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PREFACE

Introduction

Welcome to MAT1510. We hope you will enjoy studying this module.

MAT1510 is an introductory precalculus module, and serves to consolidate what you have learnt before, so that you may have a strong foundation on which to build. As the name implies, the content consists of the concepts you need in order to study calculus in future.

A wide variety of people will study this module, and we hope that it will meet their many different needs. Because students taking this module have different mathematical backgrounds, it is difficult to make assumptions about what you know. We realise that not everyone will have the required background knowledge.

Some of you may have recently studied the Unisa Mathematics Access Module, MAT0511. There is a certain amount of unavoidable overlap between this module, MAT1510, and MAT0511; the revision of the concepts involved will be helpful.

There is also some overlap between MAT1510 and school mathematics. We regard as essential background knowledge the work that is covered by both the Mathematics Access Module at Unisa (MAT0511) and the usual syllabus for secondary school mathematics. In other words we do not spend time teaching any of the topics that we expect you to have studied before.

However, depending on the school you attended, and how long ago you were in school, some of you may need to do a bit of catching up. If concepts are unfamiliar you may refer to those sections of the prescribed book identified as "assumed knowledge" in the study guide, or the MAT0511 books, or other sources.

To help you identify whether or not you know the work you are expected to know, we have provided at the beginning of each section in the study guide, a **diagnostic test** that relates to the "assumed knowledge" topics. The extent to which you can solve the problems should give you an idea of whether you know the work as well as you think you do. If you cannot easily answer the questions in the diagnostic tests, you will need to revise that section of work in the prescribed book.

If you are struggling to make sense of the work right from the start, it may be that you first need to do some additional preparation. We suggest that you contact us to discuss the problem before you waste too much time.

Prescribed Book, Study Guide and Structure of the Module

The study material for this module consists of a textbook, a study guide and several tutorial letters.

If you have read Tutorial Letter 101 you will realise that this module is based on the prescribed book, called ***Precalculus***, by James Stewart, Lothar Redlin and Saleem Watson. We are using the fifth edition, printed in 2006. The subtitle is "Mathematics for Calculus". We will study most of the first seven chapters of the textbook, which we call SRW for short. You cannot study MAT1510 without this book, so we advise you to purchase it as soon as possible.

The study guide supports the textbook. The different chapters are numbered and named to correspond to the chapter names and numbers used in SRW. So, for example, in the study guide, Chapter 1 is called FUNDAMENTALS, which is also the title of the first chapter of SRW, and Chapter 5 is called TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS to correspond to Chapter 5 of SRW, which has the same title.

Each chapter is divided into sections. At the beginning of each section you will see a block in which the following aspects are summarised.

- **Assumed knowledge**: the subsections in that section which we assume you already know (if the work is all new, no assumed knowledge is indicated)
- **Section outcomes**: the objectives for that section, i.e. the new knowledge you need to learn or new skills you need to acquire
- **Exercises**: the exercises in SRW, or additional exercises, that should help you assess whether you have attained the stated outcomes

As you begin studying the different topics, you may find that certain concepts seem familiar. It is easy to confuse **familiarity** with **understanding**, so we encourage you to work through all the sections carefully to make sure you really do understand.

It is impossible to **know that you know** mathematics, unless you have an opportunity to **show that you know**. The textbook includes **Examples** which suggest ways of tackling certain problems; as we have mentioned above there are **Exercises** that give you the opportunity to "show that you know", i.e. to **do mathematics**. These questions range from performing simple calculations to solving more complicated problems. You will need a **workbook**, such as an A4 exercise book, in which to answer the questions. Try to work neatly and systematically, so that you can refer to your solutions again at a later stage, and understand what you did. At the end of each chapter in SRW you will find a **Chapter Review**, which contains a very helpful **Concept Check** as well as additional exercises followed by a **Test**.

The **Answers** to the odd–numbered exercise questions are given at the back of the textbook. The answers to the additional exercises we include, are given at the back of the study guide. You will check your own answers to the exercise questions (we refer to this process as self–assessment). Apart from self–assessment you will also submit assignments during the course of the year, so that we can see what progress you are making. We discuss the assignment system in the first tutorial letter (Tutorial Letter 101). The main difference between the assignment questions and the exercise questions in SRW or the study guide is that the questions based on each section assess very specifically the concepts taught in that section, whereas the assignments test whether you have understood the concepts well enough to be able to **synthesise** and **apply** a variety of concepts in one problem.

When you work out a problem in one of the exercises, and check your answer, you may find that your answer is wrong. Mathematics requires plenty of concentration, and there are often many facts and processes that are applied in solving one problem, so it is unrealistic to expect that you will always be right first time. Try again if your first answer is wrong. Make sure you take into account the error symbol (see the section "To the Student" in SRW) that appears in the margin of SRW to alert you to common errors, so that you can avoid making them. It may also be that your answer is quite correct, but just presented in a different form to the answer in SRW or in the study guide. Before assuming your answer is wrong, see whether you can rewrite it in the form in which it has been given.

Remember that we use the exercises in SRW for two purposes. We use them as a diagnostic tool, to help you find out whether you need to catch up on some missing sections. These are the exercises that relate to the "assumed knowledge". We also use them to assess the understanding of new knowledge and skills. These are the exercises we note in the block at the beginning of each section.

We begin at Chapter 1 in SRW, then continue to the end of Chapter 7, leaving out Chapter 3, as well as Sections 1.9, 5.5 and 7.4. You will study Chapter 3, and most of the rest of SRW, in other mathematics modules. When you reach Chapters 5 and 6 you have the choice of studying them in the given order, or of studying Chapter 6 before Chapter 5. The reason for this is that we can approach the study of trigonometry from either an angle perspective (where we measure angles in degrees) or from the perspective of real numbers (where we measure angles in radians). If you have studied trigonometry before, at school, you will have used the angle approach, and you may then find it easier to start with Chapter 6. The two chapters are independent of each other, so it really does not matter where you begin.

Outcomes

Broad outcomes

The end–result of your studies should not only be your success in an exam. We hope that studying this module will contribute to the process of developing students who

- are effective communicators
- can identify and solve problems
- can operate independently and responsibly
- can contribute effectively to team work
- can collect, organise, analyse and critically evaluate information
- can use science and technology in an environmentally sensitive way.

A part of becoming proficient in mathematics involves good communication. **Effective communication** depends on the clear and unambiguous use of the languages that are being used. For example, we may use some combination of a spoken or written language together with aspects of technical language (think about terms such as software, gear ratio, velocity, acceleration) and "number language" (think about VAT, percentage, interest rate). So you need to understand and be proficient in mathematical language, as well as in English, which is the international language in which most sciences are communicated. Understanding English, in this context, means understanding all the familiar, everyday words as well as the meaning and purpose of the so–called "academic English" words, such as "whereas", "thus", "hence", "however", "since". We hope you will be able to recognise where the language gets in the way of your grasp of mathematical concepts, so that you can gradually learn to extract the underlying mathematics from what may seem to be complicated English sentences.

Identifying and solving problems in different contexts depends on

- careful consideration of the available information
- recognition of redundant or irrelevant information
- formulation of some statement (whether mathematical or not) that proposes a solution to the problem
- testing whether the proposed "solution" is valid.

In general, problem solving requires that we carry out several steps. SRW has some very interesting sections on problem solving. If you turn the book onto its side, you will see several sections of differently coloured pages. These sections are called “Focus on Problem Solving” and “Focus on Modeling” and they relate the concepts you will have studied to a variety of situations. Turn now to the end of Chapter 1 in SRW and read “Focus on Problem Solving: General Principles”. When you try to solve problems later, try to apply some of these problem–solving strategies. Here is a summary of the steps you would usually follow in order to solve a “word problem” (the expression we use when you first need to “translate” language statements into mathematical equations before the problem can be solved).

Guidelines for problem solving

Read the problem. List the given information.
Draw a diagram if you can.
Introduce a variable and state what the variable represents.
Summarise the information in a table, if applicable.
Write an equation.
Solve the equation. Check the solution by substituting the answer into the equation.
Check that the answer makes sense in terms of the original problem.
Check that you have answered the question that was asked.
State the solution to the problem. Make sure that you have used appropriate units.

In mathematics, we are actually applying some of these problem–solving strategies all the time, even though initially the problems we solve are extremely simple. Where possible in this module we try to relate the mathematics you will learn to real–life problems, and you will have opportunities of using mathematics to solve everyday problems. Unfortunately many problems require much more advanced mathematics (building strong houses, bridges and roads that will last, doing research on chemical compounds). Some of these problems can be solved by using the appropriate computer software. However, it is still necessary to be able to make informed judgements about the parameters involved, and not only be able to run programmes mechanically.

In the textbook many of the more interesting problems depend on using a graphing calculator, which we cannot use as a standard part of this module. If you have one it will be interesting to study some of the examples in which they are used.

Please do not think that you must be able to solve every problem in every chapter that we study. The problems vary: some are fairly straightforward while others are quite difficult. It is more important that you do the exercise questions we set, and the assignments that you need to submit. Consider the problems in the "Discovery", "Focus on Modeling" and "Focus on Problem Solving" sections as enrichment, and only spend time on them once you have completed everything else.

The ability to **learn independently**, and **take responsibility** for one's learning, is one of the prerequisites for successful distance learning. We try to provide the information you need in as learner–friendly a way as possible, and structure the material you will use in a way that will help you to study regularly and responsibly. Taking responsibility for your own progress is initially difficult for learners who are used to a more directed approach, but you will soon enjoy the freedom that distance learning offers. We trust that you will become responsible students. The first step in this process is to finish reading this introduction, read Tutorial Letter 101 carefully and take note of all the information there, then **plan a work schedule** and **stick to it**.

Many students study by means of distance education because they are in remote areas, and in such cases **group work** is usually not possible. However, the majority of you will be near an established learning centre, or will be in an area where there is some venue available where you can arrange to meet regularly and share your ideas. The textbook contains exercises which are ideal for students to tackle in groups. The answers (to the odd-numbered questions) are given at the end of the book, but you need to work out the problems for yourselves to make sure that you can find the correct answer. Different students will tackle the questions in different ways, and you can learn a lot from discussing different approaches. (Remember that you can contact the university to make arrangements to obtain addresses or telephone numbers of students in various geographic locations who are studying the same modules.) Team work is only effective if everyone in the team works, i.e. there is no room for passengers hoping for quick answers.

Critical evaluation of information is the first step towards successful problem solving. Studying mathematics teaches us to work in a systematic way, to develop ordered thinking patterns and analytical thought processes. We need to ask ourselves specific questions as we read, such as: Is this possible? Is this valid? What does this mean? What are the hidden implications of this statement? Is some of this “information” vague and unsubstantiated?

The sixth outcome may not have an immediately obvious relationship with mathematics; however, mathematical concepts are necessary to study various aspects of the life and earth sciences, which must be used in an **environmentally sensitive** way. We need mathematics to describe and interpret population growth, or the spread of the HIV virus, and these issues have scientific, economic and social implications.

Achieving these outcomes is an ongoing process, and is related to the study of many different modules; however, this module is part of the overall plan.

Specific Outcomes

Apart from these broad outcomes there are important **mathematical outcomes**. When you have completed this module we expect that you should be able to

- demonstrate a clear grasp of the concept function and the terminology domain and range of a function
- determine a new function by finding the composition of two (or three) given functions
- determine the inverse of a function, if it exists
- define the following algebraic functions: absolute value, exponential, logarithmic, and develop the skills involved in using them
- define the trigonometric functions from an angle perspective (using degrees) and from a real number perspective (using radians)
- apply the characteristics of the algebraic functions mentioned above, in manipulating expressions and solving equations and inequalities
- apply the characteristics of trigonometric functions in manipulating expressions, solving equations and proving identities
- in relation to the graph of any function f, sketch the graphs of, and obtain equations for functions whose graphs are related to the graph of f by vertical and horizontal shrinking, stretching and shifting, and by reflection in the x–axis, y–axis and the line defined by $y = x$
- sketch and interpret graphs of the algebraic and trigonometric functions mentioned above

- interpret combinations of graphs of algebraic functions, including graphs which are assumed to be familiar (i.e. straight lines, parabolas, hyperbolas and circles)
- derive a trigonometric formula for the area of a triangle; derive the sine and cosine rules
- apply the properties and rules of the algebraic and trigonometric functions mentioned to solve simple real–life problems.

Learning Strategies for Mathematics, and in Particular for this Module

It is clear that we study mathematics differently from the way in which we study many other subjects. Sometimes it seems that we study only in order to pass exams, but we hope that you will want to learn the material covered in this module for many other reasons as well. There are many books and courses which claim to teach people how to learn – unfortunately for many of us it is difficult to make the transition from **knowing how to learn** to actually **being able to learn**. An additional problem is that many books written about learning skills apply to subjects in the Humanities, and not necessarily to the Natural Sciences. Here are some suggestions that may help you learn mathematics more effectively.

- **Have the right attitude!**

  As an adult learner you approach your present studies with previously acquired attitudes. Your past attitudes and feelings towards mathematics will affect your perceptions of this study material even before you begin to work through it. If studying mathematics was a pleasant experience for you before, you are probably looking forward to this work. If you have had many years of trying to cope with poor mathematics teaching (or in some cases no qualified mathematics teacher at all), you probably have serious misgivings about being able to "make the grade".

  Negative attitudes undermine your ability to understand and enjoy what you are learning. We would like you to approach this module with an open mind. Be conscious of negative attitudes, and try to put them aside. Expect that you will have to work hard (there are no short cuts to understanding mathematical concepts) but expect also that you will ultimately master the work.

- **Practise**

  A problem many students have is that they confuse familiarity with proficiency – because they have seen the work before they think they can do it. You should not just read through the work, and because it seems familiar assume that you can apply the concepts. There is a big difference between **reading** and **doing** mathematics. Try to answer questions, solve problems or verify statements to prove to yourself that you have actually understood the concepts involved. The questions in the textbook or study guides are structured in such a way that problems are graded from simple to more difficult, and it is a good idea to tackle the simpler ones first, before dealing with more complicated ones.

A good reason to practise regularly in this way is that as you gradually get correct answers more often, you will develop confidence in your own mathematical ability. This confidence is of enormous value – when you tackle new problems you are less likely to **panic**, and more likely to **think clearly and logically**. Confidence in your own ability will help you to develop a positive attitude to your studies, and hence enjoy the work more and cope better – as you can see, this becomes an upward spiral of ever–increasing proficiency!

Mathematics can easily be compared with sport. It is simply not possible to become a better tennis player or soccer player just by watching the game. You have to become involved and **practise**.

► **Try to set aside a regular time and a suitable place for study**

Because you need to concentrate in order to make sense of abstract concepts, it is usually unproductive to try to study mathematics when you are very tired, or when there are many distractions. Adult distance learners often have to cope with the demands of work and family and many other issues as well. Be realistic about your circumstances, but try to create the best possible conditions for your studies.

As we have said before, mathematics involves **doing**, so you need to work where you can write as well as read, and you need to have plenty of rough paper handy as you work. Having said that, though, it is also a good idea to be flexible, and accept that it is possible, at times, to study on the bus, or to try various ways of solving a problem while waiting in a queue!

We mentioned earlier the importance of having the right attitude. Part of this attitude involves a willingness to give the necessary time and effort to your studies.

► **Summarise**

As you read SRW and our guide it is important that you can separate the main ideas from the discussion surrounding them. It may be helpful to make your own summary for each section once you have finished studying it. Later on, you can compare your summaries to the "Concept Check" given at the end of each chapter. You may also find it helpful to refer to your summaries when you do the assignments or prepare for the exam.

► **Think**

This may seem a strange comment, but have you thought about what thinking means? When something is being explained to you, you need to consciously understand each step of the explanation, and not just listen or read superficially. This applies in particular to distance learners. When you read the solution to a problem, you are not immediately aware of the logical processes that have been applied to reach the answer. You need to "unravel" these processes yourself, by thinking carefully about all the steps involved. Try to think about the way in which you tackle problems. When you answer questions, try to explain to yourself all the time **why** you are doing what you are doing. If you are constantly aware of the reasons for applying certain principles you are less likely to apply rules blindly and inappropriately. This requires conscious effort, since it is easy to **read** and yet not **think**. When you watch a television programme, you often absorb information "by accident" simply because you are in front of a TV set at a particular moment. To study mathematics you cannot be a *passive* reader or listener; you need to be an *active* thinker.

If problems that you tackle are initially fairly straightforward, there is the danger that you may get them right just by copying similar examples, and not really because you have thought about and understood the processes involved. Then you will find that you cannot solve more difficult problems.

► **Give yourself enough time**

How would you assess your usual reading pace? If your mother tongue is English, you may be able to read a passage out of a novel very quickly, at the rate of about 350 words per minute. Academic text is conceptually more difficult and will take longer. You may want to try this simple reading test. Find a passage out of a magazine or newspaper, and read for exactly one minute, as though you were reading in order to understand and remember what you have read. (A clock or watch with a minute hand will be useful.) Mark where you began and where you ended, and count how many words you have read.

If you have read fewer than 230 words per minute, you would be classified as a "slow" reader; between 230 and 250 words per minute puts you into the "medium" category; you are a "fast" reader if you read more than 250 words per minute.

