Tutorial letter 103/3/2017

Statistical Inference III

STA3702

Semesters 1 & 2

Department of Statistics

TRIAL EXAMINATION PAPER

1. Trial Examination Papar
2. Trial Examination Paper Solutions

1 Trial Examination Paper

INSTRUCTIONS

1. Answer all questions.
2. Show intermediate steps.

Abbreviations	
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	probability density function
cdf	cumulative distribution function
$MVUE$	Minimum Variance Unbiased Estimator
MME	Method of Moments Estimator
$CRLB$	Cramer Rao Lower Bound
pmf	probability mass function
mgf	moment generating function
re	relative efficiency

Question 1 **[Total marks=30]**

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution which belongs to the regular 1-parameter exponential family, and with probability density/mass function:

$$f(x|\theta) = a(\theta)g(x)\exp\{b(\theta)R(x)\},\ x \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

where $a(\theta) > 0$ and $g(x) > 0$.

(a) Show that another form of $f(x|\theta)$ is:

$$f(x|\theta) = \exp\{b(\theta)R(x) - a^*(\theta) + g^*(x)\}.$$

(4)

(b) Suppose that $X_1, X_2, ..., X_n$ is random sample from a distribution with pmf :

$$p(x|\theta) = \begin{cases} \theta(1-\theta)^x & \text{if } 0 < \theta < 1 \text{ and } x = 0, 1, 2, ..., \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Show that $p(x|\theta)$ belongs to the regular 1-parameter exponential family. **(4)**

(ii) What is the complete sufficient statistic for θ? **Justify your answer**. **(3)**

(iii) **Refer to part (a) above**. $E[R(X)] = \dfrac{a^{*\prime}(\theta)}{b^{\prime}(\theta)}$. Use this information to find the mean of the complete sufficient statistic for θ in part (ii) above. **(3)**

(iv) Find the maximum likelihood estimator (MLE) of θ. **(8)**

(v) Find the method of moments estimator (MME) of θ. **(4)**

(vi) Show that $\bar{X}$ is the minimum variance unbiased estimator ($MVUE$) of $\dfrac{1-\theta}{\theta}$. **(4)**

Question 2 **[Total marks=30]**

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pdf :

$$f(x|\theta) = \begin{cases} \theta \exp\{-\theta x\} & \text{if } x \geq 0 \text{ and } \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Given: $E[X] = \frac{1}{\theta}$.

(a) Prove that $\sum_{i=1}^{n} X_i$ is a minimal sufficient statistic for θ. **(8)**

(b) Suppose that $n = 200$, $\sum_{i=1}^{200} x_i = 20$, $\sum_{i=1}^{200} x_i^2 = 100$, and $\sum_{i=1}^{200} x_i^3 = 250$. Determine:

(i) the method of moments estimate (MME) of θ; **(3)**

(ii) the maximum likelihood estimate (MLE) of θ; **(9)**

(iii) the observed information, $I(\mathbf{x})$, of θ; and **(4)**

(iv) and the approximate standard errors of the estimates of θ in (i) and (ii) **(6)**

Question 3 **[Total marks=18]**

Suppose that $2,\ 1,\ 2,\ 0,\ 2,\ 2,\ 2,\ 0,\ 1,\ 2$ are ten independent observations from a population whose discrete distribution has a probability mass function:

$$p(x|\theta) = \begin{cases} \frac{e^{-\theta}\theta^x}{x!} & \text{if } \theta > 0 \text{ and } x = 0, 1, 2, ..., \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimate (MME) of θ. **(4)**

(b) Find the maximum likelihood estimate (MLE) of e^{θ}. **(9)**

(c) Find the approximate standard error for the estimate in part (a). **(5)**

Question 4 **[Total marks=22]**

A random variable X has probability density function

$$f(x|\theta) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \text{ and } \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is desired to test the null hypothesis $H_0 :\ \theta = 1$ against the alternative $H_1 :\ \theta = 2$. A random sample X_1 of size $n = 1$ is to be used. **Given:** $\displaystyle\int f(x|\theta)dx = -e^{-\theta x}$.

