Tutorial Letter 103/3/2016

Statistical Inference III

STA3702

Semesters 1 & 2

Department of Statistics

Contents

BAR CODE

1 Trial Examination Paper

INSTRUCTIONS

1. Answer all questions.
2. Show intermediate steps.

Abbreviations	
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	probability density function
cdf	cumulative distribution function
$MVUE$	Minimum Variance Unbiased Estimator
MME	Method of Moments Estimator
$CRLB$	Cramer Rao Lower Bound
pmf	probability mass function
mgf	moment generating function
re	relative efficiency

Question 1 **[Total marks= 25]**

The pdf or pmf of a random variable X belongs to the regular exponential family if it has the form:

$$f(x|\theta) = a(\theta)g(x)\exp\{\theta r(x)\},\ \ x \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \in (-\infty, \infty)$$

where $a(\theta) > 0$ and $g(x) > 0$.

(a) Show that another form of $f(x|\theta)$ is:

$$f(x|\theta)\exp\{\theta r(x) - a^*(\theta) + g^*(x)\}.$$

(**4**)

(b) Suppose that $X_1, X_2, ..., X_n$ is random sample from a distribution with pmf :

$$f(x|\theta) = \begin{cases} \theta(1-\theta)^x & \text{if } x = 0, 1, 2, ... \text{ and } \theta \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $f(x|\theta)$ belongs to the 1-parameter exponential family. **(4)**

(ii) What is the complete sufficient statistic for θ? Justify your answer. **(3)**

(iii) What is the log-likelihood function of θ? **(4)**

(iv) Find the total information for θ in the sample, $I_n(\mathbf{x})$. **(5)**

(vi) Show that the MLE of θ only depends on the sample through complete sufficient statistic.

(5)

Question 2 **[Total marks=25]**

Let $Y_1, Y_2, ..., Y_n$ be a random sample from a distribution with pdf :

$$f(y|\theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2}\exp\{-(y-\theta_1)/\theta_2\} & \text{if } y \geq \theta_1 \text{ and } \theta_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the maximum likelihood estimators of θ_1, θ_2 and θ_2^2. **(16)**

(b) Suppose that $\theta_1 = 0$. Prove that $\sum_{i=1}^{n} Y_i$ is a minimal sufficient statistic for θ_2 **(9)**

Question 3 **[Total marks=25]**

Suppose that 2, 1, 2, 0, 2, 2, 2, 0, 1, 2 are ten independent observations from a binomial experiment with $n = 4$ independent trials each with a constant probability of success θ and pmf :

$$f(x|\theta) = \begin{cases} \dfrac{4!}{(4-x)!x!}\theta^x(1-\theta)^{4-x} & \text{if } x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the method of moments estimate (MME) of θ. **(5)**

(b) Find the maximum likelihood estimate (MLE) of θ. **(7)**

(c) Find the approximate standard errors for the estimates in parts (b) and (c). **(8)**

(d) Which of the MME and the MLE is preffered and why? **(5)**

Question 4 **[Total marks=25]**

Suppose that $X_1, X_2, ..., X_n$ is a random sample from a normal distribution with pdf :

$$f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\},\ x \in (-\infty, \infty),\ \sigma^2 = E[X^2] > 0.$$

Let

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \text{ and } S^{*2} = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2.$$

(a) What is the distribution of $\frac{nS^2}{\sigma^2}$? **(3)**

(b) Show that both S^2 and S^{*2} are consistent estimators of σ^2 **(18)**

(c) Which of the two estimators of σ^2 in part (b) is better and why? **(4)**

TOTAL [100]

2 Trial Examination Paper Solutions

Question 1 **[Total marks= 25]**

(a)

$$\begin{aligned} f(x|\theta) &= a(\theta)g(x)\exp\{\theta r(x)\} \\ &= \exp\{\theta r(x) + \ln[a(\theta)] + \ln[g(x)]\} \\ &= \exp\{\theta r(x) - -\ln[a(\theta)] + \ln[g(x)]\} \\ &= \exp\left\{\theta r(x) - a^*(\theta) + g^*(x)\right\} \end{aligned}$$

where $a^*(\theta) = -\ln[a(\theta)] \in (-\infty, \infty)$ and $g^*(x) = \ln[g(x)] \in (-\infty, \infty)$. **(4)**

