Question 3 **[23]**

Suppose that $X_1, X_2, ..., X_n$ is a random sample from a distribution with probability density function:

$$f(x|\theta) = \begin{cases} \frac{1}{\theta}\exp\{-x/\theta\} & \text{if } x > 0,\ \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f(x|\theta)$ belongs to the 1-parameter exponential family. **(4)**

(b) Write down the complete sufficient statistic for θ. **(3)**

(c) Show that $\bar{X}$ is a $MVUE$ of θ **(8)**

(d) Determine the $CRLB$ for an unbiased estimator of θ. **(8)**

Question 4 **[21]**

(a) It is known that the random variable X has probability density function

$$f(x|\theta) = \begin{cases} \theta x^{(\theta-1)} & 0 < x < 1,\ \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is desired to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. A random sample X_1 of size $n = 1$ is to be used.

(i) Calculate the level of significance and the power of the test which rejects H_0 if $X_1 \geq \frac{1}{2}$. **(8)**

(ii) Consider the test which rejects H_0 if $X_1 \geq c$. Find c for which the level of significance of the test is 0.1. **(5)**

(b) Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a population with probability density in part (a). Show that the best test for the hypotheses in part (a) rejects H_0 if

$$\prod_{i=1}^{n} x_i \geq c$$

where c solves the probability equation

$$\alpha = P\left(\prod_{i=1}^{n} x_i \geq c|\theta = 1\right).$$

(8)

TOTAL: [100]

START HERE

ADDENDUM B: SECOND SEMESTER ASSIGNMENTS

B.1 Assignment 01

ONLY FOR SEMESTER 2 STUDENTS
ASSIGNMENT 01
Unique Nr.: 755093
Fixed closing date: 26 AUGUST 2016

Question 1 **[20]**

Suppose that $X_1, X_2, X_3, ..., X_n$ is a random sample from Poisson distribution with probability mass function:

$$p(x|\theta) = \begin{cases} \frac{e^{-\theta}\theta^x}{x!} & \text{if } x = 0, 1, 2, ..., \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimator (MME) of θ. (pg 39) **(4)**

(b) Find the maximum likelihood estimator (MLE) of θ. (pg 32) **(8)**

(c) Find the maximum likelihood estimators (MLE) of the following functions of θ. **Justify your answers.**

(i) e^θ **(3)**

pg 145 & 149 **(ii)** $P(X \geq 1)$. **(5)**

$$1 - P(X=0) = 1 - \frac{e^{-\theta}}{0!} = 1 - e^{-\theta}$$

Question 2 **[20]**

Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a population whose probability density function is

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1,\ \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

pg 31

(a) Find the likelihood and log likelihood functions of θ. **(4)**

(b) Show that $\prod_{i=1}^{n} X_i$ is a minimal sufficient statistic for θ. Study unit 4. pg 75? **(7)**

(c) Find the maximum likelihood estimator (MLE) of θ. Follow steps **(4)**

(d) Determine the observed information, $I(\mathbf{X})$, of θ **(5)**

$-\ddot{l}(\hat{\theta}|x)$

second derivative & put in Likelihood estimate.

Question 3 (page 50, ex 2.11.1) **[20]**

Suppose that 3, 0, 3, 0, 3, 2, 2, 0, 1, 0 are 10 independent observations from a population whose discrete distribution has a probability mass function given in the table below.

x	0	1	2	3
$P(X = x)$	$\frac{2}{3}\theta$	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

(a) Find the method of moments estimate (MME) of θ. $E(X_i) = \sum x \cdot f(x|\theta)$ **(4)**

(b) Find the maximum likelihood estimate (MLE) of θ. **(6)**

(c) Find the approximate standard errors for the estimates in parts (a) and (b). **(7)**

(d) Which of the MME and MLE is preferred and why? **(3)**

Question 4 **[20]**

Suppose that $X_1, X_2, X_3, ..., X_n$ is a random sample from a distribution with probability density function:

$$f(x|\theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2} \exp\left(-\frac{x-\theta_1}{\theta_2}\right) & \text{if } 0 < \theta_2 < \infty,\ 0 < \theta_1 \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the likelihood function of θ_1 and θ_2. one function for both **(3)**

(b) What are the joint sufficient statistics for θ_1 and θ_2? pg 78 **(5)**

pg 73 **(c)** Show that the maximum likelihood estimator (MLE) of θ_1 is $\hat{\theta}_1 = \min\{X_1, X_2, X_3, ..., X_n\}$. **(6)**

(d) Find the maximum likelihood estimator of (MLE) of θ_2. **(6)**

TOTAL: [80]

c) $l(\theta_1, \theta_2 | x) = -n\ln\theta_2 - \sum_{i=1}^{n} \left(\frac{x_i - \theta_1}{\theta_2}\right)$