If your mother tongue is not English, you can expect to take longer to read both ordinary and more complex text. Mathematical text is usually more time–consuming than other academic text, even more so for learners who have not previously *read* much mathematics. Apart from the extremely diverse nature of the students studying this module, difference in reading speeds is another reason that we do not suggest an "average" time in which to complete the module.

The matter of how much time the module takes to study is more complex if you are a "slow" reader, and at the same time have a low comprehension rate, i.e. if you cannot recall more than about 70% of the information you have read. You may benefit from reading practice, and we recommend the book by Glendinning and Holmström: *Study Reading*, as well as the book by du Toit, Heese and Orr: *Practical Guide to Reading, Thinking and Writing Skills*. Both books are available in the Unisa Library. Neither of these books deals with the reading of scientific text, but many of the general reading strategies are useful for all readers. If you plan to run the Comrades Marathon, you would probably begin by going for a gentle jog around the block, and gradually build up the distance you can cover.

Reading is much the same: you can only improve your reading skills by regular reading. Try to make time for some reading every day, such as a newspaper or magazine. It will also be helpful to use a dictionary regularly, and keep it with you to look up unfamiliar words.

- **Set realistic goals**

You will make good progress with this module if you set goals for youself. The closing dates for the assignments already create a framework in which to set target dates. Try to keep to your own schedule. If you turn to the section "How to plan your work" in Tutorial Letter 101, you will find one possible schedule we suggest. You will soon see whether you are working fast enough to keep up, or too slowly. If you do fall behind, try to get back on schedule as soon as possible – you will work better without the unnecessary stress of knowing you are behind.

Students in general, and distance learners in particular, underestimate how long it will take them to master a new concept. Remember that several steps are involved. You need to **read critically**, then you need to **test your understanding of individual concepts** and **consolidate your understanding of how different concepts relate to one another** by doing the exercises. All this takes time – lots of it!

- **Know when to give up**

When you are unable to solve a problem the first time, or the second or third time, keep on trying! Obviously we cannot tell you exactly how long to keep trying, but you will often find the right answer if you just go on long enough, and you will learn a fair amount from all the false starts. However, there will be times when you realise that you are making no progress, and then a change of strategy is called for. You could then either speak to another student, and try to solve the problem together, or you could glance at the beginning of the solution (if there is one) and then try to go on from there. If there is no solution, but only an answer, you could look at the answer and try to work backwards. If you still have difficulty with the problem, leave it for a while and come back to it at a later stage.

It has been said that intelligence is not what you know, but what you do when you do not know.

- **Try different approaches**

The examples that are worked out in the different sections are not meant to be followed as though they are recipes for all similar problems. Be willing to experiment – perhaps another method will work equally well.

It often helps to try to draw a diagram to explain to yourself how different facts relate to one another. Drawing a diagram is also a useful way of visualising a practical problem before trying to solve it.

- **Learn from your mistakes**

We expect you to make mistakes, both in your workbook and in the assignments. But if you learn from these experiences you will be able to avoid similar mistakes in future.

- **Understand – do not memorise**

  One of the best things about mathematics is that it has a logical structure. Often, understanding how a rule or formula is derived means that there is no need to memorise it, because you can work it out when you need it. However, there are certain facts or definitions which have to be memorised, so try to distinguish between the things you really do need to memorise and those you do not, and by regular practice develop confidence in your ability to reason things out if necessary.

- **Learn how to apply a given method**

  In order to do this, you first need to understand the **aim** of the method, and then understand clearly the **purpose** of each step. Then you will be able to apply the method correctly in future.

- **Be critical of your own answers**

  Question the validity of numerical answers. For example, if you calculate that someone's age is -23 years, you have probably made a mistake! Try to find the mistake yourself.

  Question the logic of your answers. When you solve a problem, read what you have written and ask yourself whether there is a logical relationship between the various steps of your solution. Or have you made deductions without having the necessary conditions? One way to avoid this is to keep the steps in your solution small, even if this means you need more of them. Also, avoid using rules or formulas that you have not understood, because the chances are that you will apply them incorrectly.

  If possible test each answer, by substituting it into the original problem, to see whether it is valid.

- **Explain to a friend**

  Trying to explain something to someone else is often a quick way to determine whether or not you have understood it yourself. The best person to use as your "pupil" is someone else studying the same work, who will be able to stop you and ask questions. This is one of the reasons that we encourage students to form study groups, and **work** together. Note the emphasis on the word "work". Study groups need active participants.

  Another useful strategy may be to put yourself in the position of the person setting the assignments, or the examiner, and set questions for one another (or even for yourself) to solve. You will then need to provide the solutions as well!

- **Work neatly**

  Ordered work suggests ordered thinking, and it will help you to become more disciplined in your reasoning if you are conscious of the need to present your work in a clear and orderly way. It also makes it much easier for you to check the steps of a solution if you can follow what you have written down.

Assessment

Your first tutorial letter (Tutorial Letter 101) gives details about the assignment system. Read the whole letter carefully, in particular the section dealing with closing dates for the submission of assignments. This letter also contains the assignment questions.

Exams will be written at the end of each year. The exam admission procedures are also explained in the first tutorial letter.

Language Matters

One of the reasons we have chosen SRW is that it is written in a straightforward way. However, because it is an American textbook, occasionally unfamiliar words, phrases and units of measure are used. Where necessary we will explain. Note that language really does **matter**. It is important that you get to grips with the specialised "language of mathematics" as soon as possible, so that poor language comprehension does not affect your understanding of the mathematics, and that poor language expression does not affect the way you write down your answers.

In the book "The Language of Science", by Orr and Schutte, two questions are asked: "Why do we need this very precise and subtle language of mathematics at all? Is it not just as easy to do mathematics using ordinary English?" These questions are answered by means of an example.

Suppose we want to express in normal English the general form we obtain when we factorise the expression

$$x^3 - y^3.$$

We would have

> The difference of the third powers of any two numbers x and y is equal to the product of the difference of these two numbers and the sum of three terms, the first of which is the square of the first number, the second the product of the two numbers, and the third the square of the second number.

This certainly seems much more complicated than the more familiar mathematical version:

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

The answer written in English is a lot more difficult to understand. We need to read it several times to make sense of what is being said. The mathematical version highlights the usefulness of specific symbols for mathematical operations.

The use of compact mathematical notation allows us to

- give statements in a general way

- be very precise in what we say
- manipulate symbols easily in order to make deductions
- present a large amount of information in a limited space.

We need to remember that mathematical symbols can be "translated" into English in specific ways, and in the process they are often abused. One of the most frequently misused signs is "=" which is the symbol for "is equal to". Hence it makes sense to write

$$x = 5$$

when we mean that the variable x is given the value 5. It makes no sense at all to write

"Joe = 10 years old".

We should write

"Joe's age is 10 years"

or

"Joe's age in years = 10".

We can check whether what we write makes sense by translating from the mathematical form to the equivalent English form. In the first case we see that it makes sense to say "x is equal to 5"; in the second we can see that it is incorrect to say "Joe is equal to 10 years old".

Another frequently used, and often misused, word, is "let". In mathematics it is used in several different ways. For example, in a problem–solving situation, we may need to find an unknown value, such as the value of an investment after five years. We then usually begin with the statement

"Let the value of the investment after five years be x rands ."

This word is also used in the following ways.

- **Let** us first simplify the expression ... (suggesting we **should** first do this before carrying on with the calculation).
- **Let** us recall the definition of a rational number ... (again suggesting that we **must** first remember some previous concept before proceeding).
- **Let** x be a real number. (This indicates that we are **assuming** a specific condition for x.)
- If we **let** $x = 4$, then $2x + 1 = 9$. (This indicates that we are **choosing** a value for x, from a variety of possible options.)

In the following Vocabulary Building List we try to give (without going into too much detail) the meanings of other words commonly used in mathematical explanations.

Word/phrase	Mathematical example
thus, i.e. accordingly, in this way	$x = 1$. **Thus** $x+3 = 4$.
therefore, i.e. for this reason	Division by zero is undefined. **Therefore** $\frac{9}{x}$ is not a real number if $x = 0$.
hence, i.e. as a result of this	We have $x = 2$. **Hence** $3x = 6$.
since ... it follows that, i.e. implies	**Since** $x = 2$ **it follows that** $3x = 6$.
if ... then ..., i.e. implies	**If** $x = 2$ **then** $3x = 6$. $x = 2$ **implies** that $3x = 6$.
consider, i.e. look at a specific case; think carefully about a concept	We first **consider** the positive values of x.
prove, i.e. show the validity of a given statement	**Prove** that $\sqrt{2}$ is an irrational number. (A proof involves building up a series of consecutive statements, for each of which there should be sufficient evidence; the final statement is the required result. It is also possible to prove a certain statement by **contradiction**. We will not deal with methods of proof in this module.)
show/verify (used in a similar way to "prove")	**Show** that $x = 2$ is not a solution of the equation. **Verify** that the angles add up to 180 degrees.
find/determine, i.e. calculate the answer (numerical or algebraic) that satisfies the particular requirements	**Find** the values of x that satisfy the equation. **Determine** the x–intercepts of the graph.
solve, i.e. find the value(s) for which a given statement is true	**Solve** for x: $2-3x = 8$.
calculate/evaluate, i.e. find a value for (Note that the "value" may be a number or a numerical or algebraic expression.)	**Calculate** $\frac{x+2}{x-5}$ if $x = 8$. **Evaluate** $3x+8$ when $x = (2a+1)$.
state, i.e. write down fully, explain clearly	**State** the Theorem of Pythagoras.
deduce, i.e. show how a result follows logically from preceding results	**Deduce** that $x^2+1 = 0$ has no real solution.
denote, i.e. indicate by means of a symbol	We **denote** the operation of addition by means of $+$.
simplify, i.e. express the answer in its simplest terms	**Simplify** the expression in brackets.
define, i.e. give in exact mathematical terms the meaning of a concept	We **define** a rational number as follows ... .
convention, a specifically agreed rule or standard	We use the BODMAS **convention** for the order of arithmetic operations.

Word/phrase	Mathematical example
express, i.e. state the answer in the given way	**Express** the answer with rational denominator.
respective, i.e. relating to a given order	The **respective** values of a, b and c are 2, 1 and 7. (This means that $a = 2$, $b = 1$ and $c = 7$.)
respectively, i.e. in the order mentioned	$x = 3$ and $x = 4$ are the solutions of the equations $3x+2 = 11$ and $5x-1 = 19$, **respectively**. (This means that $x = 3$ is the solution of $3x+2 = 11$ and $x = 4$ is the solution of $5x-1 = 19$.)
unique, i.e. the only one of its kind	The **unique** solution of $(x-3)^3 = 0$ (if x is real) is $x = 3$.
assign, i.e. allocate	In the expression $x^2+2xy+y^2$ we **assign** the value of 2 to x, and 3 to y. (This means that we replace each x with the number 2, and each y with the number 3.)
ambiguous, i.e. having more than one meaning **unambiguous**, i.e. having precisely one meaning	It is **ambiguous** to write "The solution of the equation is $x = 2,\ x = 3$". We must state this **unambiguously**, and write either "$x = 2$ and $x = 3$", or "$x = 2$ or $x = 3$".

The word **prove** is briefly explained in the Vocabulary Building List.

What is proof?

Suppose we add a few sequences of odd numbers. For example

$$\begin{aligned} 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25. \end{aligned}$$

We immediately notice that, in each case, the answer is a perfect square. Would it be correct to state that the sum of any sequence of consecutive odd numbers starting with 1 is always a perfect square?

This statement seems a reasonable conjecture, but we cannot be certain that if we add the first one hundred, or one thousand, odd numbers, we will still get a perfect square.

If we look at the pattern again, we see that

$$\begin{aligned} 1+3 &= 2^2 && \text{i.e. the sum of the first \textit{two} odd numbers is } 2^2 \\ 1+3+5 &= 3^2 && \text{i.e. the sum of the first \textit{three} odd numbers is } 3^2 \\ 1+3+5+7 &= 4^2 && \text{i.e. the sum of the first \textit{four} odd numbers is } 4^2 \\ 1+3+5+7+9 &= 5^2 && \text{i.e. the sum of the first \textit{five} odd numbers is } 5^2. \end{aligned}$$

Can we now assume that the sum of the first n odd numbers is n^2? Once again, unless we calculate the answer each time, these examples do not enable us to conclude with certainty that the sum of the first one hundred odd numbers will be 100^2, or that the sum of the first 500 odd numbers will be 500^2.

In order to be certain, we have to **prove** that the sum of the first n odd numbers is n^2, i.e.

$$1+3+5+....(2n-1) = n^2$$

for any natural number n. The most important part of this statement is that the proof must be valid **for any** n.

How do we do this? There are various possibilities. One such proof which is easy to follow uses the method of mathematical induction, which you will use in later mathematics modules. For interest you could read the discussion and the proof in Section 11.5 (10.5 in previous editions) of SRW.

Logic

In this module, although various facts are proved in SRW, you are not required to memorise any of the proofs. However, in later mathematics more attention is paid to proving important concepts, so we briefly discuss this now.

$\Rightarrow$ and $\Leftrightarrow$ are called **implication signs**.
$\Leftrightarrow$ is known as the **double implication sign.**

The two symbols $\Rightarrow$ and $\Leftrightarrow$ appear in the list of abbreviations given before Chapter 1 in SRW. You may perhaps not have seen them before. They are used in proofs or in mathematical explanations or solutions. We use $\Rightarrow$ for the "one–way" cases and $\Leftrightarrow$ for the "two–way" cases. You will notice that we use them in the study guide in the presentations of solving equations or inequalities. SRW does not present answers in this way. That does not imply that it is unimportant, but that *at this stage* you can rather concentrate on the mathematical processes, instead of on the notation and underlying logic. It is a good idea to become aware of the way in which mathematics should be written, even if you do not use this notation right now.

In the Vocabulary Building List we have the phrase "if ... then ...", which is used in *implication statements*. "If ... then ..." is an example of **"one–way" logic**, for which we use the symbol "⇒", because the **converse** of the statement is not necessarily true. In general, if a **statement** has the form

The statement $A \Rightarrow B$ (or $B \Rightarrow A$, or $A \Leftrightarrow B$) is called an **implication.**

$$A \Rightarrow B$$

(i.e.: If A is true then B is true.) then we say that the **converse of the statement** is

$$B \Rightarrow A$$

(i.e.: If B is true then A is true.).

There are many everyday examples which also illustrate that the converse of a given statement is not necessarily true. For example, consider the statement

$$\text{If } \underbrace{\text{I stand in the rain}}_{A}, \underbrace{\text{I will get wet}}_{B}.$$

Note that even though the word "then" is not always written down, it is suggested by the construction of the statement. In this example we thus have a statement of the form

$$A \Rightarrow B.$$

Let us consider the converse of this statement:

$$\text{If } \underbrace{\text{I am wet}}_{B}, \text{ then } \underbrace{\text{I was standing in the rain.}}_{A}$$

It is clear that the statement

$$B \Rightarrow A$$

is **not true**, since I could get wet from falling into a dam, or from having a bucket of water thrown over me. Since $B \Rightarrow A$ is not a valid statement, we write

We often use the words **valid** and **true** interchangeably.

$$B \not\Rightarrow A.$$

We consider another simple example.

If two odd numbers are added, the answer is an even number.

This has the form $A \Rightarrow B$, where

A is the statement "I add two odd numbers", and
B is the statement "The answer is an even number".

It is important to remember that an illustration is **not** a proof. A proof must be shown to hold **in general**.

We can *illustrate* the validity of this implication by giving an example.

> Consider the two odd numbers 1 and 3.
> Add them: $1+3=4$.
> The answer is an even number.

Now let us look at the converse of this statement.

> If I obtain an even number,
> then the two numbers that were added together are odd.

If we had decided to accept an illustration (example) as a proof, we might think that this statement is true, since

$$6 = 1 + 5$$

where 6 is even, and 1 and 5 are both odd. However, in this case $B \not\Rightarrow A$, because we can also find an example (we call this a **counter–example**) to show that even if B applies, A need not apply. We can think of

$$6 = 2 + 4$$

Now you can see why an example cannot be regarded as a proof.
To prove that an implication is **not true** it is only necessary to find **one** counter–example.

where 6 is even, and 2 and 4 are also even.

From this example we see that the converse statement is true in some cases, but not in all. Hence, in general, we say that the converse is **not true**. The statement "the converse is true" means that the converse must hold in every possible case, with no exceptions.

Where both an implication and its converse are true, we have the situation

$$A \Rightarrow B \textbf{ and } B \Rightarrow A$$

and we then say that the statements A and B are **equivalent** and we write

$$A \Leftrightarrow B.$$

Consider the following example.

$$\begin{array}{lcl} A & : & x \text{ is an even number} \\ B & : & x \text{ is divisible by } 2 \end{array}$$

We will not give a formal proof of the relationship here, but it should be easy to see that the implication

$$A \Rightarrow B$$

is true. Since x is an even number (e.g. $x = 2, 4, 6, 8, 10, \ldots$) then x is divisible by 2. It should also be clear that the implication

$$B \Rightarrow A$$

is true. Since x is divisible by 2, then x is an even number. We can thus conclude that A and B are equivalent statements, so that we can describe a certain number unambiguously either by the statement

the number is divisible by 2

or by the statement

the number is even.

