UNIVERSITY EXAMINATIONS

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STATISTICAL INFERENCE III

Duration 2 Hours 100 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT

Use of a non-programmable pocket calculator is permissible

Closed book examination

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This paper consists of 4 pages

INSTRUCTIONS

1 Answer all questions

2 Show intermediate steps

Abbreviations	
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	probability density function
cdf	cumulative distribution function
$MVUE$	Minimum Variance Unbiased Estimator
MME	Method of Moments Estimator
$CRLB$	Cramer Rao Lower Bound
pmf	probability mass function
mgf	moment generating function
re	relative efficiency

[TURN OVER]

Question 1 **[33]**

A distribution belongs to the regular 1-parameter exponential family if among other regularity conditions its *pdf* or *pmf* has the form

$$f(y|\theta) = a(\theta)g(y)\exp\{b(\theta)r(y)\},\ y \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

where $g(y) > 0$, $r(y)$ is a function of y which does not depend on θ, and $a(\theta) > 0$ and $b(\theta) \neq 0$ are real valued functions of θ. Furthermore, for this distribution (assuming the first and second derivatives of $a(\theta)$ and $b(\theta)$ exist)

$$E[r(Y)] = -\frac{a'(\theta)}{a(\theta)b'(\theta)}$$

Suppose that $Y_1, Y_2, \ldots, Y_n$ is a random sample from this distribution.

(a) Show that $f(y|\theta)$ can also be expressed as

$$f(y|\theta) = g(y)\exp\{b(\theta)r(y) - q(\theta)\},\ y \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

(2)

(b) Prove that $\sum_{i=1}^{n} r(Y_i)$ is a minimal sufficient statistic for θ **(5)**

(c) Justify that $\sum_{i=1}^{n} r(Y_i)$ is also a complete sufficient statistic for θ **(1)**

(d) Write down a function of the complete sufficient statistic for θ that is a minimum variance unbiased estimator ($MVUE$) of $E[r(Y)]$. **Justify your answer** **(3)**

(e) Suppose that $X_1, X_2, \ldots, X_n$ is random sample from a distribution with *pdf*

$$f(x|\theta) = \begin{cases} \dfrac{\theta}{(1+x)^{1+\theta}} & \text{if } x > 0 \text{ and } \theta > 0, \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that $f(x|\theta)$ belongs to the regular 1-parameter exponential family by showing that the *pdf* can be expressed as

$$f(x|\theta) = a(\theta)g(x)\exp\{b(\theta)r(x)\},\ x > 0 \text{ and } \theta > 0$$

(5)

(ii) What is the mean of the complete sufficient statistic for θ? **Justify your answer.** **(6)**

(iv) What is the maximum likelihood estimator (MLE) of θ? **Justify your answer.** **(6)**

(v) What is the method of moments estimator (MME) of θ? **Justify your answer.** **(2)**

(vi) What is the minimum variance unbiased estimator ($MVUE$) of $\frac{1}{\theta}$? **Justify your answer**

(3)

Question 2 **[30]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x|\theta) = \begin{cases} \dfrac{1}{\theta} & \text{if } 0 < x \le \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$. The probability density function of Y_n is

$$g(y|\theta) = \begin{cases} ny^{n-1}\theta^{-n} & \text{if } 0 < y \le \theta, \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the ten independent observations,

$$23,\ 23,\ 24,\ 23,\ 23,\ 23,\ 24,\ 22,\ 24,\ 25,$$

are from a population with probability density function $f(x|\theta)$.

(a) What **are** the method of moments **estimator** and **estimate** of θ? **(4)**

(b) What **are** the maximum likelihood **estimator** and **estimate** of θ? **(8)**

(c) Find the mean and the variance, in terms of θ and n, of the maximum likelihood **estimator** of θ. Is the estimator consistent? **(10)**

(d) What is an estimate of the standard error of the maximum likelihood estimate of θ? **(3)**

(e) Find an approximate 95% confidence interval for θ. **(5)**

Question 3 **[18]**

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with pdf

$$f(x|\theta_1, \theta_2) = \begin{cases} \dfrac{\theta_1}{\theta_2^{\theta_1}} x^{\theta_1 - 1} & \text{if } 0 < x \le \theta_2 \text{ and } \theta_1 > 0, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the likelihood function and the log-likelihood function of θ_1 and θ_2. **(6)**

(b) Show that the respective maximum likelihood estimators ($MLE's$) of θ_1 and θ_2 are

$$\hat{\theta}_1 = \frac{n}{n \ln X_{(n)} - \sum_{i=1}^{n} \ln X_i} \text{ and } \hat{\theta}_2 = X_{(n)} = \max\{X_1, X_2, \ldots, X_n\},$$

respectively. **Hint. First find the MLE of θ_2 and then that of θ_1.** **(12)**

[TURN OVER]

Question 4 **[19]**

A random variable X has probability density function

$$f(x|\theta) = \frac{\exp\{x-\theta\}}{(1+\exp\{x-\theta\})^2}, \quad -\infty < x < \infty \text{ and } -\infty < \theta < \infty,$$

and **cumulative** distribution function

$$F(x|\theta) = 1 - \frac{1}{1+\exp\{x-\theta\}}, \quad -\infty < x < \infty \text{ and } -\infty < \theta < \infty$$

It is desired to test the null hypothesis H_0 $\theta = 0$ against the alternative H_1 $\theta = 1$ using one random observation Y

(a) Calculate the level of significance and the power of the test which rejects H_0 if $X > 1$ **(12)**

(b) Consider the test which rejects H_0 if $X_1 \geq c$ Find c for which the level of significance of the test is 0 1 **(7)**

TOTAL [100]

First Examiner	Prof P Ndlovu
Second Examiner	Prof JO Olaomi
External Examiner	Dr EL Raath

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