Tutorial letter 201/2/2018

Statistical Inference III

STA3702

Semester 2

Department of Statistics

SOLUTIONS TO ASSIGNMENT 01

Define tomorrow.

UNISA university of south africa

Question 1 **[13]**

(a) $(i)\ \ E(\bar{Y}) = \beta\bar{x}$ and $(ii)\ \ E(\bar{Y}^2) = Var(\bar{Y}) + [E(\bar{Y})]^2 = \dfrac{\sigma^2}{n} + \beta^2\bar{x}^2$. **(1+2=3)**

(b) $\bar{Y}$ and $\bar{Y}^2$ correspond to $E(\bar{Y})$ and $E(\bar{Y}^2)$, respectively. **(1+1=2)**

(c) $\bar{Y} = \tilde{\beta}\bar{x}$ and $\bar{Y}^2 = \dfrac{\tilde{\sigma}^2}{n} + \tilde{\beta}^2\bar{x}^2$ where $\tilde{\beta}$ and $\tilde{\sigma}^2$ are method of moments estimators of β and σ^2, respectively. **(1+1=2)**

(d) $\tilde{\beta} = \dfrac{\bar{Y}}{\bar{x}} \Longrightarrow \bar{Y}^2 = \dfrac{\tilde{\sigma}^2}{n} + \tilde{\beta}^2\bar{x}^2 \Longrightarrow \dfrac{\tilde{\sigma}^2}{n} = \bar{Y}^2 - \tilde{\beta}^2\bar{x}^2 \Longrightarrow \tilde{\sigma}^2 = n(\bar{Y}^2 - \tilde{\beta}^2\bar{x}^2)$. **(1+3=4)**

(e) $E(\tilde{\beta}) = E\left(\dfrac{\bar{Y}}{\bar{x}}\right) = \dfrac{E(\bar{Y})}{\bar{x}} = \dfrac{\beta\bar{x}}{\bar{x}} = \beta$. **(2)**

Question 2 **[22]**

(a) **(4)**

(i) The likelihood function of β and σ^2 is

$$\begin{aligned} L(\beta,\sigma^2|\mathbf{x}) &= \prod_{i=1}^{n} f(y_i|\beta,\sigma^2) \\ &= \prod_{i=1}^{n}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2}\exp\left\{-\frac{1}{2\sigma^2}(y_i-\beta x_i)^2\right\} \\ &= (1/2\pi)^{n/2}(1/\sigma^2)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta x_i)^2\right\} \end{aligned}$$

(2)

(ii) The log likelihood function of β and σ^2 is

$$\begin{aligned} l(\beta,\sigma^2) &= \ln L(\beta,\sigma^2|\mathbf{x}) \\ &= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta x_i)^2 \end{aligned}$$

(2)

(b) **(16)**

Differentiating $l(\beta,\sigma^2)$ with respect to β and equating the derivative to zero obtains:

$$0 = l'(\hat{\beta},\sigma^2) = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \hat{\beta}x_i\right)x_i = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i x_i - \hat{\beta}x_i^2\right)$$

where $\hat{\beta}$ is the maximum likelihood estimator of β. The solution for $\hat{\beta}$ in the equation is $\hat{\beta} = \dfrac{\sum_{i=1}^{n} Y_i x_i}{\sum_{i=1}^{n} x_i^2}$. **(8)**

Differentiating $l(\hat{\beta}, \sigma^2)$ with respect to σ^2 and equating the derivative to zero obtains:

$$0 = l'(\hat{\beta}, \hat{\sigma}^2) = -\frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \sum_{i=1}^{n} \left(Y_i - \hat{\beta} x_i\right)^2$$

where $\hat{\sigma}^2$ is the maximum likelihood estimator of σ^2. The solution for $\hat{\sigma}^2$ in the equation is $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \hat{\beta} x_i\right)^2$. **(8)**

(c) By the invariance property of maximum likelihood estimators, the maximum likelihood estimator of σ is

$$\sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left(Y_i - \hat{\beta} x_i\right)^2}.$$

(2)

Question 3 **[20]**

(a) (i) $E(\bar{X}) = E(X) = \lambda.$ **(1)**

(ii)

$$\begin{aligned}
E(S^2) &= E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)\right] \\
&= \frac{1}{n-1}\left(\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2\right)) \\
&= \frac{1}{n-1}\left[n\left(Var(X) + [E(X)]^2\right) - n\left(Var(\bar{X}) + [E(\bar{X})]^2\right)\right] \\
&= \frac{1}{1}n-1\left[n(\lambda + \lambda^2) - n\left(\frac{\lambda}{n} + \lambda^2\right)\right] \\
&= \frac{1}{n-1}\left[n\lambda + n\lambda^2 - \lambda - n\lambda^2\right] = \lambda
\end{aligned}$$

(5)