We thus have the double implication

$$A \Rightarrow B \text{ and } B \Rightarrow A$$

> i.e. if x is an even number then it is divisible by 2,
> and if it is divisible by 2, then it is an even number

which we write as

$$A \Leftrightarrow B.$$

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Freeman, R.: *How to Learn Maths*, National Extension College, 1994.

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The Greek Alphabet

Occasionally we use letters of the Greek alphabet for variables. For interest, here is a list of the complete Greek alphabet, together with the name of each letter, and its pronounciation in the case of the more unfamiliar letters.

Capital letter	**Small letter**	**Name**	**What it sounds like**
A	α	alpha	the "ph" sounds like "f"
B	β	beta	
Γ	γ	gamma	
Δ	δ	delta	
E	ε	epsilon	
Z	ζ	zeta	
H	η	eta	
Θ	θ	theta	
I	ι	iota	"i" – ota where the "i" rhymes with sky.
K	κ	kappa	
Λ	λ	lambda	we do not sound the b (as in the word lamb)
M	μ	mu	mew
N	ν	nu	new
Ξ	ξ	xi	the "x" sounds like "ts", so that this becomes "ts–i"
O	o	omicron	
Π	π	pi	
P	ρ	rho	this sounds like "row"
Σ	σ	sigma	
T	τ	tau	this rhymes with "how"
Υ	υ	upsilon	
Φ	ϕ	phi	the "ph" sounds like "f"
X	χ	chi	rhymes with sky
Ψ	ψ	psi	we sound the "p", so that this sounds like "p–sigh"
Ω	ω	omega	

Units of Measure

You should already be familiar with metric units and the conversion of one metric unit (or combination of units) to another metric form. For example, you should be able to change a speed expressed in kilometres per hour to the speed expressed in metres per second; or be able to express concentration given in grams per millilitre as a measurement in kilograms per litre.

In SRW non–metric units are used. A list of some of the non–metric units is given below, in case you need this information to solve problems. It will not be necessary to convert from non–metric to metric units in order to solve the problems.

Linear measure
1 foot = 12 inches
1 yard = 3 feet
1 mile = 1 760 yards = 5 280 feet

Weight
1 pound = 16 ounces
1 stone = 14 pounds

Volume
1 pint = 20 fluid ounces
1 quart = 2 pints
1 gallon = 8 pints

If you need to convert a speed of say 45 miles per hour to the speed expressed in feet per second, you will then reason like this:

$$\text{45 miles per hour denotes } \frac{\text{45 miles}}{\text{1hour}}.$$

Now

$$\frac{\text{45 miles}}{\text{1 hour}} = \frac{(45 \times 5\,280) \text{ feet}}{(60 \times 60) \text{ seconds}} = 66 \text{ feet per second.}$$

Abbreviations

Some of the letters in the Greek alphabet are commonly used as abbreviations. See the list of abbreviations given before Chapter 1 in SRW. Apart from letters of the Greek alphabet, we see from the list how other letters or combinations of letters are used in a standard way to represent quantities. The last two symbols in the right hand column are used in the "language of logic", which we discuss in "Language Matters".

Conclusion

By the end of the year we hope you will have developed a solid foundation for your future studies, whether you study mathematics itself, or another discipline which relies on mathematics.

We wish you success!

1 FUNDAMENTALS

We assume that learners have studied most of the topics covered in this chapter. In each section we indicate the topics that we assume that you know and also those which we consider as new work. Make sure that you have a thorough understanding of the topics in this chapter as they will be needed for further study of this module.

1.1 REAL NUMBERS

Assumed knowledge
SRW, first part of Section 1.1:
Real numbers
Properties of real numbers
The real line
Sets and intervals

To study
SRW, the last subsection of Section 1.1,
Absolute Value and Distance.

Outcomes
After studying this section you should be able to

- evaluate absolute values
- simplify the absolute value of an algebraic expression
- write the absolute value of an algebraic expression without absolute value signs
- find the distance between two points on the number line using absolute value.

Exercises
Exercises 1.1: questions 61-69 (odd numbers)
Additional Exercises 1.1.1

Diagnostic Test 1.1

Exercises 1.1: questions 1-71 (odd numbers).

■ Absolute Value and Distance

Study this subsection in Section 1.1 of SRW.

We note from the definition of absolute value that $|0| = 0$. Thus we can write the absolute value of a as

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a \leq 0 \end{cases}$$

or as

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

since $-a = a$ when $a = 0$ (i.e. $-0 = 0$).

Take particular note of Example 7(d) as you will later need to write the absolute value of an algebraic expression without absolute value signs. Work carefully through the following example, making sure that you understand how the definition of absolute value is applied, and then try the questions from Additional Exercises 1.1.1.

Example 1.1.1

Write $|2x+3|$ without absolute value signs.

Solution

Recall the definition of the absolute value of a, namely

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Now we replace a with $2x+3$ and apply the above definition. We obtain

$$|2x+3| = \begin{cases} 2x+3 & \text{if } 2x+3 \geq 0 \\ -(2x+3) & \text{if } 2x+3 < 0 \end{cases}$$

which we can rewrite as

$$|2x+3| = \begin{cases} 2x+3 & \text{if } x \geq -\frac{3}{2} \\ -2x-3 & \text{if } x < -\frac{3}{2}. \end{cases}$$

Caution: The following is INCORRECT.

$$|2x+3| = \begin{cases} 2x+3 & \text{if } x \geq 0 \\ -2x-3 & \text{if } x < 0 \end{cases}$$

Can you explain why this is incorrect?

Additional Exercises 1.1.1

1. Write each of the following without absolute value signs.

   (a) $|x+1|$
   (b) $|3x-2|$
   (c) $|-5x+4|$
   (d) $|-4x-3|$

2. Explain why it is incorrect to write

$$|3x-2| = \begin{cases} 3x-2 & \text{if } \quad x \geq 0 \\ -(3x-2) & \text{if } \quad x < 0. \end{cases}$$

1.2 EXPONENTS AND RADICALS

Assumed knowledge	SRW, Section 1.2: The entire section, except rationalising a denominator of the form $\sqrt[n]{a^m}$ (dealt with in Examples 13(b) and (c)), i.e. Integer exponents Scientific notation Radicals Rational exponents Rationalising the denominator

Diagnostic Test 1.2

Exercises 1.2: 1-69 (odd numbers), 83, 85, 87.

1.3 ALGEBRAIC EXPRESSIONS

Assumed knowledge	SRW, Section 1.3 except Example 12: Definitions Addition, subtraction, multiplication of polynomials and other algebraic expressions Factorisation
To study	SRW, Example 12.
Outcomes	After studying this section you should be able to ► factorise algebraic expressions that contain terms with rational exponents.
Exercises	Exercises 1.3: questions 67, 69, 103

Comment

SRW uses the words "factor" and "factoring" whereas we usually use "factorise" and "factorising". The title given to Example 12 in Section 1.3 of SRW is "Factoring Expressions with Fractional Exponents". The authors mean "Factorising algebraic expressions which contain terms with rational exponents".

Diagnostic Test 1.3

Exercises 1.3: questions 1-99 (odd numbers).

■ Factorisation of Algebraic Expressions which Contain Terms with Rational Exponents

Study Example 12 in Section 1.3 of SRW.

1.4 RATIONAL EXPRESSIONS

Assumed knowledge	SRW, Section 1.4, up to Example 6: Arithmetic operations on rational expressions
To study	SRW, Compound Fractions, Rationalising the denominator or the numerator.
Outcomes	After studying this section you should be able to ► simplify compound fractions ► rationalise a denominator or numerator containing a surd by multiplying by a conjugate radical.
Exercises	Exercises 1.4: questions with odd numbers from 51 to 97

Comment

Every polynomial is an algebraic expression, but not every algebraic expression is a polynomial.

Note that a **fractional expression** is the quotient of two *algebraic expressions*, whereas a **rational expression** is the quotient of *two polynomials.* Thus a rational expression is a special type of fractional expression. All the expressions used in Examples 2 to 5 in SRW are rational expressions.

Diagnostic Test 1.4

Exercises 1.4: questions 1-49 (odd numbers).

■ Simplifying a Compound Fraction

Study Examples 6–8 in Section 1.4 of SRW. Take particular note of Examples 7 and 8 as these are the types of expressions that you need to simplify in Calculus.

■ Rationalising Denominators and Numerators

Note the application of the product formula $(a+b)(a-b) = a^2 - b^2$ in Examples 9 and 10. We include these examples here as you may need to use such techniques in order to determine whether a certain answer that you obtain is the same as that given in the book.

■ Common Errors in Addition

Avoid the common errors stated in the subsection Avoiding Common Errors, Section 1.4 of SRW.

1.5 EQUATIONS

Assumed knowledge	SRW, Section 1.5, to just before Example 13: Solving linear and quadratic equations Solving for one variable in terms of others Solving equations containing square roots Extraneous solutions
To study	SRW, Examples 13 and 14, and the additional notes below.
Outcomes	After studying this section you should be able to ► solve an equation containing terms in which fractional exponents occur ► solve an absolute value equation.
Exercises	Exercises 1.5: questions 89, 91, 93, 95, 97

Comments

The symbol $\Leftrightarrow$ is used for the first time in Section 1.5 of SRW in the table dealing with the Properties of Equality. This sign can be read in different ways. If we have

$$P \Leftrightarrow Q$$

where P and Q are two statements, then we can read this as

"P is equivalent to Q."
"If P is true then Q is true and if Q is true then P is true".
"P is true if and only if Q is true".

In the examples after the table containing the two Properties of Equality, the authors have set out the solutions without using the symbol $\Leftrightarrow$. To be mathematically quite correct, we would usually link each step in the solution and represent it as shown in the example on the next page. However, we do not expect the same degree of mathematical rigour from students at this stage, and, as SRW has done, for now you can focus on understanding the mathematical processes involved.

$$
\begin{aligned}
& 4x+7=19 \\
\Leftrightarrow\quad & 4x+7+(-7)=19+(-7) \\
\Leftrightarrow\quad & 4x=12 \\
\Leftrightarrow\quad & \frac{1}{4}\cdot 4x=\frac{1}{4}\cdot 12 \\
\Leftrightarrow\quad & x=3
\end{aligned}
$$

We can use $\Leftrightarrow$ here because the successive equations are equivalent. The use of the symbols $\Leftrightarrow$ and $\Rightarrow$ is discussed in more detail in the Preface. See the section on Logic, under the heading Language Matters.

The title of Example 3 is " Solving for One Variable in Terms of Others". This may be more familiar to you as "Changing the Subject of a Formula".

In SRW the symbol D is used for the discriminant. Another commonly used symbol for the discriminant is Δ (capital delta).

Diagnostic Test 1.5

Exercises 1.5: questions 3-87 (odd numbers), 99, 101, 107.

■ Solving an Equation Containing Terms in which Fractional Exponents Occur

Study Example 13 in Section 1.5 of SRW very carefully, taking note of all the steps involved. Make sure that you always check the answers you obtain for this type of equation to see if they are all solutions of the original equation.

Example 1.5.1

Find all the solutions of the equation $x^{\frac{3}{4}}+x^{\frac{1}{2}}-6x^{\frac{1}{4}}=0$.

Solution

$$x^{\frac{3}{4}} + x^{\frac{1}{2}} - 6x^{\frac{1}{4}} = 0$$

$$\Rightarrow \quad x^{\frac{1}{4}}\left(x^{\frac{2}{4}} + x^{\frac{1}{4}} - 6\right) = 0 \ \text{(Factor out } x^{\frac{1}{4}},\ \text{smallest exponent)}$$

$$\Rightarrow \quad x^{\frac{1}{4}}\left(x^{\frac{1}{4}} - 2\right)\left(x^{\frac{1}{4}} + 3\right) = 0 \ \text{(Factorise quadratic expression)}$$

$$\Rightarrow \quad x^{\frac{1}{4}} = 0 \ \text{ or } \ x^{\frac{1}{4}} - 2 = 0 \ \text{ or } \ x^{\frac{1}{4}} + 3 = 0 \ \text{(Zero-product property)}$$

$$\Rightarrow \quad x^{\frac{1}{4}} = 0 \ \text{ or } \ x^{\frac{1}{4}} = 2 \ \text{ or } \ x^{\frac{1}{4}} = -3$$

$$\Rightarrow \quad x = 0 \ \text{ or } \ x = 2^4 \ \text{ or } \ x = (-3)^4$$

$$\Rightarrow \quad x = 0 \ \text{ or } \ x = 16 \ \text{ or } \ x = 81$$

Check

$x = 0$:
$\text{LHS} = (0)^{\frac{3}{4}} + (0)^{\frac{1}{2}} - 6(0)^{\frac{1}{4}} = 0$
$\text{RHS} = 0$
$\text{LHS} = \text{RHS}$

$x = 16$:
$\text{LHS} = (16)^{\frac{3}{4}} + (16)^{\frac{1}{2}} - 6(16)^{\frac{1}{4}}$
$= 8 + 4 - 6(2) = 0$
$\text{RHS} = 0$
$\text{LHS} = \text{RHS}$

$x = 81$:
$\text{LHS} = (81)^{\frac{3}{4}} + (81)^{\frac{1}{2}} - 6(81)^{\frac{1}{4}}$
$= 27 + 9 - 6(3) = 18$
$\text{RHS} = 0$
$\text{LHS} \neq \text{RHS}$

Thus $x = 0$ and $x = 16$ are solutions, but $x = 81$ is not.

The only solutions are $x = 0$ and $x = 16$.

■ Absolute Value Equations

We first deduce the following property.

$$|x| = a \Leftrightarrow x = a \text{ or } x = -a$$

We reason as follows.
Suppose $|x| = a$.
If $x \geq 0$ then $|x| = x = a$.
If $x < 0$ then $|x| = -x = a$, i.e. $x = -a$.
Hence $x = a$ or $x = -a$.

Now suppose $a \geq 0$ and $x = a$ or $x = -a$.
Then according to the definition of absolute value, $|x| = a$.

The property $|x| = a \Leftrightarrow x = a$ or $x = -a$ is used in the solution of Example 14. It is better to present the solution to Example 14 in the following way.

$$|2x-5| = 3$$

$$\Leftrightarrow \quad 2x-5 = 3 \quad \text{or} \quad 2x-5 = -3$$

$$\Leftrightarrow \quad 2x = 8 \quad \text{or} \quad 2x = 2$$

$$\Leftrightarrow \quad x = 4 \quad \text{or} \quad x = 1$$

Thus the solution set is $\{1, 4\}$.

1.6 MODELING WITH EQUATIONS

Assumed knowledge	SRW, the entire Section 1.6: Modeling with equations

We expect that all students will have solved problems similar to those given in this section, either at school or in MAT0511.

The guidelines for solving word problems using equations given at the start of Section 1.6 in SRW have been adapted from the general principles in “Focus on Problem Solving” given at the end of Chapter 1 of SRW. We suggest that you read the general principles before the guidelines. Then work through Examples 1–9 identifying each of the guidelines used in solving the problems. The margin notes will help you to do this.

Note that SRW uses “rate” for “speed”. Both metric and imperial units are used in the examples.

Instead of doing a diagnostic test, try a representative selection of questions from Exercises 1.6.

1.7 INEQUALITIES

Assumed knowledge	SRW, first part of Section 1.7, and additional notes below: Solving linear and quadratic inequalities algebraically Solving a pair of simultaneous linear inequalities Solving quadratic inequalities using parabolas
To study	SRW, rest of Section 1.7, and additional notes below.
Outcomes	After studying this section you should be able to ► solve an inequality involving a quotient ► solve an inequality with three factors ► solve absolute value inequalities.
Exercises	Exercises 1.7: questions 47-75 (odd numbers) Additional Exercises 1.7.1: questions 5–8

Comments

In Example 3, Section 1.7 of SRW, it is noted that the equation

$$(x-2)(x-3)=0$$

has two solutions, namely $x=2$ and $x=3$. Since these are the only two values for which the expression $(x-2)(x-3)$ can be equal to zero, the expression can only change sign for values of x before or after these values. These numbers, i.e. 2 and 3, are called different names by different authors. They are known as **split–points, boundary points, cut points** and **critical points**. We call them split–points and the method used in Example 3 is known as the split–point method.

Diagnostic Test 1.7

Exercises 1.7: questions 7-43 (odd numbers)
Additional Exercises 1.7.1: questions 1, 3.

The following two subsections are additional notes for assumed knowledge.

■ Solving a Pair of Simultaneous Linear Inequalities

In Example 9, Section 1.7 of SRW, we have

$$5 < \tfrac{5}{9}(F - 32) < 30.$$

This is equivalent to the two inequalities

$$5 < \tfrac{5}{9}(F - 32) \quad \textbf{and} \quad \tfrac{5}{9}(F - 32) < 30.$$

In this case we do not need to consider two separate inequalities, but we work directly with $5 < \frac{5}{9}(F - 32) < 30$ since only the middle expression contains the variable F.

Now consider the following example.

Example 1.7.1

Solve

$$2x + 1 < x - 4 \leq 3x + 7$$

and sketch the solution set on a number line.

