Tutorial Letter 201/2/2016

Statistical Inference III

STA3702

Semester 2

Department of Statistics

SOLUTIONS TO ASSIGNMENT 01

BAR CODE

Question 1 **[20]**

(a) $E[X] = \theta \Longrightarrow$ the method of moments estimator (MME) of θ is

$$\tilde{\theta} = \bar{X}.$$

(4)

(b) The likelihood function of θ is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} p(x_i|\theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

and the log likelihood function of θ is

$$l(\theta) = \ln L(\theta|\mathbf{x}) = -n\theta + \sum_{i=1}^{n} x_i \ln(\theta) - \ln\left(\prod_{i=1}^{n} x_i!\right).$$

Now

$$l'(\theta) = -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = -n + \frac{n\bar{x}}{\theta}.$$

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = -n + \frac{n\bar{X}}{\hat{\theta}}.$$

The solution is $\hat{\theta} = \bar{X}$. **(8)**

(c) By the invariance property of maximum likelihood estimators the $MLE's$ of the functions are:

(i) $e^{\hat{\theta}}$; and **(3)**

(ii) $1 - e^{-\hat{\theta}}$ since $P(X \geq 1) = 1 - P(X = 0) = 1 - \bar{e}^{\theta}$. **(5)**

Question 2 **[20]**

(a) The likelihood function of θ is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \theta^n \frac{\prod_{i=1}^{n} x_i^{\theta}}{\prod_{i=1}^{n} x_i}$$

and the log likelihood function of θ is

$$l(\theta) = \ln L(\theta|\mathbf{x}) = n\ln\theta + \theta\sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln x_i.$$ **(4)**

(b)

$$\begin{aligned} L(\theta|\mathbf{x}) &= \theta^n \frac{\prod_{i=1}^n x_i^\theta}{\prod_{i=1}^n x_i} \\ &= \frac{1}{\prod_{i=1}^n x_i}\theta^n \prod_{i=1}^n x_i^\theta \\ &= m_1(\mathbf{x}) \times m_2\left(\theta, \prod_{i=1}^n x_i\right) \end{aligned}$$

where $m_1(\mathbf{x}) = \frac{1}{\prod_{i=1}^n x_i}$ and $m_2\left(\theta, \prod_{i=1}^n x_i\right) = \theta^n \prod_{i=1}^n x_i^\theta$. Hence $\prod_{i=1}^n X_i$ is a sufficient statistic for θ by the factorisation theorem.

Now let $Y_1, Y_2, Y_3, ..., Y_n$ be another random sample from the same distribution. Then

$$\frac{L(\theta|\mathbf{x})}{L(\theta|\mathbf{y})} = \left(\prod_{i=1}^n \frac{y_i}{x_i}\right)\left(\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i}\right)^\theta$$

is independent of θ if $\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i} = 1$ equivalently if $\prod_{i=1}^n y_i = \prod_{i=1}^n x_i$. This means that $\prod_{i=1}^n X_i$ is a minimal sufficient statistic for θ. **(7)**

(c) Note that

$$l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i.$$

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} + \sum_{i=1}^n \ln(X_i).$$

The solution is

$$\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln(X_i)}.$$

(4)

(d) Note that $l''(\theta) = -\frac{n}{\theta^2}$. Hence the observed information of θ is

$$I(\mathbf{X}) = -l''(\hat{\theta}) = \frac{n}{\hat{\theta}^2} = \frac{1}{n}\left(\sum_{i=1}^n \ln(X_i)\right)^2.$$ **(5)**

Question 3 **[20]**

(a)

$$\begin{aligned} E(X) &= \sum_{x=0}^3 xP(X=x) \\ &= 0 \times \frac{2}{3}\theta + 1 \times \frac{1}{3}\theta + 2 \times \frac{2}{3}(1-\theta) + 3 \times \frac{1}{3}(1-\theta) \\ &= \frac{7}{3} - 2\theta \end{aligned}$$

This means $\theta = \frac{1}{2}\left(\frac{7}{3} - E(X)\right)$ and that the method of moments estimate (MME) of θ is

$$\tilde{\theta} = \frac{1}{2}\left(\frac{7}{3} - \bar{x}\right) = \frac{1}{2}\left(\frac{7}{3} - 1.4\right) = \frac{7}{15} = 0.4667$$

since from the data $\bar{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = \frac{14}{10} = 1.4.$ **(4)**

(b) The likelihood function of θ is

$$\begin{aligned} L(\theta|\mathbf{x}) &= \left(\frac{2}{3}\theta\right)^4 \times \left(\frac{1}{3}\theta\right) \times \left[\frac{2}{3}(1-\theta)\right]^2 \times \left[\frac{1}{3}(1-\theta)\right]^3 \\ &= \frac{64}{59059}\theta^5(1-\theta)^5 \end{aligned}$$

