Tutorial letter 201/2/2018

Statistical Inference II

STA2602

Semester 2

Department of Statistics

Solutions to Assignment 1

QUESTION 1 **[25]**

(a) **(6)**

$$\begin{aligned} MSE(\hat{\theta}) &= E[(\hat{\theta}-\theta)^2] \\ &= E[[(\hat{\theta}-E(\hat{\theta}))+(E(\hat{\theta})-\theta)]^2] \\ &= E[(\hat{\theta}-E(\hat{\theta}))^2+2(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta)+(E(\hat{\theta})-\theta))^2] \\ &= E[(\hat{\theta}-E(\hat{\theta}))^2]+2(E(\hat{\theta})-E(\hat{\theta}))(E(\hat{\theta})-\theta)+(E(\hat{\theta})-\theta))^2 \\ &= E[(\hat{\theta}-E(\hat{\theta}))^2]+0+(E(\hat{\theta})-\theta))^2 \\ &= V(\hat{\theta})+[B(\hat{\theta})]^2 \end{aligned}$$

(b) **(19)**

(i) $E(\hat{\theta}_3)=\alpha E(\hat{\theta}_1)+(1-\alpha)E(\hat{\theta}_2)=\alpha\theta+(1-\alpha)\theta=\theta.$ (3)

(ii) $V(\hat{\theta}_3)=\alpha^2 V(\hat{\theta}_1)+(1-\alpha)^2 V(\hat{\theta}_2)=[\alpha^2+(1-\alpha)^2]\sigma^2=(2\alpha^2-2\alpha+1)\sigma^2.$ (4)

(iii) We find the value of α which solves the equation

$$0=\frac{\partial V(\hat{\theta}_3)}{\partial\alpha}=(4\alpha-2)\sigma^2.$$

The solution is $\alpha=\frac{1}{2}$ minimizes $V(\hat{\theta}_3)$. (5)

(iv) $V(\hat{\theta}_3)=\alpha^2 V(\hat{\theta}_1)+(1-\alpha)^2 V(\hat{\theta}_2)=\alpha^2\sigma^2+3(1-\alpha)^2\sigma^2=[4\alpha^2-6\alpha+3]\sigma^2.$ (3)
Hence the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_3$ is

$$eff(\hat{\theta}_1,\hat{\theta}_3)=\frac{V(\hat{\theta}_1)}{V(\hat{\theta}_3)}=\frac{1}{4\alpha^2-6\alpha+3}.$$

(2)

(v) $eff(\hat{\theta}_1,\hat{\theta}_3)\Big|_{\alpha=1/2}=\frac{1}{4\alpha^2-6\alpha+3}\Big|_{\alpha=1/2}=1.$ (2)

QUESTION 2 **[8]**

$E\left(\frac{n\hat{\theta}_n}{\theta}\right)=n \Longrightarrow \frac{n}{\theta}E(\hat{\theta})=n \Longrightarrow E(\hat{\theta}_n)=\theta$. Hence $\lim_{n\to} E(\hat{\theta}_n)=\theta.$ (3)

$V\left(\frac{n\hat{\theta}_n}{\theta}\right)=2n \Longrightarrow \frac{n^2}{\theta^2}V(\hat{\theta}_n)=2n \Longrightarrow V(\hat{\theta}_n)=\frac{2}{n}\theta^2$. Hence $\lim_{n\to\infty} V(\hat{\theta}_n)=0.$ (5)

That $\lim_{n\to} E(\hat{\theta}_n)=\theta$ and $\lim_{n\to\infty} V(\hat{\theta}_n)=0$ implies that $\hat{\theta}_n$ is a consistent estimator of θ.

QUESTION 3 **[7]**

(a) **(3)**

$$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} p(x_i, \beta) = \prod_{i=1}^{n} \beta(1-\beta)^{x-1} \\ &= \beta(1-\beta)^{x_1-1} \times \beta(1-\beta)^{x_2-1} \times ... \times \beta(1-\beta)^{x_n-1} \\ &= \beta^n(1-\beta)^{\sum_{i=1}^{n} x_i - n} \end{aligned}$$

(b) **(4)**

From part (a) we have $L(\beta) = \beta^n(1-\beta)^{\sum_{i=1}^{n} x_i - n} = g\left(\sum_{i=1}^{n} x_i, \beta\right) \times h(x_1, x_2, ..., x_n)$ where $g\left(\sum_{i=1}^{n} x_i, \beta\right) = \beta^n(1-\beta)^{\sum_{i=1}^{n} x_i - n}$ **(depends on the sample only through** $\sum_{i=1}^{n} x_i$) and $h(x_1, x_2, ..., x_n = 1$ (**independent of** β). This means, by the factorization criterion, $\sum_{i=1}^{n} X_i$ is a sufficient statistic for β.

QUESTION 4 **[10]**

(a) **(3)**

$E(X_i) = \dfrac{1}{\beta}$ means $\beta = \dfrac{1}{E(X_i)}$. The method-of-moments estimator is $\tilde{\beta}$ which is obtained by replacing $E(X_i)$ in $\beta = \dfrac{1}{E(X_i)}$ with a corresponding sample moment which in this case is $\bar{X}$. Hence $\tilde{\beta} = \dfrac{1}{\bar{X}}$ is the method-of-moments estimator of β.