Solution

We again have two inequalities, namely

$$2x + 1 < x - 4 \quad \textbf{and} \quad x - 4 \leq 3x + 7.$$

Since all three expressions in $2x + 1 < x - 4 \leq 3x + 7$ involve the variable x, it is impossible to solve directly as was done in Example 9 in SRW. We solve the two inequalities $2x + 1 < x - 4$ and $x - 4 \leq 3x + 7$ separately. We must thus find the values of x that satisfy **both** inequalities.

Now

$$2x + 1 < x - 4$$

$\Leftrightarrow$ $x < -5.$ Subtract x and 1 from both sides.

Also

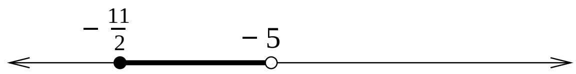
$$x - 4 \leq 3x + 7$$

$\Leftrightarrow$ $-11 \leq 2x$ Subtract x and 7 from both sides.

$$\Leftrightarrow \qquad 2x \geq -11$$

$$\Leftrightarrow \qquad x \geq -\tfrac{11}{2}. \qquad \text{Divide both sides by 2.}$$

Thus the solution is $x < -5$ **and** $x \geq -\frac{11}{2}$, i.e. $-\frac{11}{2} \leq x < -5$. The solution set is $\left[-\frac{11}{2}, -5\right)$ which is represented on the number line below.



■ Solving a Quadratic Inequality Using a Parabola

In Example 3, Section 1.7 of SRW, a quadratic inequality was solved using a table of signs. We now investigate how we can use a parabola to solve the same inequality.

We assume that you have studied parabolas. If not, come back to this example once you have studied Section 2.5 of SRW.

Example 1.7.2

Solve the inequality
$$x^2 - 5x + 6 \leq 0$$
using a rough sketch of the parabola defined by $y = x^2 - 5x + 6$.

Solution

We consider the parabola defined by
$$y = x^2 - 5x + 6.$$

We obtain the x–intercepts of the parabola by solving $x^2 - 5x + 6 = 0$.

We have
$$x^2 - 5x + 6 = (x-3)(x-2) = 0$$
and thus $x = 3$ or $x = 2$. Hence the parabola cuts the x–axis at 2 and 3.

The parabola is concave up since the coefficient of x^2 is positive.

A rough sketch of the parabola is shown in Figure 1.7.1. Note that we need not find the y–intercept or vertex as these are not relevant to the problem.

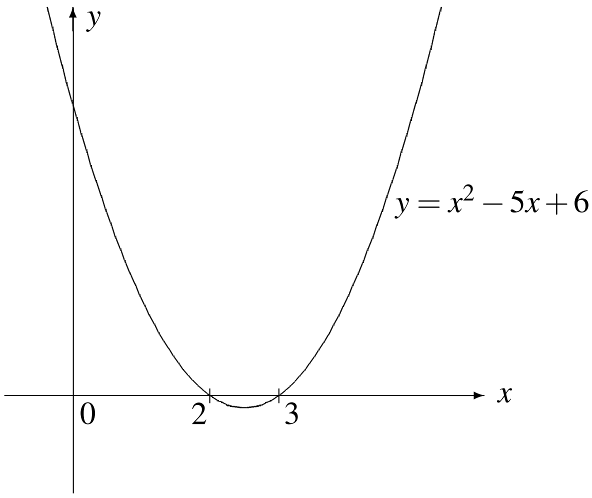


Figure 1.7.1

Now we want to find the values of x such that

$$x^2 - 5x + 6 \leq 0$$

i.e. such that the graph of $y = x^2 - 5x + 6$ lies below or on the x–axis.

From the sketch we see that the parabola lies below the x–axis for $2 < x < 3$ and cuts the x–axis, i.e. is on the x–axis, at $x = 2$ and at $x = 3$.

Thus the solution of the inequality is $2 \leq x \leq 3$.

■ Solving an Inequality which Contains a Quotient

First read the "Guidelines for solving non–linear inequalities" after Example 3, Section 1.7 of SRW and the cautionary note at the bottom of that page before you study Example 4. Notice the importance of applying Step 1 in the solution. Avoid the common error of "cross multiplying" across the inequality sign, i.e. DO NOT WRITE

$$\text{If } \frac{1+x}{1-x} \geq 1 \text{ then } 1+x \geq 1-x$$

unless you are certain that $1-x > 0$ since this statement is only true if $1-x > 0$. See Rule 3 in Rules for Inequalities, Section 1.7, of SRW. You can reason as follows.

$$\frac{1+x}{1-x} \geq 1$$

$$\Leftrightarrow \qquad 1+x \geq 1-x \text{ and } 1-x > 0$$

$$\text{or} \qquad 1+x \leq 1-x \text{ and } 1-x < 0$$

However, it is more complicated to solve the problem this way. Rather use the method given in the solution to Example 4.

In the solution to Example 4, the last inequality, namely $\frac{2x}{1-x} \geq 0$, was solved by using a table of signs. This inequality can also be solved by using a sketch of a parabola, as shown below.

Use of a parabola

We first notice that $\frac{2x}{1-x}$ is undefined if $1-x=0$, i.e. if $x=1$, and hence $x=1$ cannot be one of the solutions.

Secondly, we note that for $x \neq 1$, the term $1-x$ can be either positive or negative depending on the values of x, but $(1-x)^2$ is always positive. Thus for $x \neq 1$ we can multiply both sides of

$$\frac{2x}{1-x} \geq 0$$

by $(1-x)^2$ and obtain

$$\frac{(1-x)^2 \cdot 2x}{1-x} \geq 0.$$

After cancelling we obtain

$$2x(1-x) \geq 0.$$

Thus instead of solving $\frac{2x}{1-x} \geq 0$, we solve $2x(1-x) \geq 0$ for $x \neq 1$.

Consider the parabola defined by $y = 2x(1-x)$, i.e. by $y = -2x^2 + 2x$.

The x–intercepts are 0 and 1 and the parabola is concave down. Figure 1.7.2 is a rough sketch of this parabola.

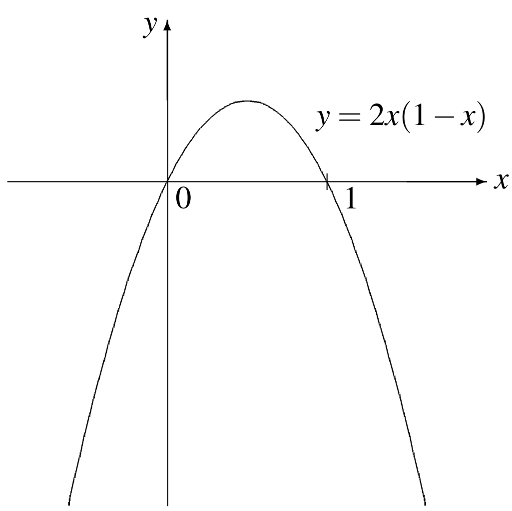


Figure 1.7.2

From Figure 1.7.2 we see that the parabola lies above or on the x–axis if $0 \leq x \leq 1$. Thus the solution of $2x(1-x) \geq 0$ is $0 \leq x \leq 1$, but the solution of $\dfrac{2x}{1-x} \geq 0$ is $0 \leq x < 1$ since we must exclude $x = 1$. This is the same as the solution obtained when we use a table of signs.

■ Solving an Inequality with Three Factors

Study Example 5, Section 1.7 of SRW.

■ Absolute Value Inequalities

Study Examples 6 and 7, Section 1.7 of SRW. The two examples given are simple and use directly the properties of absolute value inequalities given in the table above them. Note that in the table, $c > 0$. We now look at some more complicated examples.

Example 1.7.3

Solve the inequality

$$\frac{1}{|1-2x|} > 3.$$

Solution

The inequality is undefined if $1 - 2x = 0$ as division by zero is undefined.

Thus for $x \neq \frac{1}{2}$,

$$\frac{1}{|1-2x|} > 3$$

$$\Leftrightarrow \quad 1 > 3|1-2x| \qquad \text{Note that } |1-2x| > 0.$$

$$\Leftrightarrow \quad |1-2x| < \tfrac{1}{3}$$

$$\Leftrightarrow \quad -\tfrac{1}{3} < 1-2x < \tfrac{1}{3} \qquad \text{Note: } |x| < c \Leftrightarrow -c < x < c.$$

$$\Leftrightarrow \quad -\tfrac{4}{3} < -2x < -\tfrac{2}{3}$$

$$\Leftrightarrow \quad \tfrac{4}{3} > 2x > \tfrac{2}{3} \qquad \text{Note the change of the direction of the inequality signs.}$$

$$\Leftrightarrow \quad \tfrac{2}{3} > x > \tfrac{1}{3}.$$

Hence the solution is $\frac{2}{3} > x > \frac{1}{3}$ and $x \neq \frac{1}{2}$, which we write as

$$\tfrac{1}{3} < x < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < x < \tfrac{2}{3}.$$

Example 1.7.4

Solve the inequality

$$\frac{x}{|2-x|} - 2 \leq 0.$$

Solution

The inequality is undefined for $x = 2$.

Thus for $x \neq 2$,

$$\frac{x}{|2-x|} - 2 \leq 0$$

$$\Leftrightarrow \qquad \frac{x}{|2-x|} \leq 2$$

$$\Leftrightarrow \qquad x \leq 2\,|2-x|$$

$$\Leftrightarrow \qquad |2-x| \geq \frac{x}{2}.$$

Since we do not know whether x is negative or positive we cannot apply the properties of absolute value inequalities given in Section 1.7 of SRW. We now make use of the definition of absolute value.

We note that

$$|2-x| = \begin{cases} 2-x & \text{if} \quad x < 2 \\ x-2 & \text{if} \quad x > 2. \end{cases}$$

If $x < 2$ then

$$|2-x| \geq \frac{x}{2}$$

$$\Leftrightarrow \qquad 2-x \geq \frac{x}{2}$$

$$\Leftrightarrow \qquad 2 \geq \frac{3x}{2}$$

$$\Leftrightarrow \qquad x \leq \frac{4}{3}.$$

If $x > 2$ then

$$
\begin{aligned}
& |2-x| \geq \frac{x}{2} \\
\Leftrightarrow \quad & x-2 \geq \frac{x}{2} \\
\Leftrightarrow \quad & \frac{x}{2} \geq 2 \\
\Leftrightarrow \quad & x \geq 4.
\end{aligned}
$$

Thus

$$x \leq \tfrac{4}{3} \text{ or } x \geq 4.$$

Additional Exercises 1.7.1

1–2 Solve the inequality. Express the solution in interval notation and sketch the solution set on a number line.

1. $-x-1 < 2x+1 \leq x+2$
2. $x \leq 2x+1 \leq 3x-2$

3–4 Use a rough sketch of an appropriate parabola to solve the inequality.

3. $x^2+5x+6 \geq 0$
4. $3+x-2x^2 > 0$

5–8 Solve the inequality.

5. $\dfrac{3}{|x-2|} > 1$
6. $\dfrac{2}{|3-2x|} \leq 1$
7. $\dfrac{x}{|2x-1|} > 2$
8. $\dfrac{1}{|x-5|} \leq \dfrac{2}{x}$

1.8 COORDINATE GEOMETRY

Assumed knowledge	SRW, all of Section 1.8 except Example 6 and Symmetry: Coordinate plane Graphing regions in the coordinate plane Knowing and applying the distance and midpoint formulas Sketching a graph by plotting points Finding x– and y–intercepts Sketching circles Finding and identifying the equation of a circle
To study	SRW, Section 1.8, Example 6, Symmetry and additional notes below.
Outcomes	After studying this section you should be able to ► sketch the graph of an equation involving absolute value ► give and apply the definitions of symmetry with respect to the y–axis, the x–axis and the origin ► sketch graphs using symmetry.
Exercises	Exercises 1.8: questions 69-79 (odd numbers)

Diagnostic Test 1.8

Exercises 1.8: Do a representative selection of odd numbered questions from 1 to 65 and 81 to 97.

■ Sketching the Graph of an Equation Involving Absolute Value

Study Example 6 in Section 1.8 of SRW and the following example.

Example 1.8.1

Use a table of values to sketch the graph of $y = |x+3|$.

Solution

We set up a table of values.

x	$y = \|x+3\|$	(x,y)
-5	2	$(-5,2)$
-4	1	$(-4,1)$
-3	0	$(-3,0)$
-2	1	$(-2,1)$
-1	2	$(-1,2)$
0	3	$(0,3)$
1	4	$(1,4)$
2	5	$(2,5)$

We plot the points whose coordinates are given in the table and join them.

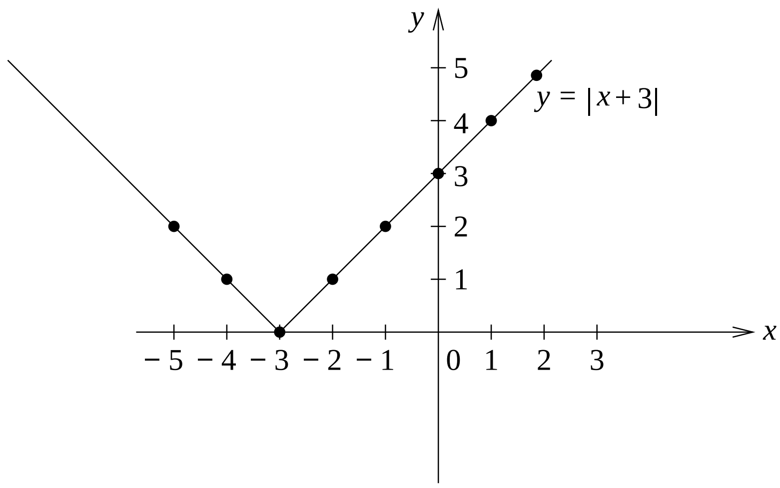


Figure 1.8.1

■ **Symmetry**

Study the relevant subsection in Section 1.8 of SRW.

1.9 GRAPHING CALCULATORS AND COMPUTERS

You may omit this section. However, if you have a graphing calculator or a computer and a graphing package then you may find this section very informative.

1.10 LINES

Assumed knowledge	SRW, Section 1.10: The entire section, i.e. Slope of a line The point–slope form of the equation of a line The slope–intercept form of the equation of a line Vertical and horizontal lines The general equation of a line Parallel and perpendicular lines Applications of linear equations

Diagnostic Test 1.10

Exercises 1.10: Try a representative selection of odd numbered questions.

1.11 MODELING VARIATION

To study	SRW, Section 1.11.
Outcomes	After studying this section you should be able to ► know and apply the concepts of direct variation, inverse variation and joint variation.
Exercises	Exercises 1.11: questions 1-29 (odd numbers), 35, 37, 41

Work through the entire section. If you have studied MAT0511 you will have studied the subsections on direct and inverse variation. However, make sure that you understand this section by doing the exercises referred to above.

1 REVIEW AND TEST

Now assess yourself by attempting the questions in the Concept Check, trying a selection of questions from the Exercises and doing the Test at the end of Chapter 1 in SRW.

2 FUNCTIONS

One of the basic concepts in mathematics is that of a function. In this chapter we study functions and their graphs, transformations and extreme values of functions, ways in which to combine functions and one–to–one functions and their inverses.

2.1 WHAT IS A FUNCTION?

Assumed knowledge	SRW, Section 2.1: The entire section, i.e. Definition of a function Domain and range of a function Evaluating a function Representations of functions

Comments

- The definition of a function given in Section 2.1 of SRW is given in terms of a rule. You may also have come across the following more formal definition.

  Definition of a function

  A function f between two sets A and B is a set of ordered pairs of $A \times B$ in which each element of A is paired with a unique element of B.

Notation

- If f is a function then we use D_f and R_f, respectively, to denote the domain and range of f.

Abbreviation

- We should distinguish between a function f (which is a rule) and the number $f(x)$, which is the value of f at x.

  Nonetheless, it is common to abbreviate an expression such as

  $$\text{the function } f \text{ defined by } f(x) = x^2 + x$$

  to

  $$\text{the function } f(x) = x^2 + x.$$

  We will also use this abbreviation in our study guide.

Natural domain

- If the domain of a function is not stated, then we use the convention that the domain is the set of all real numbers for which the formula makes sense and defines a real number. Some authors call this domain the **natural domain**.

Error

- In the last line above Example 6, Section 2.1, SRW, change $\{x \mid x \neq 0\}$ to $\{x \mid x \geq 0\}$, i.e. the domain of g is $\{x \mid x \geq 0\}$.

Diagnostic Test 2.1

Exercises 2.1: questions 1–57 (odd numbers).

Applications

In the set of exercises for Section 2.1 the applied problems are grouped together under the heading "Applications". Try to answer the odd–numbered questions under this heading.

2.2 GRAPHS OF FUNCTIONS

Assumed knowledge SRW, Section 2.2, except Examples 5–8:
Graphs of constant, linear, quadratic, cubic and square root functions
Graph of a semi–circle
Getting information from the graph of a function
The vertical line test
Equations that define functions

To study SRW, Section 2.2, Examples 5–8.

Outcomes After studying this section you should be able to

- sketch the graph of a reciprocal function
- sketch the graph of a piecewise–defined function (including the absolute value function)
- define a suitable piecewise–defined function according to given information.