The log likelihood function of θ is

$$l(\theta) = \ln\left(\frac{64}{59059}\right) + 5\ln\theta + 5\ln(1-\theta)$$

and

$$l'(\theta) = \frac{5}{\theta} - \frac{5}{1-\theta}.$$

The maximum likelihood estimate of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{5}{\hat{\theta}} - \frac{5}{1-\hat{\theta}}.$$

The solution is $\hat{\theta} = \frac{5}{10} = 0.5$ **(6)**

(c) Note that $l''(\theta) = -\frac{5}{\theta} - \frac{5}{(1-\theta)^2}$. Hence for the MME, $l''(\tilde{\theta}) = -22.9559 - 17.5803 = -40.5362$ and

$$Var(\tilde{\theta}) \approx -\frac{1}{l''(\tilde{\theta})} = 0.0247 \text{ and } se(\tilde{\theta}) = \sqrt{Var(\tilde{\theta})} \approx \sqrt{0.0247} = 0.1571.$$

Similarly for the MLE,

$$Var(\hat{\theta}) \approx -\frac{1}{l''(\hat{\theta})} = 0.025 \text{ and } se(\hat{\theta}) = \sqrt{Var(\hat{\theta})} \approx \sqrt{0.025} = 0.1581.$$

(7)

(d) Base on only the standard errors of the estimates, the MME is preferred because it has a smaller standard error. **(3)**

Question 4 **[20]**

(a) The likelihood function of θ_1 and θ_2 is

$$L(\theta_1,\theta_2|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta_1,\theta_2) = \begin{cases} \theta_2^{-n}\exp\left(-\sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2}\right) & \text{if } 0<\theta_2<\infty,\ 0<\theta_1\le \text{all the } x_i's, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$L(\theta_1,\theta_2|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta_1,\theta_2) = \begin{cases} \theta_2^{-n}\exp\left(-\sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2}\right) & \text{if } 0<\theta_2<\infty,\ 0<\theta_1\le x_{(1)}, \\ 0 & \text{otherwise} \end{cases}$$

where $x_{(1)} = \min\{x_1, x_2, x_3, ..., x_n\}$. **(3)**

(b) Let $I(\theta_1,\theta_2|\mathbf{x}) = \begin{cases} 1 & \text{if } 0<\theta_2<\infty,\ 0<\theta_1\le x_{(1)}, \\ 0 & \text{otherwise.} \end{cases}$

Then

$$\begin{aligned} L(\theta_1,\theta_2|\mathbf{x}) &= \prod_{i=1}^{n} f(x_i|\theta_1,\theta_2) = \theta_2^{-n}\exp\left(-\sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2}\right)I(\theta_1,\theta_2|\mathbf{x}) \\ &= m_1(\mathbf{x}) \times m_2\left(\theta_1,\theta_2,x_{(1)},\sum_{i=1}^{n}x_i\right) \end{aligned}$$

where $m_1(\mathbf{x}) = 1$ and $m_2\left(\theta_1,\theta_2,x_{(1)},\sum_{i=1}^{n}x_i\right) = \theta_2^{-n}\exp\left(-\sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2}\right)I(\theta_1,\theta_2|\mathbf{x})$. Hence $X_{(1)}$ and $\sum_{i=1}^{n}X_i$ are the joint sufficient statistics for θ_1 and θ_2 by the factorisation theorem. **(5)**

(c) Note that the log likelihood function of θ_1 and θ_2 is

$$\begin{aligned} l(\theta_1,\theta_2) &= \ln L(\theta_1,\theta_2|\mathbf{x}) \\ &= \begin{cases} -n\ln\theta_2 - \sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2} & \text{if } 0<\theta_2<\infty,\ 0<\theta_1\le x_{(1)}, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

which is a strictly increasing function of θ_1 as $\theta_2 > 0$. This is shown formally by

$$\frac{\partial l(\theta_1,\theta_2)}{\partial\theta_1} = \frac{n}{\theta_2} > 0 \text{ since } \theta_2 > 0.$$

The conclusion is that for any $\theta_2 > 0$, $l(\theta_1,\theta_2)$ attains a maximum at the maximum value of θ_1 which is $x_{(1)}$. Hence, the maximum likelihood estimator (MLE) of θ_1 is $\hat{\theta}_1 = \min\{X_1, X_2, ..., X_n\} = X_{(1)}$. **(6)**

(d) Consider $l(\hat{\theta}_1,\theta_2)$ whose derivative with respect to θ_2 is

$$l'(\hat{\theta}_1,\theta_2) = -\frac{n}{\theta_2} + \frac{1}{\theta_2^2}\sum_{i=1}^{n}(x_i-\hat{\theta}_1).$$

The MLE of θ_2 is $\hat{\theta}_2$ which solves the equation

$$0 = l'(\hat{\theta}_1,\hat{\theta}_2) = \frac{n}{\hat{\theta}_2} + \frac{1}{\hat{\theta}_2^2}\sum_{i=1}^{n}(X_i-\hat{\theta}_1).$$

The solution is

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\theta}_1) = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}).$$

(6)

TOTAL: [80]