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x_i!}$. (2)

Hence for any $\theta > 0.1$, $r(0.1, \theta|\mathbf{x}) = \dfrac{L(0.1|\mathbf{x})}{L(\theta|\mathbf{x})} = \left(\dfrac{0.1}{\theta}\right)^{\sum_{i=1}^{10} x_i} e^{-10(0.1-\theta)}$. (3)

According to the Neyman-Pearson theorem, the best test of the hypotheses rejects H_0 if

$$\begin{aligned} r(0.1, \theta|\mathbf{x}) &= \left(\frac{0.1}{\theta}\right)^{\sum_{i=1}^{10} x_i} e^{-10(0.1-\theta)} \le k \in (0, 1] \\ &\equiv \text{ if } (\ln 0.1 - \ln\theta)\sum_{i=1}^{10} x_i \le \ln k + 10(0.1-\theta) = k^* \\ &\equiv \text{ if } \sum_{i=1}^{10} x_i \ge \frac{k^*}{\ln 0.1 - \ln\theta} = c \end{aligned}$$

where c solves the probability equation $\alpha = P\left(\sum_{i=1}^{10} X_i \ge c|\theta = 0.1\right)$. (8)

(b) $\alpha = P\left(\sum_{i=1}^{10} X_i \ge 1|\theta = 0.1\right) = 1 - P\left(\sum_{i=1}^{10} X_i < 1|\theta = 0.1\right) = 1 - e^{-10\times 0.1} = 0.6321.$ **(2)**

Question 4 **[25]**

(a) **(13)**

The likelihood function of θ is: $L(\theta|\mathbf{x}) = \theta^{\sum_{i=1}^{10} x_i}(1-\theta)^{10-\sum_{i=1}^{10} x_i}$. (2)

Hence, $r(0.5, 0.1|\mathbf{x}) = \dfrac{L(0.5|\mathbf{x})}{L(0.1|\mathbf{x})} = \left(\dfrac{5}{9}\right)^{10} 9^{\sum_{i=1}^{10} x_i}$. (3)

According to the Neyman-Pearson theorem, the best test of the hypotheses rejects H_0 if

$$\begin{aligned} r(0.5, 0.1|\mathbf{x}) &= \left(\frac{5}{9}\right)^{10} 9^{\sum_{i=1}^{10}} \leq k \in (0, 1] \\ &\equiv \text{ if } 9^{\sum_{i=1}^{10} x_i} \leq \left(\frac{9}{5}\right)^{10} k = k^* \\ &\equiv \text{ if } \sum_{i=1}^{10} x_i \leq \ln k^* / \ln 9 = c \end{aligned}$$

where c solves the probability equation $\alpha = P\left(\sum_{i=1}^{10} X_i \leq c|\theta = 0.5\right)$. (8)

(b) **(12)**

(i) $\alpha = P\left(\sum_{i=1}^{10} X_i \leq 1|\theta = 0.5\right) = P\left(\sum_{i=1}^{10} X_i = 0|\theta = 0.5\right) = (0.5)^{10} + 10 \times (0.5)^{10} = 0.0107.$ **(6)**

(ii) The probability of the type II error is $\beta = P\left(\sum_{i=1}^{10} X_i \geq 2|\theta = 0.1\right)$

$= 1 - P\left(\sum_{i=1}^{10} X_i \leq 1|\theta = 0.1\right) = 1 - [(.9)^{10} + 10 \times 0.1 \times (.9)^9] = 1 - 0.7361.$

(5)

Hence the power of the test is $1 - \beta = 0.7361$ **(1)**

TOTAL: [95]