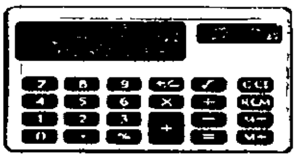
UNIVERSITY EXAMINATIONS



UNIVERSITEITSEKSAMENS

UNISA university of south africa

STA3702

May/June 2017

STATISTICAL INFERENCE III

Duration 2 Hours

100 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT

Use of a non-programmable pocket calculator is permissible

Closed book examination

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This paper consists of 4 pages

INSTRUCTIONS

1 Answer all questions

2 Show intermediate steps

Abbreviations	
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	probability density function
cdf	cumulative distribution function
$MVUE$	Minimum Variance Unbiased Estimator
MME	Method of Moments Estimator
$CRLB$	Cramer Rao Lower Bound
pmf	probability mass function
mgf	moment generating function
re	relative efficiency

[TURN OVER]

Question 1 **[30]**

A distribution belongs to the regular 1-parameter exponential family if among other regularity conditions its pdf or pmf has the form

$$f(x|\theta) = a(\theta)g(x)\exp\{b(\theta)r(x)\},\ x \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

where $g(x) > 0$, $r(x)$ is a function of x which does not depend on θ, and $a(\theta) > 0$ and $b(\theta) \neq 0$ are real valued functions of θ Furthermore, for this distribution (assuming the first and second derivatives of $a(\theta)$ and $b(\theta)$ exist)

$$E[r(X)] = -\frac{a'(\theta)}{a(\theta)b'(\theta)}$$

(a) Write down the complete sufficient statistic for θ in terms of a random sample $X_1, X_2, \ldots, X_n$ from the distribution. **Justify your answer.** **(1+1)**

(b) Write down a function of the complete sufficient statistic for θ (in terms of a random sample $X_1, X_2, \ldots, X_n$ from the distribution) that is a minimum variance unbiased estimator ($MVUE$) of $E[r(X)]$ **Justify your answer.** **(2+1)**

(c) Suppose that X is random sample (of size $n = 1$) from a distribution with pmf

$$p(x|\theta) = \begin{cases} \binom{m}{x}\theta^x(1-\theta)^{m-x} & \text{if } 0 < \theta < 1 \text{ and } x = 0, 1, 2, \ldots, m, \\ 0 & \text{elsewhere} \end{cases}$$

(i) Show that $p(x|\theta)$ belongs to the regular 1-parameter exponential family by showing that the pmf can be expressed as

$$p(x|\theta) = a(\theta)g(x)\exp\{b(\theta)r(x)\},\ x \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

(6)

(ii) What is the complete sufficient statistic for θ? **Justify your answer.** **(3)**

(iii) What is the mean of the complete sufficient statistic for θ in part (ii) above? **Justify your answer.** **(5)**

(iv) What is the maximum likelihood estimator (MLE) of θ? **Justify your answer.** **(6)**

(v) What is the method of moments estimator (MME) of θ? **Justify your answer.** **(2)**

(vi) What is the minimum variance unbiased estimator ($MVUE$) of θ? **Justify your answer.** **(3)**

Question 2 **[30]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x|\theta) = \begin{cases} \dfrac{\theta}{(1+x)^{1+\theta}} & \text{if } x > 0 \text{ and } \theta > 0, \\ 0 & \text{otherwise} \end{cases}$$

Given $E[\ln(1+X)] = \dfrac{1}{\theta}$

(a) Prove that $\prod_{i=1}^{n}(1+X_i)$ is a minimal sufficient statistic for θ **(8)**

(b) Suppose that $n = 200$, $\sum_{i=1}^{200} \ln(1+x_i) = 20$, $\sum_{i=1}^{200} 2\ln(1+x_i) = 40$, and $\sum_{i=1}^{200} 3\ln(1+x_i) = 60$ Determine

- **(i)** the method of moments estimate (MME) of θ, **(3)**
- **(ii)** the maximum likelihood estimate (MLE) of θ, **(9)**
- **(iii)** the observed information, $I(\mathbf{x})$, of θ, and **(4)**
- **(iv)** and the approximate standard errors of the estimates of θ in (i) and (ii) **(6)**

Question 3 **[18]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from the **inverse Gaussian** distribution with pdf

$$f(x|\theta_1, \theta_2) = \begin{cases} \left(\dfrac{\theta_2}{2\pi}\right)^{1/2} x^{-3/2} \exp\left\{-\dfrac{1}{2}\theta_2\left(\dfrac{\sqrt{x}}{\theta_1} - \dfrac{1}{\sqrt{x}}\right)^2\right\} & \text{if } x > 0, \theta_1 > 0 \text{ and } \theta_2 > 0, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the likelihood function and the log-likelihood function of θ_1 and θ_2 **(6)**

(b) Show that the respective maximum likelihood estimators $(MLE's)$ of θ_1 and θ_2 are

$$\hat{\theta}_1 = \bar{X} \text{ and } \hat{\theta}_2 = \frac{n}{\sum_{i=1}^{n}\left(\dfrac{\sqrt{x_i}}{\hat{\theta}_1} - \dfrac{1}{\sqrt{x_i}}\right)^2}$$

Hint: First find the MLE of θ_1 and then that of θ_2. **(12)**

Question 4 **[22]**

A random variable X has probability density function

$$f(x|\theta) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0 \text{ and } \theta > 0, \\ 0 & \text{otherwise} \end{cases}$$

It is desired to test the null hypothesis H_0 $\theta = 1$ against the alternative H_1 $\theta = 2$ A random sample X_1 of size $n = 1$ is to be used **Given·** $\int f(x|\theta)dx = -e^{-\theta x}$

(a) Calculate the level of significance and the power of the test which rejects H_0 if $X_1 \geq \frac{1}{2}$ **(6+9)**

(b) Consider the test which rejects H_0 if $X_1 \geq c$ Find c for which the level of significance of the test is 0 1

(7)

TOTAL [100]