Exercises Exercises 2.2: questions 13–21, 37–49 (odd numbers), 53

Comment

You may omit Example 2 in Section 2.2 of SRW. Try to do Example 4 without using a graphing calculator. You should be able to sketch the graph of a semi–circle without using a graphing calculator.

Diagnostic Test 2.2

Exercises 2.2: questions 1–11 (odd numbers), 23, 25, 55–71 (odd numbers), 79, 81.

■ Piecewise–defined Functions

Study Examples 5–8 in Section 2.2 of SRW.

■ Applications

To test your ability to apply the concepts involved in Section 2.2, do the odd–numbered exercises under the heading "Applications" given in Exercises 2.2 of SRW.

2.3 INCREASING AND DECREASING FUNCTIONS; AVERAGE RATE OF CHANGE

To study	SRW, Section 2.3.
Outcomes	After studying this section you should be able to ► recognise the intervals on which a function increases or decreases ► calculate the average rate of change of a function ► apply the concept of average rate of change to functions that model real–world situations.
Exercises	Exercises 2.3: questions 1, 3, 13–35 (odd numbers)

Work through the entire section.

2.4 TRANSFORMATIONS OF FUNCTIONS

To study SRW, Section 2.4, and additional notes.

Outcomes After studying this section you should be able to

- use the graph of $y = f(x)$ to sketch the graphs of $y = f(x) + a$ and $y = f(x+a)$, where $a \in \mathbb{R}$
- use the graph of $y = f(x)$ to sketch the graphs of $y = -f(x)$ and $y = f(-x)$
- use the graph of $y = f(x)$ to sketch the graph of $y = af(x)$, for $a > 1$ and for $0 < a < 1$
- use the graph of $y = f(x)$ to sketch the graph of $y = f(ax)$, for $a > 1$ and for $0 < a < 1$
- use the graph of $y = f(x)$ to sketch the graph of $y = |f(x)|$
- perform in succession several of the processes listed above to obtain, from the graph of $y = f(x)$, the graph of a related function
- recognise from an equation of the form $y = g(x) = af[b(x-c)] + d$; where $a, b, c, d \in \mathbb{R}$, $a \neq 0$ and $b \neq 0$, and f and g are functions; how the graph of g has been constructed from that of f
- determine whether a function f is even, odd or neither
- sketch the graph of an even or odd function by using symmetry
- use the graph of $y = f(x) = |x|$ to sketch the graphs of absolute value functions defined by $g(x) = af(bx-c) + d$; $a, b, c, d \in \mathbb{R}$, $a \neq 0$ and $b \neq 0$.

Exercises Exercises 2.4: questions 1–47 (odd numbers), 53, 55, 61, 63, 69, 71
Additional Exercises 2.4.1: questions 1–10

In question 21 you must sketch the graph of $y = \frac{1}{x}$. This is a hyperbola and we assume that you can draw hyperbolas.

■ Shifting, Reflecting, Shrinking and Stretching of Graphs

We refer to these techniques as the principles of transformation of functions.
There is one further technique, namely reflection of the graph of a one–to–one function in the line $y = x$ to obtain the graph of the inverse function. This will be dealt with in Section 2.8.

Study the first four subsections in Section 2.4 of SRW which deal with techniques to obtain the graph of a function from the graph of some related function. These techniques are very important as you will use them to sketch the graphs of various functions, including absolute value, quadratic, exponential, logarithmic and trigonometric functions. Therefore make sure that you know how to apply these techniques. Once you have completed these subsections in SRW, study the two examples below. The first one shows the difference between obtaining the graphs of

$$y = g(x) = f(a(x+b)) \text{ and } y = k(x) = f(ax+b)$$

from the graph of $y = f(x)$; and the second shows how to obtain the graph of $y = |f(x)|$ from the graph of $y = f(x)$.

Example 2.4.1

Explain how you would obtain the graph of

(a) $y = g(x) = f(3(x+1))$

(b) $y = k(x) = f(3x+1)$

from the graph of $y = f(x)$.

Solution

(a) We begin by *shrinking the graph of f horizontally by a factor of $\frac{1}{3}$* to obtain the graph of

$$y = h(x) = f(3x).$$

Then we *shift the resulting graph one unit to the left* to obtain the graph of

$$y = g(x) = h(x+1) = f(3(x+1)).$$

(b) We begin by *shifting the graph of f one unit to the left* to obtain the graph of

$$y = j(x) = f(x+1).$$

Then we *shrink the resulting graph horizontally by a factor of $\frac{1}{3}$* to obtain the graph of

$$y = g(x) = j(3x) = f(3x+1).$$

Alternative solution

Note that if we rewrite $3(x+1)$ as $3x+3$, we can apply the technique in the solution to (b) above to solve (a); and if we rewrite $3x+1$ as $3\left(x+\frac{1}{3}\right)$ then we can apply the technique used in (a) above to solve (b).

(a) Since $f(3(x+1)) = f(3x+3)$, we begin by *shifting the graph of f three units to the left* to obtain the graph of

$$y = r(x) = f(x+3).$$

Then we *shrink the resulting graph horizontally by a factor of* $\frac{1}{3}$ to obtain the graph of

$$y = g(x) = r(3x) = f(3x+3) = f(3(x+1)).$$

(b) Since $f(3x+1) = f\left(3\left(x+\frac{1}{3}\right)\right)$ we begin by *shrinking the graph of f horizontally by a factor of* $\frac{1}{3}$ to obtain the graph of

$$y = h(x) = f(3x).$$

We then *shift the resulting graph* $\frac{1}{3}$ *of a unit to the left* to obtain the graph of

$$y = k(x) = h\left(x+\tfrac{1}{3}\right) = f\left(3\left(x+\tfrac{1}{3}\right)\right) = f(3x+1).$$

Example 2.4.2

Consider the graph of $y = f(x)$ sketched in Figure 2.4.1. Sketch the graph of $y = |f(x)|$.

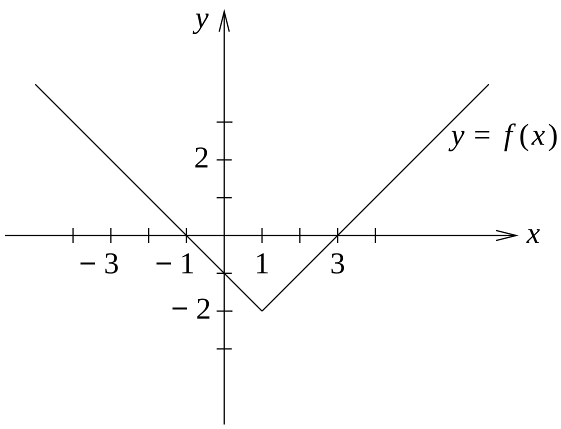


Figure 2.4.1

Solution

Applying the definition of absolute value we have

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

From Figure 2.4.1 we have

$$f(x) \geq 0 \text{ when } x \leq -1 \text{ or } x \geq 3$$

and

$$f(x) < 0 \text{ when } -1 < x < 3.$$

Thus

$$|f(x)| = \begin{cases} f(x) & \text{when} \quad x \le -1 \text{ or } x \ge 3 \\ -f(x) & \text{when} \quad -1 < x < 3. \end{cases}$$

Thus for $x \le -1$ or $x \ge 3$ the graph of $y = |f(x)|$ is the same as that for $y = f(x)$. For $-1 < x < 3$ the graph of $y = |f(x)|$ is the same as that for $y = -f(x)$. Now, for $-1 < x < 3$, we obtain the graph of $y = -f(x)$ by reflecting the graph of $y = f(x)$ in the x–axis. Figure 2.4.2 shows the graph of $y = |f(x)|$.

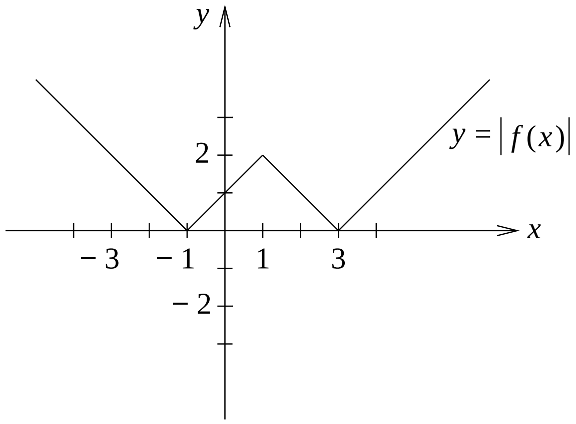


Figure 2.4.2

■ Even and Odd Functions

Study the subsection "Even and Odd Functions" in Section 2.4 of SRW.

■ Graphs of Absolute–value Functions

In Example 6 in Section 2.2 of SRW the graph of $y = f(x) = |x|$ was drawn by considering the function f as a piecewise–defined function. The point at which the two lines meet is called the **salient point**. Thus the salient point of the graph of $y = |x|$ is $(0,0)$.

Salient point

Before drawing the graphs of a few related absolute–value functions we consider two implications of the absolute–value property $|ax| = |a|\,|x|$, for $a \in \mathbb{R}$.

► When $a = -1$ we have $|-x| = |x|$. (1)

► $|ax| = |a|\,|x| = \begin{cases} a|x| & \text{if} \quad a > 0 \\ -a|x| & \text{if} \quad a < 0. \end{cases}$ (2)

Suppose f is the function defined by $y = f(x) = |x|$.

- From (1) it follows that $|-x| = |x|$ and so $f(-x) = f(x)$.

  This means that f is an even function and thus the graph of f is symmetric with respect to the y–axis.

- From (2) it follows that

$$\begin{aligned} y &= f(ax) = |ax| = |a|\ |x| = |a|\, f(x) \quad \text{for } a \neq 0 \\ &= \begin{cases} af(x) & \text{if} \quad a > 0 \\ -af(x) & \text{if} \quad a < 0. \end{cases} \end{aligned}$$

  Consequently the graphs of $y = f(ax)$ and $y = |a|\, f(x)$ are the same. Thus we need only consider $a > 0$ and since $f(ax) = af(x)$ we see that horizontal shrinking coincides with vertical stretching and horizontal stretching coincides with vertical shrinking.

Example 2.4.3

Apply the principles of transformation of functions to sketch (without doing any further calculations) the graphs of the functions given below. In each case sketch the graphs on the same system of axes.

(a) $y = f(x) = |x|$; $y = g(x) = |x| + 3$; $y = h(x) = |x + 2|$

(b) $y = f(x) = |x|$; $y = g(x) = |2x| = 2|x| = |-2x|$;
$y = h(x) = \left|\frac{1}{2}x\right| = \frac{1}{2}|x| = \left|-\frac{1}{2}x\right|$

(c) $y = f(x) = |x|$; $y = g(x) = -|x|$

(d) $y = f(x) = |x|$; $y = g(x) = -2|x - 3| + 1$

Solution

(a)

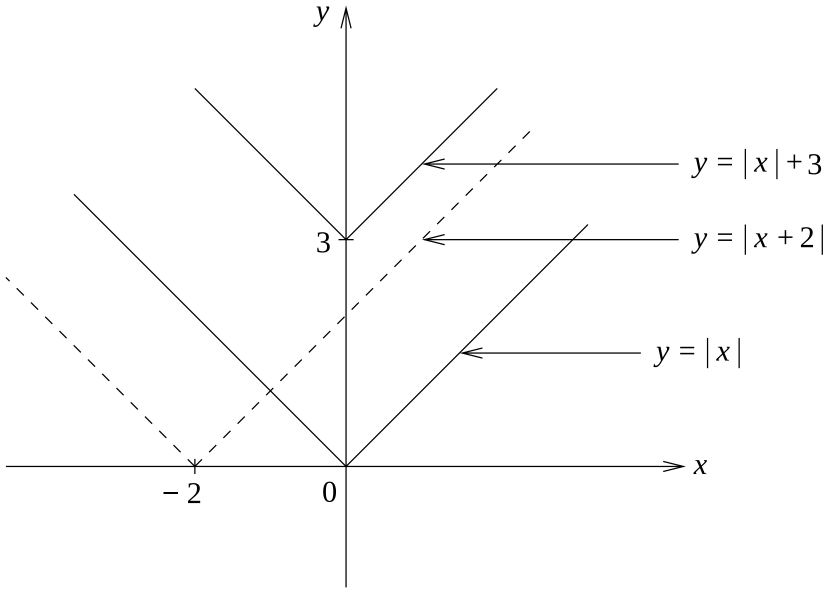


Figure 2.4.3

We note that $g(x) = f(x) + 3$ and $h(x) = f(x+2)$. Thus the graph of g is obtained by shifting the graph of f upwards by three units, and the graph of h is obtained by shifting the graph of f two units to the left.

(b)

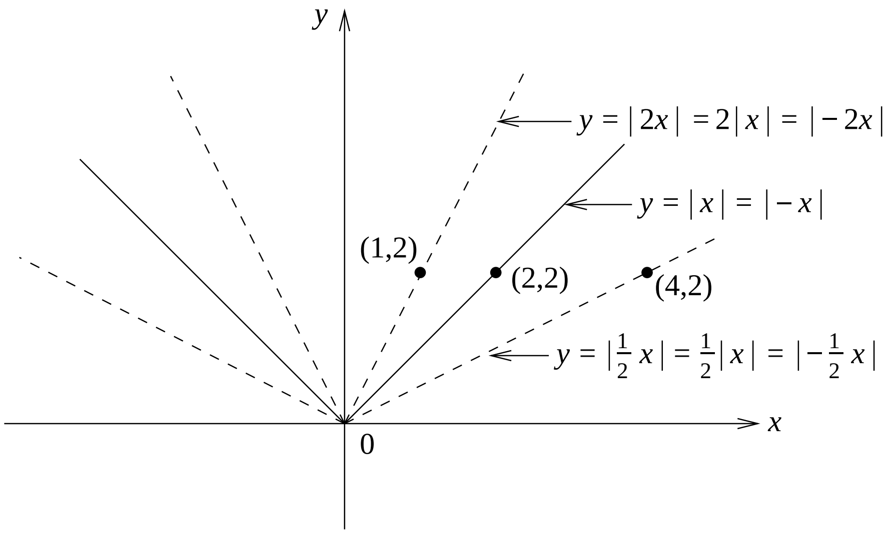


Figure 2.4.4

We note that

$$g(x) = f(2x) = f(-2x) = 2f(x)$$

and

$$h(x) = f\left(\tfrac{1}{2}x\right) = f\left(-\tfrac{1}{2}x\right) = \tfrac{1}{2}f(x).$$

We see that the graph of g is obtained by shrinking the graph of f horizontally by a factor of half, which **in this specific case** (i.e. the absolute–value function) is the same as stretching the graph of f vertically by a factor of two.

Also, the graph of h is obtained by stretching the graph of f horizontally by a factor of two, which, again **in this specific case**, is the same as shrinking the graph of f vertically by a factor of half.

The slopes of the lines defined by $y = f(x)$, $y = g(x)$ and $y = h(x)$ are respectively

$$1,\ 2 \text{ and } \tfrac{1}{2} \text{ for } x \geq 0$$

and

$$-1, -2 \text{ and } -\tfrac{1}{2} \text{ for } x < 0.$$

(c)

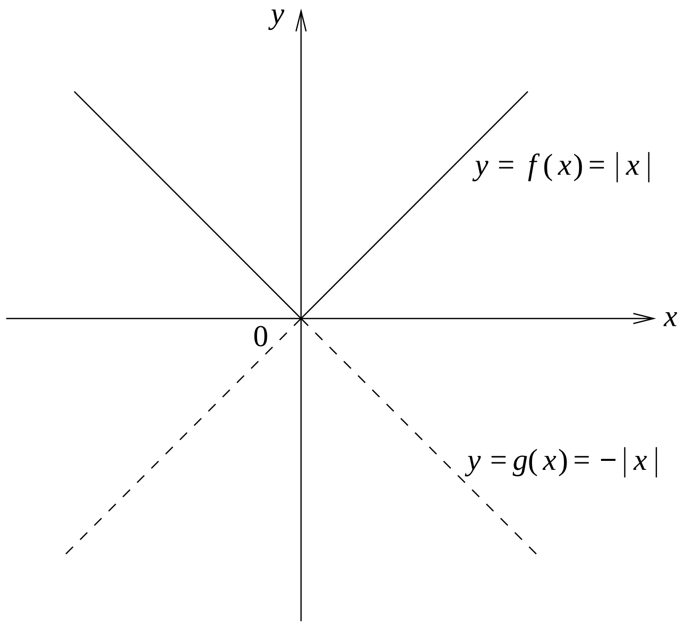


Figure 2.4.5

We note that $g(x) = -f(x)$ and so the graph of g is obtained by reflecting the graph of f in the x–axis.

(d) In Figure 2.4.6 below we have sketched all the intermediate graphs before obtaining the final graph. You need not show all these graphs unless you are specifically asked to do so. Note that $g(x) = -2f(x-3)+1$. The following steps are involved in drawing the graph of g.