(b) **(7)**

From **QUESTION 3** the log-likelihood function is

$$l(\beta) = \ln L(\beta) = n \ln \beta = \ln(1-\beta)\left(\sum_{i=1}^{n} x_i - n\right). \qquad (1)$$

The maximum likelihood estimate of β is $\hat{\beta}$ which solves the equation

$$0 = l'(\hat{\beta}) = \left.\frac{\partial l(\beta)}{\partial \beta}\right|_{\beta=\hat{\beta}} = \frac{n}{\hat{\beta}} - \frac{1}{1-\hat{\beta}}\left(\sum_{i=1}^{n} x_i - n\right) = \frac{n}{\hat{\beta}} - \frac{n}{1-\hat{\beta}}(\bar{x}-1). \qquad (2)$$

The solution is $\hat{\beta} = \dfrac{1}{\bar{x}}$. Hence the maximum likelihood estimator of β is $\hat{\beta} = \dfrac{1}{\bar{X}}$. (4)

QUESTION 5 **[20]**

(a) **(6)**

$$\begin{aligned} E(\bar{Y}) &= E(\beta\bar{x} + \bar{\epsilon}) \\ &= E(\beta\bar{x}) + E(\bar{\epsilon}) \\ &= \beta\bar{x} + 0 \Longrightarrow E\left(\frac{\bar{Y}}{\bar{x}}\right) = \beta. \end{aligned}$$

Hence $\frac{\bar{Y}}{\bar{x}}$ is an unbiased estimator of β. (4)

$V\left(\frac{\bar{Y}}{\bar{x}}\right) = \frac{1}{\bar{x}^2}V(\bar{Y}) = \frac{\sigma^2}{n\bar{x}^2}$. (1)

(b) **(7)**

The likelihood of the sample is

$$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} f(y_i, \beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2} \\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2} \end{aligned}$$

(2)

and the log-likelihood is

$$l(\beta) = \ln L(\beta) = -n/2\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2.$$

(1)

The maximum likelihood estimate of β is $\hat{\beta}$ which solves the equation

$$0 = l'(\hat{\beta}) = \left.\frac{\partial l(\beta_0)}{\partial\beta}\right|_{\beta=\hat{\beta}} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i(y_i - \hat{\beta}).$$

(2)

The solution is $\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$. Hence the maximum likelihood estimator is $\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2} \neq \frac{\bar{Y}}{\bar{x}}$. (2)

(c) **(7)**

The least squares estimator of β is $\hat{\beta}$ which solves the equation

$$0 = \left.\frac{\partial SSE}{\partial \beta}\right|_{\beta=\hat{\beta}} = -2\sum_{i=1}^{n} x_i(y_i - \hat{\beta}).$$

(4)

The solution is $\hat{\beta} = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$. Hence the least squares estimator of β is is the same as the maximum likelihood estimator of β. (3)

QUESTION 6 **[30]**

(a) **(6)**

$$S_{xx} = \sum_{i=1}^{10} x_i^2 - \frac{1}{10}\left(\sum_{i=1}^{10} x_i\right)^2 = 7688 - \frac{(268)^2}{10} = 485.6,$$ (2)

$$S_{yy} = \sum_{i=1}^{10} y_i^2 - \frac{1}{10}\left(\sum_{i=1}^{10} y_i\right)^2 = 83.8733 - \frac{(27.73)^2}{10} = 6.97801,$$ **(2)**

and $S_{xy} = \sum_{i=1}^{10} x_i y_i - \frac{1}{10}\left(\sum_{i=1}^{10} x_i\right)\left(\sum_{i=1}^{10} y_i\right) = 800.62 - \frac{(268)(27.73}{10} = 57.456.$ **(2)**

(b) **(5)**

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{57.456}{\sqrt{(485.6)(6.97801)}} = 0.99.$$ (3)

The correlation coefficient is positive and large (**close to 1**). This means there is a strong positive linear relationship between the yearly number of sales people (x) and the yearly sales revenue ($y \times 1000$). (2)

(c) **(6)**

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{57.456}{485.6} = 0.1183,$$ (3)

and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = \frac{27.73}{10} - (0.118319605)\frac{268}{10} = -0.3984.$ (3)

(**The fitted model is** $\hat{y} = -0.398 + 0.1183x$.)

(d) **(3)**

$$SSE = S_{yy} - \hat{\beta}_1 S_{xy} = 6.97801 - (0.118319605)(57.456) = 0.1798.$$ (2)

Hence the estimate of σ^2 is $s^2 = \frac{SSE}{5-2} = \frac{0.179838775}{8} = 0.02248.$ (1)

(e) **(2)**

The estimate of $V(\hat{\beta}_1)$ is $s^2_{\hat{\beta}_1} = \frac{s^2}{S_{xx}} = \frac{0.022479847}{485.6} = 0.000046.$

(f) **(8)**

The test statistic is $t = \dfrac{\hat{\beta}_1}{s_{\hat{\beta}_1}} = \dfrac{0.1183}{0.0068} = 17.3871.$ (3)

The decision rule is "Reject H_0 if $|t| > t_{0.025} = 2.306$ (**obtained from the** $t - tables$ **using 8 degrees of freedom**)" or "Reject H_0 if either $t < -t_{0.025} = -2.306$ or $t > t_{0.025} = 2.306$." (3)

Since $t = 17.3871 > t_{0.025} = 2.306$ we reject H_0 and conclude, at the 0.05 level of significance, that there is a strong linear between the yearly number of sales people (x) and the yearly sales revenue ($y \times 1000$). (2)

TOTAL: [100]