(iii)

$$\begin{aligned}
E(\bar{S}^2) &= E\left[\frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)\right] \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2\right)) \\
&= \frac{1}{n}\left[n\left(Var(X) + [E(X)]^2\right) - n\left(Var(\bar{X}) + [E(\bar{X})]^2\right)\right] \\
&= \frac{1}{1}n\left[n(\lambda + \lambda^2) - n\left(\frac{\lambda}{n} + \lambda^2\right)\right] \\
&= \frac{1}{n}\left[n\lambda + n\lambda^2 - \lambda - n\lambda^2\right] = \frac{n-1}{n}\lambda
\end{aligned}$$

(5)

(b) $(i)\ \ Var(\bar{X}) = \dfrac{Var(X)}{n} = \dfrac{\lambda}{n}$. **(1)**

$(ii)\ \ E(\lambda - \bar{S}^2) = \lambda - E(\bar{S}^2) = \lambda - \dfrac{n-1}{n}\lambda = \dfrac{\lambda}{n}$. **(2)**

(c) (i) That $E(\bar{X}) = \lambda$ and $\lim\limits_{n\to\infty} Var(\bar{X}) = \lim\limits_{n\to\infty} \dfrac{\lambda}{n} = 0$ implies that $\bar{X}$ is a consistent estimator of λ.

(3)

$(ii)\ \ \lim\limits_{n\to\infty} E(\bar{S}^2) = \lambda - \lim\limits_{n\to\infty} \dfrac{\lambda}{n} = \lambda$. **(3)**

Question 4 **[35]**

(a) **(16)**

(i) The likelihood function of θ is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta} \left(\prod_{i=1}^{n} x_i\right)^{-1}$$

and the log likelihood function of θ is

$$l(\theta) = \ln L(\theta|\mathbf{x}) = n\ln\theta - \theta\sum_{i=1}^{n}\ln x_i - \ln\prod_{i=1}^{n} x_i.$$

Now

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n}\ln x_i \text{ and } l''(\theta) = -\frac{n}{\theta^2} \Longrightarrow \mathbf{I}(\theta) = -l''(\theta) = \frac{n}{\theta^2}.$$

(8)

(ii) The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} - \sum_{i=1}^{n}\ln X_i.$$

The solution is $\hat{\theta} = \dfrac{n}{\sum_{i=1}^{n}\ln X_i}$.

$$\mathbf{I}(\mathbf{x}) = -l''(\hat{\theta}) = \frac{n}{\hat{\theta}^2}.$$

(5)

(iii) The Cramer-Rao lower bound is $\left[\dfrac{\partial\theta}{\partial\theta}\right]^2 / \mathbf{I}(\theta) = \dfrac{\theta^2}{n}$. **(3)**

(b) **(7)**

From part $(a)(i)$ above:

$$\begin{aligned} L(\theta|\mathbf{x}) &= \left(\prod_{i=1}^{n} x_i\right)^{-1} \theta^n \left(\prod_{i=1}^{n} x_i\right)^{-\theta} \\ &= m_1(\mathbf{x}) \times m_2\left(\theta, \prod_{i=1}^{n} x_i\right) \end{aligned}$$

where $m_1(\mathbf{x}) = \left(\prod_{i=1}^n x_i\right)^{-1}$ and $m_2\left(\theta, \prod_{i=1}^n x_i\right) = \theta^n \left(\prod_{i=1}^n x_i\right)^{-\theta}$. Hence $\prod_{i=1}^n X_i$ is a sufficient statistic for θ by the factorisation theorem. **(3)**

Now let $Y_1, Y_2, Y_3, ..., Y_n$ be another random sample from the same distribution. Then

$$\frac{L(\theta|\mathbf{x})}{L(\theta|\mathbf{y})} = \left(\prod_{i=1}^{n} \frac{x_i}{y_i}\right)^{-1} \left(\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i}\right)^{-\theta}$$

is independent of θ if $\prod_{i=1}^n x_i / \prod_{i=1}^n y_i = 1$ equivalently if $\prod_{i=1}^n y_i = \prod_{i=1}^n x_i$. This means that $\prod_{i=1}^n X_i$ is a minimal sufficient statistic for θ. **(4)**

(c) **(12)**

The maximum likelihood estimate of $\hat{\theta}$ is $\hat{\theta} = \dfrac{10}{\sum_{i=1}^{10} \ln x_i} = \dfrac{10}{8.4028} = 1.1901.$ **(4)**

The standard error of $\hat{\theta}$ is $se(\hat{\theta}) = \sqrt{1/I(\mathbf{x})} = \sqrt{\hat{\theta}^2/10} = 0.1416.$ **(4)**

The 95% confidence interval of θ is $1.1901 \pm 1.96 \times .1416 = [0.9125; 1.4677].$ **(4)**

TOTAL: [90]