1. We sketch the graph of f.
2. We shift the graph of f three units to the right to obtain the graph of $y = f(x-3)$.
3. We stretch the graph of $y = f(x-3)$ vertically by a factor of two to obtain the graph of $y = 2f(x-3)$.
4. We reflect the graph of $y = 2f(x-3)$ in the x–axis to obtain the graph of $y = -2f(x-3)$.
5. Finally we shift the graph of $y = -2f(x-3)$ one unit upwards to obtain the graph of $y = -2f(x-3)+1$.

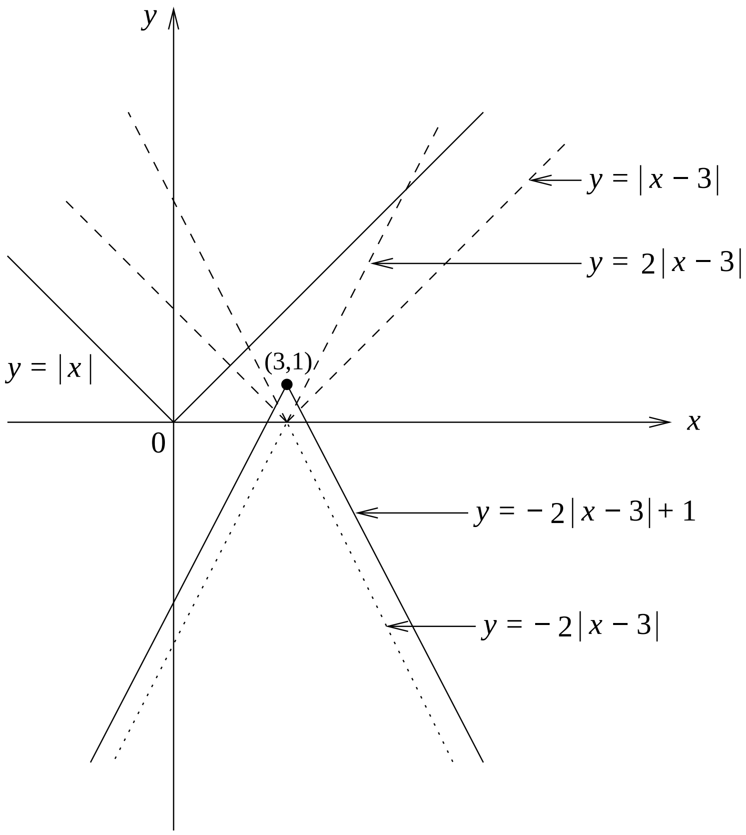


Figure 2.4.6

From Figure 2.4.6 we note that $(3,1)$ is the salient point of the graph of $y = g(x) = -2|x-3|+1$.

Example 2.4.3 leads us to several facts about the graph of f, where f is defined by $y = f(x) = a|x-p| + q$.

- We see that the salient point of the graph is (p,q).
- If $a > 0$ then the graph opens upwards and if $a < 0$ then the graph opens downwards.
- The slopes of the two lines making up the graph are equal to $\pm|a|$.

Example 2.4.4

Apply the principles of transformation of functions to sketch the graph of $y = g(x) = |1-2x| - 3$ using the graph of $y = f(x) = |x|$.

Solution

We have

$$\begin{aligned} g(x) &= |1-2x| - 3 \\ &= \left|-2\left(x-\tfrac{1}{2}\right)\right| - 3 \\ &= 2\left|x-\tfrac{1}{2}\right| - 3. \end{aligned}$$

We go through the following steps to sketch the graph of $y = g(x)$.

1. Sketch the graph of $y = f(x) = |x|$.
2. Stretch the graph of f vertically by a factor of 2 to obtain the graph of $y = 2|x|$.
3. Shift the graph of $y = 2|x|$ half a unit to the right to obtain the graph of $y = 2\left|x-\frac{1}{2}\right|$.
4. Shift the graph of $y = 2\left|x-\frac{1}{2}\right|$ three units downwards to obtain the graph of $y = 2\left|x-\frac{1}{2}\right| - 3$.

The graph is sketched in Figure 2.4.7.

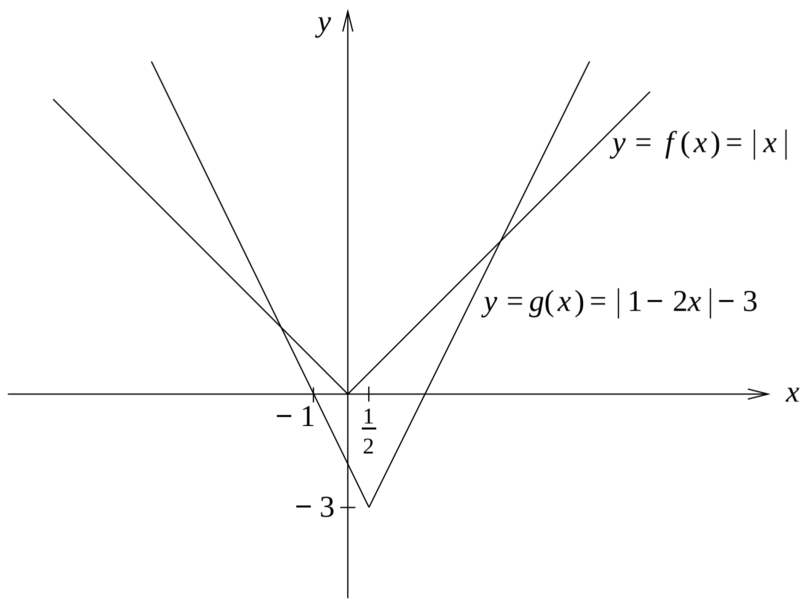


Figure 2.4.7

We could also have reasoned like this.

Rewrite the equation that defines g in the form $y = a|x - p| + q$. Thus

$$y = g(x) = 2\left|x - \tfrac{1}{2}\right| - 3.$$

From the equation we see that

- the salient point of the graph of g is $(p, q) = \left(\frac{1}{2}, -3\right)$
- the graph opens upwards since $a = 2 > 0$
- the gradient of the right section of the graph is 2 and the gradient of the left section is -2.

Additional Exercises 2.4.1

1. Determine whether the function $f(x) = -\frac{1}{x}$ is even or odd and use symmetry to sketch its graph.

2. Without drawing any graphs, explain how you could use the graph of f to sketch the graphs of

   (a) $y = g(x) = f\left(\frac{1}{4}(x+1)\right)$

   (b) $y = h(x) = f\left(\frac{1}{4}x+1\right)$.

3–7 Apply the principles of transformation of functions to sketch (without further calculations) the graphs of the given functions on the same system of axes.

3. $y = f(x) = |x|;\ y = g(x) = |x| - 2;\ y = h(x) = |x-4|$

4. $y = f(x) = |x|;\ y = g(x) = |-3x|;\ y = h(x) = -3|x|$

5. $y = f(x) = |x|;\ y = g(x) = 3|x+1| - 2$

6. $y = f(x) = |x|;\ y = g(x) = |2x-4|$

   (Hint: Note that $|2x-4| = |2(x-2)| = 2|x-2|$.)

7. $y = f(x) = |x|;\ y = g(x) = -|2x-4| + 3$

   (Use the hint given in question 6.)

8–10 Explain without drawing any graphs how you would obtain the graph of the given function g from the function f defined by $y = f(x) = |x|$.

8. $y = g(x) = 3|x+1| - 2$

9. $y = g(x) = -|2x-6| + 1$

   (See the hint given in question 6.)

10. $y = g(x) = |1-x| + 2$

    (Note: $1 - x = -(x-1)$.)

2.5 QUADRATIC FUNCTIONS; MAXIMA AND MINIMA

Assumed knowledge	SRW, Section 2.5: The whole section excluding the subsection on "Using Graphing Devices to Find Extreme Values", i.e. Completing the square to write the quadratic function $f(x) = ax^2 + bx + c$ in the standard form $f(x) = a(x-h)^2 + k$ Drawing the graph of a quadratic function of the form $f(x) = a(x-h)^2 + k$ Finding the maximum or minimum value of a quadratic function Solving applied maximum and minimum problems

Comments

- The authors of SRW have left out certain steps and explanations when they completed the square. Study, for example, the solution to Example 1 in Section 2.5. If we include all the steps we have the following.

$$f(x) = 2x^2 - 12x + 23$$

$$= 2\left(x^2 - 6x\right) + 23$$

Factor out the coefficient of x^2, namely 2, from the x-terms.

Brackets are also known as parentheses.

$$= 2\left(x^2 - 6x + 3^2 - 3^2\right) + 23$$

Inside the brackets, add and subtract the square of half the coefficient of x.

$$= 2\left(x^2 - 6x + 9\right) + 2 \times \left(-3^2\right) + 23$$

Take -3^2 out of the brackets. Remember to multiply this by 2 since the expression in brackets is multiplied by 2.

$$= 2(x-3)^2 - 18 + 23$$

Complete the square inside the brackets.

$$= 2(x-3)^2 + 5$$

- Note how the authors apply the principles of transformation of functions when they sketch parabolas.
- You may omit the subsection of Section 2.5 which deals with the use of graphing devices to find extreme values.

Diagnostic Test 2.5

Exercises 2.5: questions 1-43, 59-65 (odd numbers).

The following example is an application of an extreme value problem to a combination of graphs.

Example 2.5.1

Figure 2.5.1 shows the graphs of the functions f and g defined by

$$y = f(x) = 2x^2 - 8x + 6$$

and

$$y = g(x) = -2x + 6.$$

The graphs intersect at $(0,6)$ and $(3,0)$. M and N are points that lie on the line and parabola respectively, and the line joining them is parallel to the y–axis.

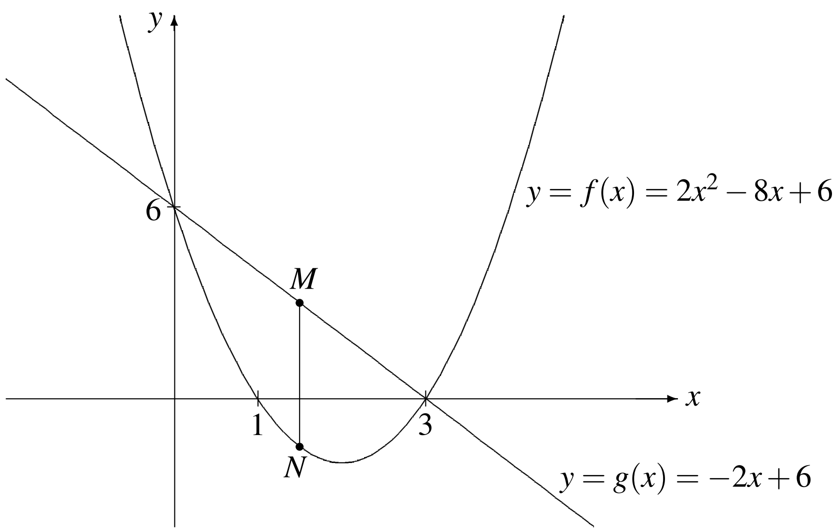


Figure2.5.1

Determine the maximum length of MN if $0 < x < 3$, and find the value of x at which this occurs.

Solution

For $0 < x < 3$ the graph of g lies above the graph of f. The vertical distance between corresponding points on these graphs is given by the difference between the y–values of these points. Thus the length of MN is given by

$$\begin{aligned} d(x) &= g(x) - f(x) \\ &= -2x + 6 - \left(2x^2 - 8x + 6\right) \\ &= -2x^2 + 6x. \end{aligned}$$

The length of MN is thus given by the function d, which is a **quadratic** function of x. (Do not confuse this function with the quadratic function f.)

As x varies from 0 to 3 so the length of MN changes from very short to longer. At some stage it becomes shorter again. Thus the length varies and at some point it will be at its maximum.

The function d has the form

$$d(x) = ax^2 + bx + c$$

where $a = -2,\ b = 6$ and $c = 0$.

Thus, according to SRW, d has a maximum value since $a < 0$ and the maximum value occurs at

$$x = \frac{-b}{2a} = \frac{-6}{-4} = \frac{3}{2}.$$

The maximum value is

$$d\left(\frac{3}{2}\right) = -2\left(\frac{9}{4}\right) + 6\left(\frac{3}{2}\right) = -\frac{9}{2} + \frac{18}{2} = \frac{9}{2} = 4\frac{1}{2}.$$

Thus the maximum length of MN is $4\frac{1}{2}$ units and this occurs at $x = 1\frac{1}{2}$.

We could also reason as follows.

We know that the maximum (or minimum) value of any quadratic function is the y–coordinate of the vertex of its graph. Thus we can also determine the vertex of the graph of d. We have

$$\begin{aligned} d(x) &= -2x^2 + 6x \\ &= -2\left(x^2 - 3x\right) \\ &= -2\left(x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right) \\ &= -2\left(x^2 - 3x + \left(\tfrac{3}{2}\right)^2\right) + 2 \times \left(\tfrac{3}{2}\right)^2 \\ &= -2\left(x - \tfrac{3}{2}\right)^2 + \tfrac{9}{2}. \end{aligned}$$

Thus the vertex of the graph of d is $\left(\frac{3}{2}, \frac{9}{2}\right)$. Hence the maximum value of d is $\frac{9}{2}$, and this occurs at $x = \frac{3}{2}$.

Consequently the maximum length of MN is $4\frac{1}{2}$ units and this occurs at $x = 1\frac{1}{2}$.

2.6 MODELING WITH FUNCTIONS

To Study SRW, the entire Section 2.6, except Example 4.

Outcomes After studying this section you should be able to

- find a function that models a real–life situation
- use the model, once it is found, to analyse properties of the object or process in the real–life situation
- model and solve applied maximum and minimum problems.

Exercises Exercises 2.6: questions 1–23 (odd numbers)

Comment

Study the Guidelines for Modeling with Functions carefully, and apply these steps every time you need to solve a problem like those given in this section. You may omit Example 4.

2.7 COMBINING FUNCTIONS

Assumed knowledge	SRW, first two pages of Section 2.7: Sum, difference, product and quotient of functions and their domains The graph of the sum of two functions by graphical addition
To study	SRW , the rest of Section 2.7.
Outcomes	After studying this section you should be able to ► determine the composition of functions ► specify the domain of a composite function ► define the separate functions that have been used in obtaining a composite function ► set up and interpret a composite function in "application" problems.
Exercises	Exercises 2.7: questions 17–53 (odd numbers), 59

Comment

On p. 21 of this study guide we introduced the notation D_f to denote the domain of a function f. According to this notation we can rewrite the table on the first page of Section 2.7, SRW, as follows.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains D_f and D_g. Then the functions $f+g$, $f-g$, fg and f/g are defined as follows.

$(f+g)(x) = f(x)+g(x)$	$D_{f+g} = D_f \cap D_g$
$(f-g)(x) = f(x)-g(x)$	$D_{f-g} = D_f \cap D_g$
$(fg)(x) = f(x)\,g(x)$	$D_{fg} = D_f \cap D_g$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	$D_{\frac{f}{g}} = D_f \cap D_g - \{x \in D_g \mid g(x) = 0\}$

This notation makes it quite clear to which domain we are referring. For example, in order to avoid confusion, we could begin the solution of Example 1(a) in Section 2.7 of SRW as follows.

(a) $D_f = \{x \in \mathbb{R} \mid x \neq 2\}$ and $D_g = \{x \in \mathbb{R} \mid x \geq 0\}$.
$D_f \cap D_g = \{x \in \mathbb{R} \mid x \geq 0 \text{ and } x \neq 2\}$.

Thus, we have

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) = \frac{1}{x-2} + \sqrt{x} \\ D_{f+g} &= \{x \in \mathbb{R} \mid x \geq 0 \text{ and } x \neq 2\}, \text{ etc.} \end{aligned}$$

Diagnostic Test 2.7

Exercises 2.7: questions 1,3,5,7,9.

■ Composition of Functions

Work through this subsection of Section 2.7 in SRW carefully, making sure that you understand each step in the solutions of the examples. Take particular note of the following comments.

- ► According to the definition of the composition $f \circ g$ of f and g we have

  $$D_{f \circ g} = \left\{x \in D_g \mid g(x) \in D_f\right\}.$$

  You can determine this set by finding all values of x for which $g(x)$ **and** $(f \circ g)(x)$ are defined. Thus, when you find the domain of $f \circ g$, you must not only consider the values of x for which $f \circ g$ is defined, but also the values of x for which g is defined. See, for example, Example 4, Section 2.7, SRW.

- ► In general

  $$(g \circ f)(x) \neq (f \circ g)(x).$$

  See Example 3, Section 2.7, SRW.

- ► When we decompose a function into two or more functions, the functions may not necessarily be uniquely defined. Consider the following example.

Example 2.7.1

Suppose k is the function defined by $k(x) = \sqrt{x^2+1}$.

(a) Determine functions f and g such that $(f \circ g)(x) = k(x)$.

(b) Determine functions f, g and h such that $(f \circ g \circ h)(x) = k(x)$.

Solution

We consider the processes that have been applied to x in order to arrive at the equation

$$y = \sqrt{x^2+1}.$$

We see that

x has first been squared
1 has been added to x^2
the square root of x^2+1 has been determined.

(a) We have three processes above but we are looking for two functions f and g, whose composite is k. We combine the processes in *two different ways* which shows that the functions f and g are not uniquely defined.

