Tutorial Letter 201/1/2014

Statistical Inference III

STA3702

Semester 1

Department of Statistics

SOLUTIONS TO ASSIGNMENT 01

BAR CODE

QUESTION 1 **[Total marks= 20]**

(a) $\theta = E(T) = E(aT_1 + bT_2) = aE(T_1) + bE(T_2) = a\theta + b\theta \Longrightarrow a + b = 1.$ **(4)**

(b) If $Var(T_1) = 2Var(T_2)$, then

$$\begin{aligned} Var(T) &= a^2Var(T_1) + b^2Var(T_2) \\ &= 2a^2Var(T_2) + b^2Var(T_2) = (2a^2 + b^2)Var(T_2) \end{aligned}$$

(5)

(c) Minimize $Var(T)$ by choice of a and b subject to the constraint $a + b = 1$.
Now, $Var(T) = \left(2a^2 + 1 - 2a + a^2\right) Var(T_2) = (3a^2 - 2a + 1)Var(T_2)$ upon using the constraint $b = 1 - a$.
Furthermore, $0 = \dfrac{\partial Var(T)}{\partial a} = (6a - 2)Var(T_2) \Longrightarrow a = \frac{2}{6} = \frac{1}{3}$ and $b = 1 - \frac{1}{3} = \frac{2}{3}$ are a and b which minimizes $Var(T)$. **(7)**

(d) $Var(T) = \left(\frac{3}{9} - \frac{2}{3} + 1\right) Var(T_2) = \frac{2}{3}Var(T_2) \Longrightarrow re(T_2, T) = \dfrac{Var(T)}{Var(T_2)} = \frac{2}{3} \Longrightarrow T$ is better than T_2 since it has the smaller variance. **(4)**

Question 2 **[Total marks= 20]**

(a) $MME: \; E(X) = \mu \Longrightarrow MME$ of μ is $\bar{x}$. **(4)**

$MLE: L(\mu) = \prod_{i=1}^{n} f(x_i|\mu) = (2\pi)^{-n/2} e^{-\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$ and

$l(\mu) = \ln L(\mu) = -\dfrac{n}{2}\ln(2\pi) - \dfrac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2 \Longrightarrow l'(\mu) = \sum_{i=1}^{n}(x_i - \mu) \Longrightarrow$ the MLE of μ is $\hat{\mu}$ which solves $\sum_{i=1}^{n}(x_i - \hat{\mu}) = 0$ and the solution is $\hat{\mu} = \bar{x}$ which is also the MME of μ. **(12)**

(b) The MLE of $\mu^3$is $\bar{x}^3$ by the invariance property of $MLE's$. **(4)**

Question 3 **[Total marks=13]**

(a) $l''(\mu) = -n \Longrightarrow I_n(\mu) = -E[l''(\mu)] = -E(-n) = n.$ **(4)**

(b) $Var(\hat{\mu}) = Var(\bar{X}) = \frac{1}{n}Var(X) = \frac{1}{n}.$ **(2)**

(c) $g(\mu) = E(\hat{\mu}) = E(\bar{X}) = \mu \Longrightarrow g'(\mu) = 1 \Longrightarrow CRLB = \dfrac{[g'^2}{I_n(\mu)} = \frac{1}{n} = Var(\hat{\mu}).$ **(4)**

(d) Since $E(\hat{\mu}) = \mu$ and $Var(\hat{\mu}) = CRLB$, this means $\hat{\mu}$ is a $MVUE$ of μ. **(3)**

Question 4 **[Total marks=25]**

(a) $\bar{X} = \frac{4}{6} = \frac{2}{3}$. **(2)**

(b) $E(X) = n\theta = 2\theta \Longrightarrow \theta = \frac{1}{2}E(X) \Longrightarrow MME$ of θ is $\tilde{\theta} = \frac{1}{2}\bar{X} = \frac{4}{12} = \frac{1}{3}$. **(4)**

(c) $L(\theta) = \prod_{i=1}^{6} p(x_i|\theta) = \theta^4(1-\theta)^8 \prod_{i=1}^{6} \frac{2}{2 \times (2 - x_i)} \Longrightarrow$

$l(\theta) = \ln L(\theta) = \sum_{i=1}^{6} \ln \left\{ \frac{2}{2 \times (2 - x_i)} \right\} + 4\ln(\theta) + 8\ln(1-\theta) \Longrightarrow l'(\theta) = \frac{4}{\theta} - \frac{8}{1-\theta} \Longrightarrow$ the MLE of θ is $\hat{\theta}$ which solves $\frac{4}{\hat{\theta}} - \frac{8}{1-\hat{\theta}} = 0$ and the solution is $\hat{\theta} = \frac{1}{3}$. **(8)**

(d) MME and MLE: $I_6(\tilde{\theta}) = I_6(\hat{\theta}) = -\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\theta=1/3} = \left[\frac{4}{\theta^2} + \frac{8}{(1-\theta)^2}\right]_{\theta=1/3} = 54$ since $\tilde{\theta} = \hat{\theta} = \frac{1}{3}$. Hence $Var(\tilde{\theta}) = Var(\hat{\theta}) \approx \frac{1}{I_6(\tilde{\theta})} = \frac{1}{54}$ and $se(\tilde{\theta}) = se(\hat{\theta}) = \sqrt{\frac{1}{54}} = 0.1361$. **(8)**

(e) Neither since both are unbiased and have equal standard errors. **(3)**

Question 5 **[Total marks=22]**

Since $E(X) = \theta$, then the MME of θ is $\tilde{\theta} = \bar{X}$. **(4)**

$Var(\tilde{\theta}) = Var(\bar{X}) = \frac{1}{n}Var(X) = \frac{\theta}{n}$. **(2)**

$\ln f(x|\theta) = -\ln(x!) - \theta + x\ln(\theta)$ and $\frac{\partial^2 \ln f(x|\theta)}{\partial\theta^2} = -\frac{x}{\theta^2}$. Thus, the Fisher information for θ is

$$\begin{aligned} I_n(\theta) &= -nE\left\{\frac{\partial^2 \ln f(x|\theta)}{\partial\theta^2}\right\} \\ &= nE\left\{\frac{X}{\theta^2}\right\} = n\frac{\theta}{\theta^2} = \frac{n}{\theta}. \end{aligned}$$

Now, for an unbiased estimator of θ, $g(\theta) = \theta$ and $g'(\theta)$=1. Hence,

$$CRLB = \frac{[g'^2}{I_n(\theta)} = \frac{1}{I_n(\theta)} = \frac{\theta}{n}.$$

(8)

$E(\tilde{\theta}) = E(\bar{X}) = \theta$ and $Var(\tilde{\theta}) = CRLB$ implies that $\tilde{\theta}$ is a $MVUE$ of θ. **(3)**

$Var(\bar{X}) = \frac{\theta}{n} = \frac{1}{n}E(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2 = E(\bar{X}^2) - \theta^2 \Longrightarrow$

$E(\bar{X}^2) - \frac{1}{n}E(\bar{X}) = E(\bar{X}^2 - \frac{1}{n}\bar{X}) = \theta^2 \Longrightarrow$

$\bar{X}^2 - \frac{1}{n}\bar{X}$ is a $MVUE$ of θ^2 since the estimator is unbiased and is a function of a $MVUE$ of θ.

(5)

TOTAL [100]