(a) Calculate (i) the level of significance and (ii) the power of the test which rejects H_0 if $X_1 \geq \frac{1}{2}$. **(15)**

(b) Consider the test which rejects H_0 if $X_1 \geq c$. Find c for which the level of significance of the test is 0.1. **(7)**

TOTAL [100]

2 Trial Examination Solutions

Question 1 **[Total marks=30]**

(a)

$$\begin{aligned} f(x|\theta) &= a(\theta)g(x)\exp\{b(\theta)R(x)\},\ x\in\chi\subset(-\infty,\infty) \text{ and } \theta\in\Theta\subset(-\infty,\infty) \\ &= \exp\{b(\theta)R(x)+\ln a(\theta)+\ln g(x)\} \\ &= \exp\{b(\theta)R(x)-a^*(\theta)+g^*(x)\} \end{aligned}$$

where $a^* = -\ln a(\theta)$ and $g^* = \ln g(x)$. **(4)**

(b) (i)

$$\begin{aligned} p(x|\theta) &= \theta(1-\theta)^x,\ x\in\{0,1,2,...\}\subset(-\infty,\infty),\ \theta\in(0,1)\subset(-\infty,\infty) \\ &= \exp\{x\ln(1-\theta)+\ln\theta\} = \exp\{b(\theta)R(x)-a^*(\theta)+g^*(x)\} \end{aligned}$$

where $a^*(\theta) = -\ln\theta$, $b(\theta) = \ln(1-\theta)$, $R(x) = x$ and $g^*(x) = 1$. Hence from the definition of a distribution that belongs to the regular exponential family and part (a), $p(x|\theta)$ is a member of the regular exponential family. **(4)**

(ii) $\sum_{i=1}^{n} R(X_i) = \sum_{i=1}^{n} X_i$ since $f(x|\theta)$ belongs to the 1-parameter exponential family. **(3)**

(iii) $E\left[\sum_{i=1}^{n} R(X_i)\right] = E\left[\sum_{i=1}^{n} X_i\right] = nE[X] = n\dfrac{a^{*'}(\theta)}{b'(\theta)} = n\dfrac{-1/\theta}{-1/(1-\theta)} = n\dfrac{1-\theta}{\theta}$. **(3)**

(iv) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \theta^n(1-\theta)^{\sum_{i=1}^{n} x_i} \Longrightarrow$$

the log-likelihood function of θ is

$$l(\theta) = \ln L(\theta) = n\ln\theta + \ln(1-\theta)\sum_{i=1}^{n} x_i \Longrightarrow$$

$$l'(\theta) = \frac{n}{\theta} - \frac{1}{1-\theta}\sum_{i=1}^{n} x_i.$$

(4)

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} - \frac{1}{1-\hat{\theta}}\sum_{i=1}^{n} x_i \text{ and the solution is } \hat{\theta} = \frac{1}{1+\bar{x}}.$$

(4)

(v) From part (iii), $E[X] = \dfrac{1-\theta}{\theta} \Longrightarrow \theta = \dfrac{1}{1+E[X]} \Longrightarrow$ the MME of θ is $\tilde{\theta} = \dfrac{1}{1+\bar{x}}$.
(4)

(vi) $\bar{X}$ is a function of a complete sufficient statistic (see part (ii)) and $E[X] = \dfrac{1-\theta}{\theta}$ (see part (v)). **(4)**

Question 2 **[Total marks=30]**

(a) The joint density function is:

$$L(\mathbf{x}|\theta) = \theta^n \exp\left\{-\sum_{i=1}^{n} \theta x_i\right\} = m_1(\mathbf{x}) \times m_2\left(\sum_{i=1}^{n} x_i, \theta\right)$$

where $m_1(\mathbf{x}) = 1$ and $m_2\left(\sum_{i=1}^{n} x_i, \theta\right) = \theta^n \exp\{-\sum_{i=1}^{n} \theta x_i\}$. Hence, $\sum_{i=1}^{n} X_i$ is a sufficient statistic for θ by the factorization theorem. **(3)**