1. We combine the first two processes and let g be the function defined by $g(x) = x^2+1$. We consider the third process and let f be the function defined by $f(x) = \sqrt{x}$. We now check to see whether these two functions satisfy the condition that

$$(f \circ g)(x) = k(x).$$

Check:
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(x^2+1\right) \\ &= \sqrt{x^2+1} \\ &= k(x)\end{aligned}$$

2. We consider the first process and let the function g be defined by $g(x) = x^2$. We now combine the last two processes and let the function f be defined by $f(x) = \sqrt{x+1}$. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(x^2\right) \\ &= \sqrt{x^2+1}.\end{aligned}$$

(b) In this case we are looking for three functions f, g and h whose composite is k. We consider each of the processes in turn and define the functions f, g and h by

$$h(x) = x^2, \;\; g(x) = x+1 \;\text{ and }\; f(x) = \sqrt{x}.$$

Then

$$\begin{aligned}(f \circ g \circ h)(x) &= (f \circ g)(h(x)) \\ &= (f \circ g)\left(x^2\right) \\ &= f\left(g\left(x^2\right)\right) \\ &= f\left(x^2+1\right) \\ &= \sqrt{x^2+1}.\end{aligned}$$

In this case the functions are uniquely defined.

2.8 ONE−TO−ONE FUNCTIONS AND THEIR INVERSES

To study SRW, the entire Section 2.8, and the additional notes below.

Outcomes After studying this section you should be able to

- ► determine whether a function is one–to–one by applying the definition of a one–to–one function
- ► determine whether a function is one–to–one by applying the horizontal line test to the graph of the function
- ► use the inverse function property to determine whether two given functions are inverses of each other
- ► determine the equation of the inverse function of a given one–to–one function
- ► restrict the domain of a function that is not one–to–one so that the resulting function is one–to-one
- ► obtain the graph of the inverse function f^{-1} by reflecting the graph of the function f in the line $y = x$.

Exercises Exercises 2.8: questions 1–53 (odd numbers), 65, 67, 69
Additional Exercises 2.7.1: questions 1–3

■ One–to–one Functions

Read the first two pages of Section 2.8 in SRW and take note of the following comments.

- ▶ If you use the definition of a one–to–one function to show that a function f is one–to–one, then you can show that for all $x_1,\, x_2 \in D_f$

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

- ▶ If you use the horizontal line test to show that a function is (or is not) one–to–one please **DO NOT** just say “By the horizontal line test the function is (or is not) one–to–one”. You must include your reasoning as is done in Solution 2 of Examples 1 and 2.

■ The Inverse of a Function

Read the rest of Section 2.8 in SRW and note the following.

- ▶ By using our notation for the domain and range of a function we can rewrite the last two lines above Example 4 as

$$\begin{aligned} D_{f^{-1}} &= R_f \\ R_{f^{-1}} &= D_f. \end{aligned}$$

- ▶ From the definition and property of inverse functions it follows that the inverse function f^{-1} of the function f is one–to–one and that the inverse of f^{-1}, i.e. $\left(f^{-1}\right)^{-1}$, is f.

- ▶ If you want to show that a function g is the inverse of a function f then you **must** check **two** conditions, namely

$$(g \circ f)(x) = x \text{ for every } x \in D_f$$

and

$$(f \circ g)(x) = x \text{ for every } x \in D_g.$$

- ▶ Once you have found the inverse function f^{-1} of a function f you should check that it satisfies the two conditions above to make sure that you have not made a mistake in determining the equation that defines f^{-1}. Check that

$$\left(f^{-1} \circ f\right)(x) = x \text{ for every } x \in D_f$$

and

$$\left(f \circ f^{-1}\right)(x) = x \text{ for every } x \in D_{f^{-1}}.$$

► In the solution to Example 8(c) it is important to keep in mind that since $y = \sqrt{x-2}$ we have $y \geq 0$. Then, when the inverse function is determined, namely $f^{-1}(x) = x^2 + 2$, we must have $x \geq 0$. Note that the function $g(x) = x^2 + 2$, where $D_g = \mathbb{R}$, is **not** the inverse of f, because g is not one–to–one. (Why not? Consider the graph of g.)

■ Inverse Functions on Restricted Domains

After Example 2 and in Figure 5 it was shown that it is possible to restrict the domain of a function which is not one–to–one so that the resulting function, which we call the **restricted function**, is one–to–one. We now consider two examples in which we restrict the domain of a function which is not one–to–one and then find the inverse of the restricted function.

Example 2.8.1

The graph of $y = f(x) = 2(x-1)^2 + 3$ is given in Figure 2.8.1.

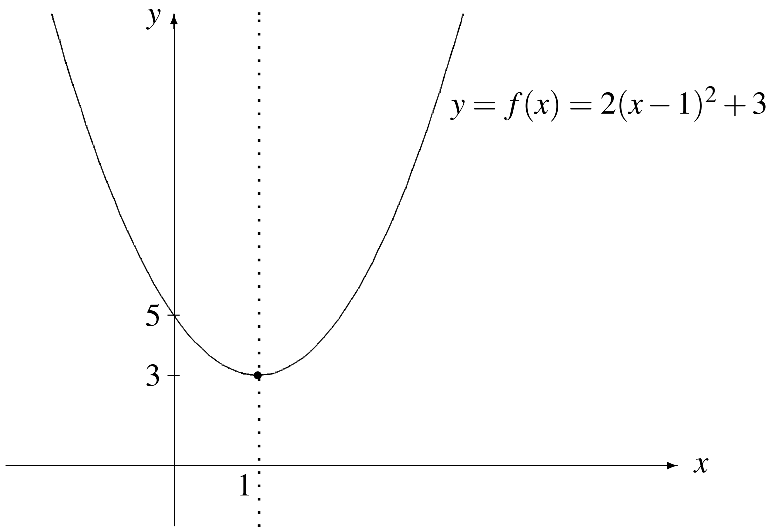


Figure 2.8.1

Restrict the domain of f to D_{f_r} so that the restricted function f_r is one–to–one. Find the inverse function f_r^{-1}. Also give the domain and range of f_r and f_r^{-1}.

Solution

We first note that the parabola is symmetric about the line $x = 1$. If we restrict the domain to $\{x \in \mathbb{R} \mid x \geq 1\}$ we see that f is one–to–one on this domain, since there is no horizontal line which intersects this part of the graph more than once.

We denote the function which is restricted to the domain $\{x \in \mathbb{R} \mid x \geq 1\}$ by f_r. Thus we have

$$f_r(x) = f(x) \text{ for } x \in D_{f_r} = \{x \in \mathbb{R} \mid x \geq 1\}.$$

From the graph we see that

$$R_{f_r} = R_f = \{y \in \mathbb{R} \mid y \geq 3\}.$$

Now

$$\begin{aligned}
& & y &= 2(x-1)^2 + 3 \\
&\Leftrightarrow & 2(x-1)^2 &= y - 3 \\
&\Leftrightarrow & (x-1)^2 &= \frac{y-3}{2} \\
&\Leftrightarrow & (x-1) &= \pm\sqrt{\frac{y-3}{2}} \\
&\Leftrightarrow & x &= 1 \pm \sqrt{\frac{y-3}{2}}. \\
&\text{Thus} & x &= 1 + \sqrt{\frac{y-3}{2}}. && \text{Since } x \geq 1.
\end{aligned}$$

We interchange x and y and obtain

$$y = 1 + \sqrt{\frac{x-3}{2}}.$$

Thus

$$f_r^{-1}(x) = 1 + \sqrt{\frac{x-3}{2}}.$$

We see that

$$D_{f_r^{-1}} = \{x \in \mathbb{R} \mid x \geq 3\} = R_{f_r} \text{ and } R_{f_r^{-1}} = \{y \in \mathbb{R} \mid y \geq 1\} = D_{f_r}.$$

Check:

For every $x \in D_{f_r}$

$$\begin{aligned}
\left(f_r^{-1} \circ f_r\right)(x) &= f_r^{-1}\left(f_r(x)\right) \\
&= f_r^{-1}\left(2(x-1)^2+3\right) \\
&= 1+\sqrt{\frac{2(x-1)^2+3-3}{2}} \\
&= 1+\sqrt{\frac{2(x-1)^2}{2}} \\
&= 1+\sqrt{(x-1)^2} \\
&= 1+x-1 \\
&= x.
\end{aligned}$$

Also for every $x \in D_{f_r^{-1}}$

$$\begin{aligned}
\left(f_r \circ f_r^{-1}\right)(x) &= f_r\left(f_r^{-1}(x)\right) \\
&= f_r\left(1+\sqrt{\frac{x-3}{2}}\right) \\
&= 2\left(1+\sqrt{\frac{x-3}{2}}-1\right)^2+3 \\
&= 2\left(\sqrt{\frac{x-3}{2}}\right)^2+3 \\
&= 2\left(\frac{x-3}{2}\right)+3 \\
&= x-3+3 \\
&= x.
\end{aligned}$$

Alternative

We can also restrict the domain to $\{x \in \mathbb{R} \mid x \leq 1\}$. In this case we have

$$f_r(x) = f(x) \quad \text{for } x \in D_{f_r} = \{x \in \mathbb{R} \mid x \leq 1\}.$$

Again

$$R_{f_r} = \{y \in \mathbb{R} \mid y \geq 3\}.$$

Following the same procedure as that above we have

$$y = 2(x-1)^2+3$$

$$\Leftrightarrow \qquad x = 1 \pm \sqrt{\frac{y-3}{2}}.$$

Now since $x \leq 1$ we choose

$$x = 1 - \sqrt{\frac{y-3}{2}}.$$

Consequently $f_r^{-1}(x) = 1 - \sqrt{\frac{x-3}{2}}$.

Also

$$D_{f_r^{-1}} = \{x \in \mathbb{R} \mid x \geq 3\} = R_{f_r} \text{ and } R_{f_r^{-1}} = \{y \in \mathbb{R} \mid y \leq 1\} = D_{f_r}.$$

Check for yourself that

$$\begin{aligned}\left(f_r^{-1} \circ f_r\right)(x) &= x \text{ for every } x \in D_{f_r} \\ &\textbf{and} \\ \left(f_r \circ f_r^{-1}\right)(x) &= x \text{ for every } x \in D_{f_r^{-1}}.\end{aligned}$$

Example 2.8.2

The graph of $y = f(x) = -2|x+1| + 2$ is given in Figure 2.8.2.

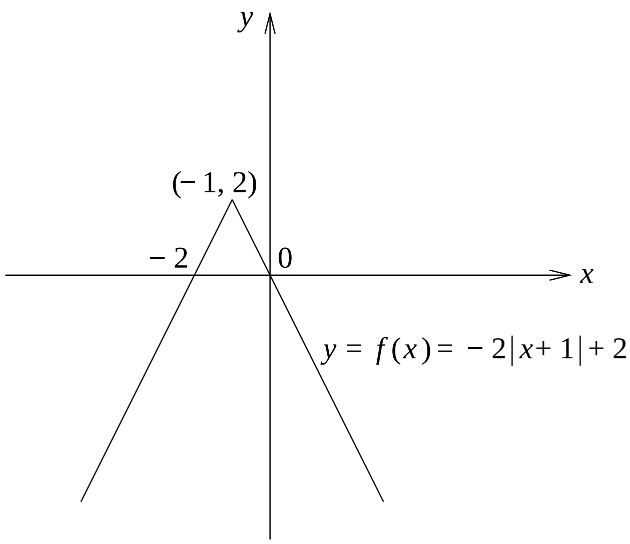


Figure 2.8.2

Restrict the domain of f to D_{f_r} so that the restricted function f_r is one–to–one. Find the inverse f_r^{-1} of the restricted function. Also give $D_{f_r^{-1}}$ and $R_{f_r^{-1}}$.

Solution

The axis of symmetry of the graph of f is the line $x = -1$. We choose the restricted domain to be

$$D_{f_r} = \{x \in \mathbb{R} \mid x \leq -1\}.$$

From the graph we see that

$$R_{f_r} = \{y \in \mathbb{R} \mid y \leq 2\}.$$

Now

$$f_r(x) = f(x) = -2|x+1| + 2 \text{ for } x \in D_{f_r}.$$

Since $x \leq -1$ we have

$$f_r(x) = -2(-x-1) + 2 = 2x + 4.$$

Now for $x \leq -1$ and $y \leq 2$

$$\begin{array}{lrcl} & y &=& 2x+4 \\ \Leftrightarrow & 2x &=& y-4 \\ \Leftrightarrow & x &=& \dfrac{y-4}{2}. \end{array}$$

We interchange x and y and for $y \leq -1$ and $x \leq 2$ we obtain

$$y = \frac{x-4}{2}.$$

Hence

$$f_r^{-1}(x) = \frac{x-4}{2} \text{and}$$

$$D_{f_r^{-1}} = \{x \in \mathbb{R} \mid x \leq 2\} \text{ and } R_{f_r^{-1}} = \{y \in \mathbb{R} \mid y \leq -1\}.$$

Check:

For every $x \in D_{f_r}$

$$\begin{aligned} \left(f_r^{-1} \circ f_r\right)(x) &= f_r^{-1}(f_r(x)) \\ &= f_r^{-1}(2x+4) \\ &= \frac{2x+4-4}{2} \\ &= x. \end{aligned}$$

Also for every $x \in D_{f_r^{-1}}$

$$\begin{aligned}
\left(f_r \circ f_r^{-1}\right)(x) &= f_r\left(f_r^{-1}(x)\right) \\
&= f_r\left(\frac{x-4}{2}\right) \\
&= 2\left(\frac{x-4}{2}\right)+4 \\
&= x-4+4 \\
&= x.
\end{aligned}$$

For interest we give the graphs of f_r and f_r^{-1} on the same system of axes in Figure 2.8.3.

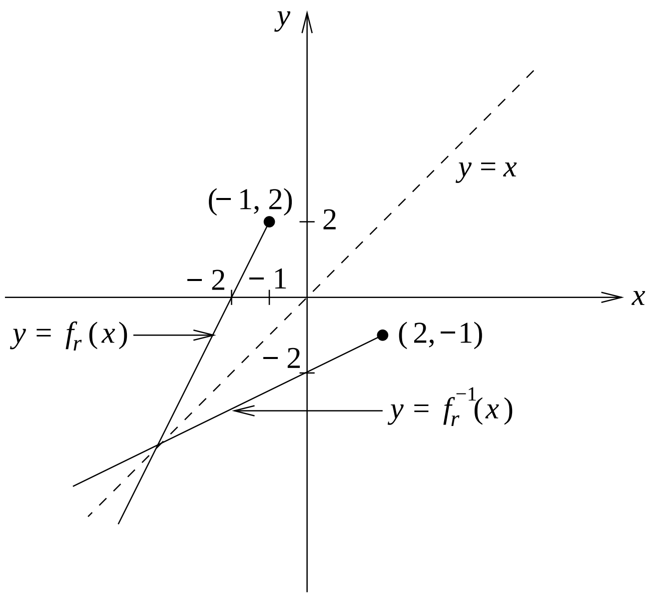


Figure 2.8.3

It is quite clear from Figure 2.8.3 that the graph of $y = f_r^{-1}(x)$ is the reflection of the graph of $y = f_r(x)$ in the line $y = x$.

As an exercise do the same problem with $D_{f_r} = \{x \in \mathbb{R} \mid x \geq -1\}$.

Additional Exercises 2.8.1

1. The composition of functions is associative, i.e.

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

   Use this property to show that the inverse function f^{-1} of a one–to–one function f is unique.

   Hint: Suppose f has an inverse function other than f^{-1}, which we call g. Show that $g = f^{-1}$.

2. The function g is defined by

$$y = g(x) = -\tfrac{1}{3}(x+3)^2 + 1.$$

   Restrict the domain of g so that the restricted function g_r is one–to–one. Find the inverse g_r^{-1} and write down the domain and range of both g_r and g_r^{-1}. Check that g_r^{-1} is the inverse of g_r.

3. Answer the question in Example 2.8.2 by letting $D_{f_r} = \{x \in \mathbb{R} \mid x \geq -1\}$. Sketch the graphs of $y = f_r(x)$ and $y = f_r^{-1}(x)$ on the same system of axes.

2.9 COMBINATION OF GRAPHS

We have included this additional section in the study guide and it does not relate to a specific section in SRW. Since you have now studied absolute value functions we look briefly at a few problems which deal with combinations of the graphs of absolute value functions and some other familiar functions.

To study	The additional notes below.
Outcomes	After studying this section you should be able to ► interpret the information presented by combinations of graphs of absolute value functions and other familiar functions, such as lines, parabolas, etc.
Exercises	Additional Exercises: 2.9.1: questions 1–4

■ Graphs of Absolute Value Functions and other Functions

Example 2.9.1

Suppose the functions f, g and h are defined by

$$y = f(x) = |2x-3|;\ \ y = g(x) = 4 \ \text{ and } \ y = h(x) = x.$$

(a) Draw the graphs of the three functions on the same system of axes.

(b) Find the points of intersection of the graphs of

- (i) f and g
- (ii) f and h.

(c) Use the graphs to answer the following questions.

- (i) For which values of x is $|2x-3| \geq 4$?
- (ii) For which values of x is $|2x-3| < x$?

(d) What is the vertical distance between the graphs of f and g at $x = 1$?