(b) (i)

$$\begin{aligned} f(x|\theta) &= \theta(1-\theta)^x \\ &= \exp\{x\ln(1-\theta) + \ln\theta\} \\ &= \exp\left\{\eta x + \ln(1-e^\eta)\right\} \\ &= \exp\left\{\eta r(x) - a^*(\eta) + g^*(x)\right\} \end{aligned}$$

where $r(x) = x$, $\eta = \ln(1-\theta)$, $a^*(\eta) = \ln(1-e^\eta) \in (-\infty, \infty)$ and $g^*(x) = 1$. **(4)**

(ii) $\sum_{i=1}^n r(X_i) = \sum_{i=1}^n X_i$ since $f(x|\theta)$ belongs to the 1-parameter exponential family. **(3)**

(iii) The likelihood function is $L(\theta) = \prod_{i=1}^n f(x_i|\theta) = \theta^n(1-\theta)^{\sum_{i=1}^n x_i} \Longrightarrow$ the log-likelihood function of θ is $l(\theta) = \ln L(\theta) = n\ln\theta + \sum_{i=1}^n x_i \ln(1-\theta)$. **(4)**

(iv) $l'(\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1-\theta} \Longrightarrow l''(\theta) = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^n x_i}{(1-\theta)^2}$. Hence the total information for θ in the sample is

$$I_n(\mathbf{x}) = -l''(\theta) = \frac{n}{\theta^2} + \frac{\sum_{i=1}^n x_i}{(1-\theta)^2} = \frac{n}{\theta^2} + \frac{n\bar{x}}{(1-\theta)^2}$$

(5)

(vi) The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} - \frac{\sum_{i=1}^n x_i}{1-\hat{\theta}}.$$

Clearly, the solution will depend on the sample only through the complete sufficient statistic $\sum_{i=1}^n X_i$. (The solution is $\hat{\theta} = \frac{1}{1+\bar{x}}$.) **(5)**

Question 2 **[Total marks= 25]**

(a) The likelihood function of (θ_1, θ_2) is:

$$\begin{aligned} L(\theta_1, \theta_2) &= \prod_{i=1}^n f(y_i|\theta_1, \theta_2) \\ &= \begin{cases} \theta_2^{-n}\exp\{-\sum_{i=1}^n (y_i - \theta_1)/\theta_2\} & \text{if } y_{(1)} = \min(y_i) \geq \theta_1 \text{ and } \theta_2 > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The log-likelihood function of (θ_1, θ_2) is:

$$l(\theta_1,\theta_2) = \ln L(\theta_1,\theta_2) = \begin{cases} -n\ln\theta_2 - \sum_{i=1}^n (y_i - \theta_1)/\theta_2 & \text{if } y_{(1)} \geq \theta_1 \text{ and } \theta_2 > 0, \\ -\infty & \text{otherwise.} \end{cases}$$

$l(\theta_1, \theta_2)$ increases as θ_1 varies from its 0 to $y_{(1)}$. Hence the MLE of θ_1 is $y_{(1)}$. **(7)**

$$l(y_{(1)},\theta_2) = -n\ln\theta_2 - \sum_{i=1}^n (y_i - y_{(1)})/\theta_2 \Longrightarrow l'(y_{(1)},\theta_2) = -\frac{n}{\theta_2} + \sum_{i=1}^n (y_i - y_{(1)})/\theta_2^2.$$

The MLE of θ_2 is $\hat{\theta}_2$ which solves the equation:

$$0 = l'(y_{(1)},\hat{\theta}_2) = -\frac{n}{\hat{\theta}_2} + \sum_{i=1}^n (y_i - y_{(1)})/\hat{\theta}_2^2.$$

The solution is $\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^n (y_i - y_{(1)})$. **(6)**

By the invariance property of $MLE's$, the MLE of θ_2^2 is:

$$(\hat{\theta}_2)^2 = \left(\frac{1}{n}\sum_{i=1}^n (y_i - y_{(1)})\right)^2.$$

(3)

(b) If $\theta_1 = 0$, then the likelihood function of θ_2 is:

$$L(\theta_2|\mathbf{y}) = \theta_2^{-n}\exp\{-\sum_{i=1}^n y_i/\theta_2\} = m_1(\mathbf{y}) \times m_2(\sum_{i=1}^n y_i, \theta_2)$$

where $m_1(\mathbf{y}) = 1$ and $m_2(\sum_{i=1}^n, \theta_2) = \theta_2^{-n}\exp\{-\sum_{i=1}^n y_i/\theta_2\}$. Hence, $\sum_{i=1}^n Y_i$ is a sufficient statistic for θ_2 by the factorization theorem. **(4)**