Let $Y_1, Y_2, ..., Y_n$ be a random sample from the same sampled distribution. Then

$$\frac{L(\mathbf{y}|\theta)}{L(\mathbf{x}|\theta)} = \exp\left\{\theta\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right)\right\}$$

which is independent of θ if $(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i) = 0$ equivalently if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. This means $\sum_{i=1}^{n} Y_i$ is a minimal sufficient statistic for θ. **(5)**

(b)

(i) $E[X] = \frac{1}{\theta}$ (given) $\Longrightarrow \theta = \frac{1}{E[X]}$ the MME of θ is $\tilde{\theta} = \dfrac{1}{\bar{x}} = \dfrac{1}{20/200} = 10.$ **(3)**

(ii) The likelihood function of θ is

$$L(\theta) = \theta^n \exp\left\{-\sum_{i=1}^{n} x_i \theta\right\}$$

and the log-likelihood function of θ is

$$l(\theta) = n\ln(\theta) - \theta \sum_{i=1}^{n} x_i.$$

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} - \sum_{i=1}^{n} x_i.$$

The solution is $\hat{\theta} = \frac{1}{\bar{x}} = 10$ **(9)**

(iii) $l''(\theta) = -\dfrac{n}{\theta^2}$ Hence $I(\mathbf{x}) = -l''(\hat{\theta}) = \dfrac{n}{\hat{\theta}^2} = \dfrac{200}{100} = 2.$ **(4)**

(iv) Since $\hat{\theta} = \tilde{\theta}$, the standard errors of both estimates are both equal to

$$se = \sqrt{1/I(\mathbf{x})} = \sqrt{1/2} = 0.70711.$$

(6)

Question 3 **[Total marks=18]**

(a) The sampled distribution is Poisson hence $E[X] = \theta$. Thus the MME of θ is

$$\tilde{\theta} = \bar{x} = 1.4.$$

(4)

(b) The likelihood function of θ is:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{10} p(x_i|\theta) \\ &= \left(\frac{e^{-\theta}\theta^0}{0!}\right)^2 \times \left(\frac{e^{-\theta}\theta^1}{1!}\right)^2 \times \left(\frac{e^{-\theta}\theta^2}{6!}\right)^6 = 2^{-6}\theta^{14}e^{-10\theta} \end{aligned}$$

The log-likelihood function is:

$$l(\theta) = \ln L(\theta) = constant + 14\ln\theta - 10\theta \Longrightarrow l'(\theta) = \frac{14}{\theta} - 10.$$

The MLE of θ is $\hat{\theta}$ which solves the equation:

$$0 = l'(\hat{\theta}) = \frac{14}{\hat{\theta}} - 10.$$

The solution is $\hat{\theta} = \frac{14}{10} = 1.4.$

By the invariance property of $mle's$, the mle of e^{θ} is $e^{\hat{\theta}} = 4.0552.$

(9)

(c) $l''(\theta) = -\frac{14}{\theta^2}$. Hence $Var[\tilde{\theta}] \approx \dfrac{1}{-l''(1.4)} = \dfrac{1}{7.14286} = 0.14$ and $se(\tilde{\theta}) \approx \sqrt{0.14} = 0.37417.$
(5)

Question 4 **[Total marks=22]**

(a)

(i) The level of significance of the test is

$$\alpha = P\left(X_1 \geq \frac{1}{2}|\theta = 1\right) = \int_{1/2}^{\infty} e^{-x}dx = \left[-e^{-x}\right]_{1/2}^{\infty} = e^{-.5} = 0.6065.$$

(6)

(ii) The probability of the type II error is

$$\beta = P\left(X_1 \leq \frac{1}{2}|\theta = 2\right) = \int_0^{1/2} 2e^{-2x} = \left[-e^{-2x}\right]_0^{1/2} = 1 - e^{-1} = 0.63212.$$

(6)

Hence the power of the test is

$$Power = 1 - \beta = 0.3679.$$

(3)

(b) $0.1 = P(X_1 \geq c|\theta = 1) = \int_c^{\infty} e^{-x} dx = \left[-e^{-x}\right]_c^{\infty} = e^{-c}$. This means

$$0.1 = e^{-c} \Longrightarrow c = -\ln(0.1) = 2.3025.$$

(7)

TOTAL [100]