$-\sum(x_i - \theta_1)$ needs to be as big as possible

$\sum(x_i - \theta_1)$ needs to be as small as possible

$\therefore \theta_1$ needs to be as big as possible but it cannot be higher than x_1 as $\theta_1 \leq x$

$\therefore \hat{\theta}_1 = \min(X_1, X_2, X_3, \ldots, X_n)$

B.2 Assignment 02

ONLY FOR SEMESTER 2 STUDENTS
ASSIGNMENT 02
Unique Nr.: 715755
Fixed closing date: 30 SEPTEMBER 2016

Question 1 **[20]**

A distribution belongs to the regular exponential family if among other regularity conditions its pdf or pmf has the form:

$$f(x|\theta) = g(x)\exp\{p(\theta)K(x) - q(\theta)\},\ \theta \in \Theta \in (\infty, \infty).$$

Furthermore, for this distribution $E[K(X)] = \dfrac{q'(\theta)}{p'(\theta)}$.

(a) Show that the other form of the above pdf is:

$$f(x|\theta) = \exp\{p(\theta)K(x) + q^*(\theta) + g^*(x)\},\ \theta \in \Theta \in (\infty, \infty).$$

(4)

(b) Write down the complete sufficient statistic for θ. **(4)**

(c) Show that the MLE of θ only depends on the sample through the complete sufficient statistic for θ. **(6)**

(d) Find the $MVUE$ of $E[K(X)] = \dfrac{q'(\theta}{p'(\theta)}$. **(6)**

minimum variance unbiased estimation

Question 2 **[36]**

Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a population whose probability mass function is

$$p(x|\theta) = \begin{cases} \theta(1-\theta)^{x-1} & \text{if } x = 1,2,3,...\ 0 < \theta < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $p(x|\theta)$ belongs to the 1-parameter exponential family. **(5)**

google **(b)** Write down the complete sufficient statistic for θ. **(4)**

(c) Find the mean of the complete sufficient statistic for θ. **(2)**

(d) Does the MLE of θ depend on the sample through the complete sufficient statistic for θ? Give reasons for your answer. **(6)**

(e) Is the MLE of $\frac{1}{\theta}$ also a $MVUE$ of $\frac{1}{\theta}$? If not find a $MVUE$ of $\frac{1}{\theta}$. **(7)**

(f) Suppose that $n = 200$, $\sum_{i=1}^{200} x_i = 400$, $\sum_{i=1}^{200} x_i^2 = 100$, and $\sum_{i=1}^{200} x_i^3 = 250$. Determine:

(i) the observed information, $I(\mathbf{x})$, of θ; and **(6)**

(ii) and the approximate 95% confidence interval for θ. **(6)**

Question 3 **[23]**

Suppose that $X_1, X_2, ..., X_n$ is a random sample from a distribution with probability density function:

$-1-\theta$

$$f(x|\theta) = \begin{cases} \theta(1+x)^{-(1+\theta)} & \text{if } x > 0,\ \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $f(x|\theta)$ belongs to the 1-parameter exponential family. **(4)**

(b) Write down the complete sufficient statistic for θ. **(3)**

(c) Determine the $CRLB$ for an unbiased estimator of $\frac{1}{\theta}$. pg 62 **(8)**

(d) Find the $MVUE$ of $\frac{1}{\theta}$. **(8)**

Question 4 **[21]**

(a) It is known that the random variable X has probability density function

$$f(x|\theta) = \begin{cases} \theta x^{(\theta-1)} & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is desired to test the null hypothesis $H_0 : \theta = 2$ against the alternative $H_1 : \theta = 1$. A random sample X_1 of size $n = 1$ is to be used.

(i) Calculate the level of significance and the power of the test which rejects H_0 if $X_1 \leq \frac{1}{2}$. **(7)**

(ii) Consider the test which rejects H_0 if $X_1 \leq c$. Find c for which the level of significance of the test is 0.1. **(5)**

(b) Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a population with probability density in part (a). Show that the best test for the hypotheses in part (a) rejects H_0 if

$$\prod_{i=1}^{n} x_i \leq c$$

where c solves the probability equation

$$\alpha = P\left(\prod_{i=1}^{n} x_i \leq c|\theta = 2\right).$$

(8)

TOTAL: [100]

d) $L(\theta|\underline{x}) = \theta(1+\sum_{i=1}^{n} x_i)^{-(1+\theta)}$

$\ln L(\theta|x) = \ln\theta + \ln \frac{1+\sum_{i=1}^{n} x_i}{(1+\theta)}$