Let $X_1, X_2, ..., X_n$ be a random sample from the same sampled distribution. Then

$$\frac{L(\theta_2|\mathbf{y})}{L(\theta_2|\mathbf{x})} = \exp\left\{\frac{1}{\theta_2}\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right\}$$

which is independent of θ_2 if $(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i) = 0$ equivalently if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. This means $\sum_{i=1}^n Y_i$ is a minimal sufficient statistic for θ. **(5)**

Question 3 **[Total marks=25]**

(a) $E[X] = 4\theta \Longrightarrow \theta = \frac{1}{4}E[X] \Longrightarrow$ the MME of θ is

$$\tilde{\theta} = \frac{1}{4}\bar{x} = \frac{1}{4}(14/10) = 0.35$$

(5)

(b) The likelihood function of θ is:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{10} \frac{4!}{(4-x_i)!x!}\theta^{x_i}(1-\theta)^{4-x_i} \\ &= \prod_{i=1}^{10} \frac{4!}{(4-x_i)!x_i!}\theta^{\sum_{i=1}^{10} x_i}(1-\theta)^{40-\sum_{i=1}^{10} x_i} \\ &= \prod_{i=1}^{10} \frac{4!}{(4-x_i)!x_i!}\theta^{14}(1-\theta)^{40-14} \\ &= \prod_{i=1}^{10} \frac{4!}{(4-x_i)!x_i!}\theta^{14}(1-\theta)^{26} \end{aligned}$$

The log-likelihood function is:

$$l(\theta) = \ln L(\theta) = constant + 14\ln\theta + 26\ln(1-\theta) \Longrightarrow l'(\theta) = \frac{14}{\theta} - \frac{26}{1-\theta}.$$

The MLE of θ is $\hat{\theta}$ which solves the equation:

$$0 = l'(\hat{\theta}) = \frac{14}{\hat{\theta}} - \frac{26}{1-\hat{\theta}}.$$

The solution is $\hat{\theta} = \frac{14}{40} = 0.35$. **(7)**

(c) $l''(\theta) = -\frac{14}{\theta^2} - \frac{26}{(1-\theta)^2} \Longrightarrow$

$$Var[\tilde{\theta}] = Var[\hat{\theta}] \approx \frac{1}{-l''(0.35)} = \frac{1}{114.2857 + 61.5385} = \frac{1}{175.8242} = 0.0057.$$

Hence

$$se(\tilde{\theta}) = se(\hat{\theta}) \approx \sqrt{0.0057} = 0.0754.$$

(8)

(d) Both because there are unbiased and have equal standard errors. Equivalently, the MME and the MLE of θ are the same. **(5)**

Question 4 **[Total marks=25]**

(a) $\frac{nS^2}{\sigma^2} \sim \chi_n^2$ **(3)**

(b) Consistency of S^2: $E[\frac{nS^2}{\sigma^2}] = n$ and $Var[\frac{nS^2}{\sigma^2} = 2n$ implies the following.

$E[S^2] = n\frac{\sigma^2}{n} = \sigma^2 \Longrightarrow$ S^2 is an unbiased estimator of σ^2. Furthermore,

$Var[S^2] = 2n\frac{\sigma^4}{n^2} = 2\frac{\sigma^4}{n}$ which turns to zero as n turns to infinity. Hence S^2 is a consistent estimator of σ^2.

Consistency of S^{*2}: $E[\frac{(n-1)S^{*2}}{\sigma^2}] = n$ and $Var[\frac{(n-1)S^{*2}}{\sigma^2} = 2n$ implies the following.

$E[S^{*2}] = n\frac{\sigma^2}{n-1} \Longrightarrow$ S^{*2} is a biased estimator of σ^2 with bias $\frac{\sigma^2}{n-1}$ which turns to zero as n turns to infinity. Furthermore,

$Var[S^{*2}] = 2n\frac{\sigma^4}{(n-1)^2}$ which turns to zero as n turns to infinity. Hence S^{*2} is a consistent estimator of σ^2.

(18)

(c) $MSE(S^2) = Var[S^2] = \frac{2\sigma^4}{n}$ and

$$MSE(S^{*2} = Var[S^{*2}] + [Bias(S^{*2})]^2 = \frac{2n+1}{(n-1)^2}\sigma^4 > MSE(S^2)$$

since $\frac{2n+1}{(n-1)^2} > \frac{1}{n}$. **(4)**

TOTAL [100]