(e) Restrict the domain of f such that $D_{f_r} = \left\{x \in \mathbb{R} \mid x \leq \frac{3}{2}\right\}$. State why f_r is one–to–one and find the function f_r^{-1}. Also give $D_{f_r^{-1}}$ and $R_{f_r^{-1}}$.

(f) Sketch the graphs of $y = f_r(x)$ and $y = f_r^{-1}(x)$ on the same system of axes.

Solution

(a) We rewrite the equation defining f in order to draw its graph:

$$y = f(x) = |2x - 3| = 2\left|x - \tfrac{3}{2}\right|.$$

The three graphs are given in Figure 2.9.1.

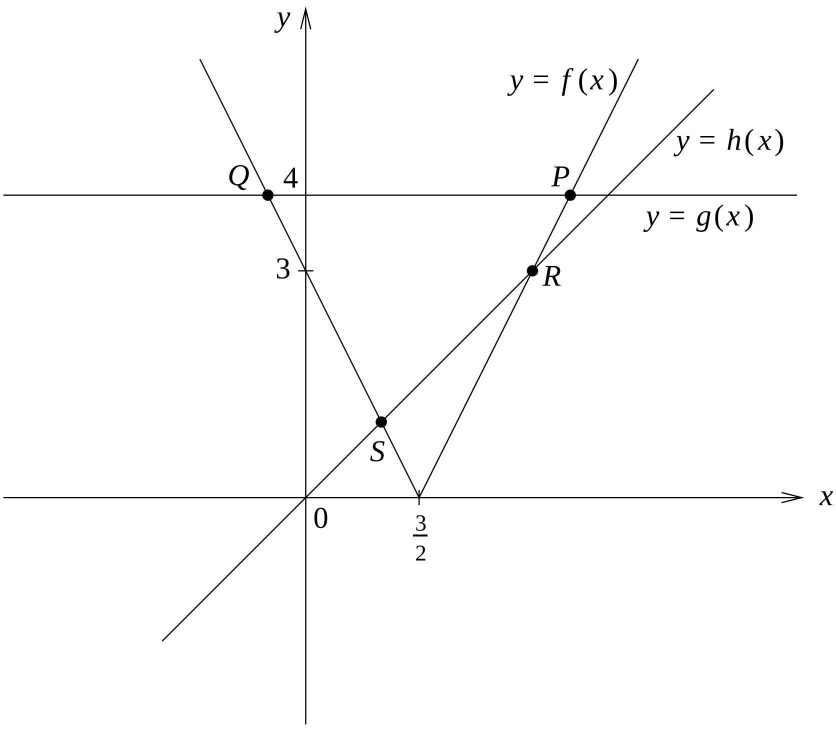


Figure 2.9.1

(b) Since

$$|2x-3| = \begin{cases} 2x-3 & \text{if } 2x-3 \geq 0 \\ -(2x-3) & \text{if } 2x-3 < 0 \end{cases}$$
$$= \begin{cases} 2x-3 & \text{if } x \geq \frac{3}{2} \\ -2x+3 & \text{if } x < \frac{3}{2} \end{cases}$$

we must find the points of intersection of the graphs of f and g and of f and h by using the two different formulas for $|2x-3|$.

(i) If $x \geq \frac{3}{2}$ then $P(x,y)$, which is one point of intersection of the graphs of f and g, must satisfy

$$\left.\begin{array}{l} y = 2x-3 \\ y = 4 \end{array}\right\}$$

simultaneously.

Thus $2x-3=4$, i.e. $2x=7$, i.e. $x=\frac{7}{2}$.

Hence $P=\left(\frac{7}{2},4\right)$.

If $x<\frac{3}{2}$ then $Q(x,y)$, the other point of intersection of the graphs of f and g, must satisfy

$$\left.\begin{array}{l} y = -2x+3 \\ y = 4 \end{array}\right\}$$

simultaneously.

Thus $-2x+3=4$, i.e. $-2x=1$, i.e. $x=-\frac{1}{2}$.

Hence $Q=\left(-\frac{1}{2},4\right)$.

(ii) If $x \geq \frac{3}{2}$, then $R(x,y)$, which is one point of intersection of the graphs of f and h, must satisfy

$$\left.\begin{array}{l} y = 2x-3 \\ y = x \end{array}\right\}$$

simultaneously.

Thus $2x-3=x$, i.e. $x=3$ and hence $y=3$.

So $R=(3,3)$.

If $x<\frac{3}{2}$ then $S(x,y)$, the other point of intersection of the graphs of f and h, must satisfy

$$\left.\begin{array}{l} y = -2x+3 \\ y = x \end{array}\right\}$$

simultaneously.

Hence $-2x+3=x$, i.e. $x=1$ and hence $y=1$.

Thus $S=(1,1)$.

These points of intersection are indicated in Figure 2.9.1.

(c) (i) In order to use the graphs to answer the question we must express

$$|2x-3| \geq 4 \tag{1}$$

as an inequality involving f, g and/or h.

The inequality (1) can be expressed as

$$f(x) \geq g(x).$$

Thus we must determine the values of x for which the graph of f lies above or on the graph of g. From Figure 2.9.1 we see that

$$f(x) \geq g(x) \text{ if } x \leq -\tfrac{1}{2} \text{ or } x \geq \tfrac{7}{2}.$$

Remember that the x–coordinate of Q is $-\frac{1}{2}$ and the x–coordinate of P is $\frac{7}{2}$.

(ii) The inequality $|2x-3| < x$ can be expressed as $f(x) < h(x)$. Now the values of x for which $f(x) < h(x)$ are the values of x for which the graph of f lies below the graph of h. From Figure 2.9.1 we see that

$$f(x) < h(x) \text{ if } 1 < x < 3.$$

(d) At $x = 1$ the graph of g lies above the graph of f. Thus the vertical distance between them is given by $d(x)$ where

$$\begin{aligned} d(x) &= g(x) - f(x) \\ &= 4 - |2x-3|. \end{aligned}$$

Hence the vertical distance when $x = 1$ is

$$\begin{aligned} d(1) &= 4 - |2-3| \\ &= 4 - |-1| \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

(e) If $x \leq \frac{3}{2}$ then

$$f_r(x) = |2x-3| = -2x+3.$$

Also $D_{f_r} = \left\{x \in \mathbb{R} \mid x \leq \frac{3}{2}\right\}$ and $R_{f_r} = \{y \in \mathbb{R} \mid y \geq 0\}$.

Now

$$\begin{aligned} & & y &= -2x+3 \\ &\Leftrightarrow & 2x &= 3-y \\ &\Leftrightarrow & x &= \frac{3-y}{2}. \end{aligned}$$

When we interchange x and y we obtain

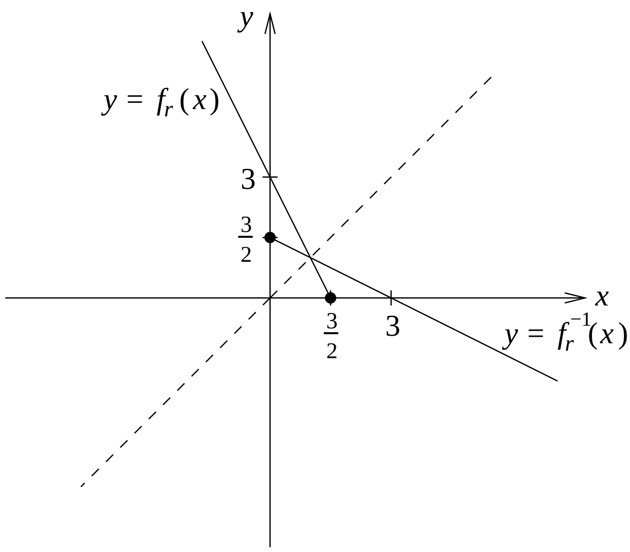
$$y = \frac{3-x}{2}.$$

Hence

$$f_r^{-1}(x) = \frac{3-x}{2}.$$

Now $D_{f_r^{-1}} = R_{f_r} = \{x \in \mathbb{R} \mid x \geq 0\}$ and $R_{f_r^{-1}} = D_{f_r} = \left\{y \in \mathbb{R} \mid y \leq \frac{3}{2}\right\}$.

(f)



Example 2.9.2

Consider the graphs sketched in Figure 2.9.2.

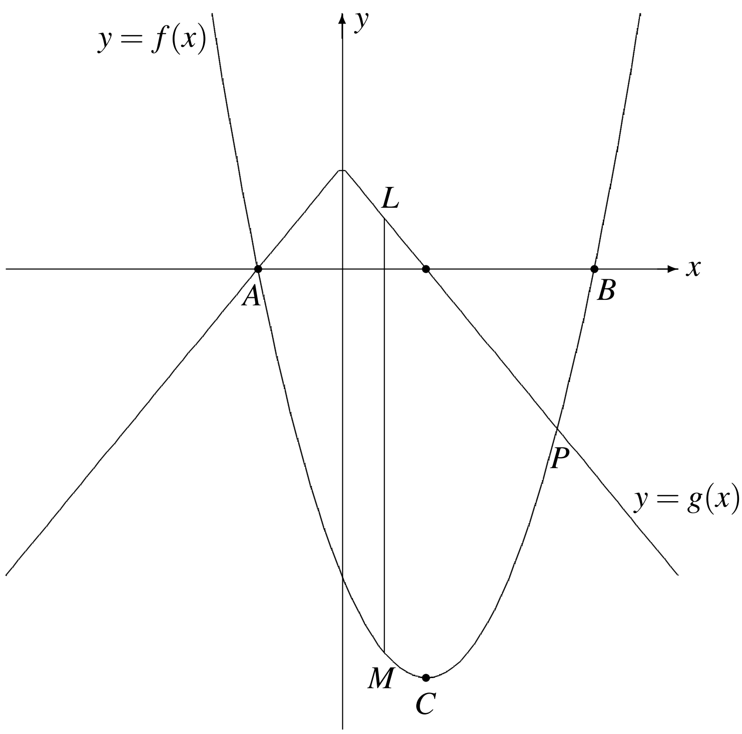


Figure 2.9.2

The function f is defined by

$$y = f(x) = \frac{1}{2}x^2 - 2x - 6$$

and the function g is defined by

$$y = g(x) = a|x| + 2.$$

LM is parallel to the y–axis and C is the vertex of the parabola.

(a) Calculate the x–intercepts of the graph of f.

(b) Use the x–intercepts of the parabola to find the coordinates of C.

(c) Determine the value of a.

(d) What is the maximum length of LM, if LM lies between A and P? How far away is the line LM from B when it is at its maximum length?

(e) From the graph determine the values of x such that $f(x)\,g(x) \geq 0$?

(f) Restrict the domain of f such that the restricted function f_r is one–to–one. Find the inverse f_r^{-1} of the restricted function f_r. Also give $D_{f_r^{-1}}$ and $R_{f_r^{-1}}$.

Solution

(a) We have

$$y = f(x) = \frac{1}{2}x^2 - 2x - 6.$$

Let $y = 0$. Then

$$\begin{aligned} & \tfrac{1}{2}x^2 - 2x - 6 = 0 \\ \Leftrightarrow\quad & x^2 - 4x - 12 = 0 \\ \Leftrightarrow\quad & (x-6)(x+2) = 0 \\ \Leftrightarrow\quad & x = 6 \text{ or } x = -2. \end{aligned}$$

Hence the x–intercepts are -2 and 6.

(b) The graph is symmetrical about the line through C, parallel to the y–axis. Consequently the line through C cuts the x–axis half way between the two x–intercepts, and hence the x–coordinate of C is 2. Substituting $x = 2$ into the equation that defines f gives

$$\begin{aligned} y &= \tfrac{1}{2}(2)^2 - 2(2) - 6 \\ &= 2 - 4 - 6 \\ &= -8. \end{aligned}$$

The coordinates of C are thus $(2, -8)$.

(c) We have

$$g(x) = a|x| + 2.$$

The graph of g cuts the x–axis at A. From (a) we have $A = (-2,0)$. We substitute $(-2,0)$ into the equation for g.

$$\begin{aligned} & 0 = a|-2| + 2 \\ \Leftrightarrow\quad & 0 = 2a + 2 \\ \Leftrightarrow\quad & 2a = -2 \\ \Leftrightarrow\quad & a = -1 \end{aligned}$$

(d) For values of x between the points A and P, L lies above M, and hence the length of LM is given by $d(x) = g(x) - f(x)$. Now

$$d(x) = g(x) - f(x) = (-|x| + 2) - \left(\tfrac{1}{2}x^2 - 2x - 6\right).$$

Since $x < 0$ or $x \geq 0$ between A and P we consider separately $x < 0$ and $x \geq 0$.

Case 1: $x < 0 \Rightarrow |x| = -x$

Then

$$\begin{aligned} d(x) = g(x) - f(x) &= (-(-x) + 2) - \left(\tfrac{1}{2}x^2 - 2x - 6\right) \\ &= x + 2 - \tfrac{1}{2}x^2 + 2x + 6 \\ &= -\tfrac{1}{2}x^2 + 3x + 8. \qquad (1) \end{aligned}$$

Case 2: $x \geq 0 \Rightarrow |x| = x$

Then

$$\begin{aligned} d(x) = g(x) - f(x) &= (-x + 2) - \left(\tfrac{1}{2}x^2 - 2x - 6\right) \\ &= -x + 2 - \tfrac{1}{2}x^2 + 2x + 6 \\ &= -\tfrac{1}{2}x^2 + x + 8. \qquad (2) \end{aligned}$$

The parabolas have a *highest* point because the coefficient of x^2 is negative.

The length of LM is thus given by equation (1) or (2). These equations represent parabolas, with a highest point. At the highest point, $d(x)$ reaches its maximum value. The maximum value of $d(x)$ is the y–coordinate of the vertex of the parabola that represents the function d. We write $d(x)$ in the form $y = a(x-h)^2 + k$.

For (1):

$$\begin{aligned} d(x) &= -\tfrac{1}{2}\left(x^2-6x\right)+8 \\ &= -\tfrac{1}{2}\left(x^2-6x+9\right)+\tfrac{1}{2}(9)+8 \\ &= -\tfrac{1}{2}(x-3)^2+12\tfrac{1}{2}. \end{aligned}$$

The vertex (h,k) is thus $\left(3,12\frac{1}{2}\right)$; hence the maximum length of LM is $12\frac{1}{2}$ units. But this occurs when $x=3$, and for this case we have considered only values of x such that $x<0$. Hence this is not a possible value of $d(x)$.

For (2):

$$\begin{aligned} d(x) &= -\tfrac{1}{2}\left(x^2-2x\right)+8 \\ &= -\tfrac{1}{2}\left(x^2-2x+1\right)+\tfrac{1}{2}+8 \\ &= -\tfrac{1}{2}(x-1)^2+8\tfrac{1}{2}. \end{aligned}$$

Hence $(h,k)=\left(1,8\frac{1}{2}\right)$ and thus the maximum length of LM is $8\frac{1}{2}$ units. This occurs when $x=1$, i.e. when the line LM is 5 units away from B.

(e) $f(x)g(x)\geq 0$ when the graphs of f and g are simultaneously above, or simultaneously below, or on, the x–axis. The graphs are never simultaneously above the x–axis. Both graphs are below the x–axis between the point of intersection of the graph of g with the positive x–axis, and B. The positive x–intercept of the graph of g is 2 (by symmetry, since $A=(-2,0)$). Also, the graphs cut the x–axis at -2, 2 and 6.

Hence $f(x)g(x)\geq 0 \Leftrightarrow x=-2$ or $2\leq x\leq 6$.

(f) The line of symmetry of the graph of f is the line $x=2$.

We can consider the restricted domains

$$\{x\in\mathbb{R} \mid x\leq 2\} \text{ or } \{x\in\mathbb{R} \mid x\geq 2\}.$$

We consider $D_{f_r}=\{x\in\mathbb{R} \mid x\geq 2\}$.

Thus $f_r(x)=f(x)$ for $x\in D_{f_r}$.

We also have $R_{f_r}=\{y\in\mathbb{R} \mid y\geq -8\}$.

Now

$$\begin{aligned}
& y = \tfrac{1}{2}x^2 - 2x - 6 \\
\Leftrightarrow \quad & y = \tfrac{1}{2}\left(x^2 - 4x\right) - 6 \\
\Leftrightarrow \quad & y = \tfrac{1}{2}\left(x^2 - 4x + 4 - 4\right) - 6 \\
\Leftrightarrow \quad & y = \tfrac{1}{2}\left(x^2 - 4x + 4\right) - \tfrac{1}{2}(4) - 6 \\
\Leftrightarrow \quad & y = \tfrac{1}{2}(x-2)^2 - 8 \\
\Leftrightarrow \quad & \tfrac{1}{2}(x-2)^2 = y + 8 \\
\Leftrightarrow \quad & (x-2)^2 = 2(y+8) \\
\Leftrightarrow \quad & x - 2 = \pm\sqrt{2(y+8)} \\
\Leftrightarrow \quad & x = 2 \pm \sqrt{2(y+8)}.
\end{aligned}$$

Since $x \geq 2$ we choose

$$x = 2 + \sqrt{2(y+8)}.$$

We interchange x and y and obtain

$$y = 2 + \sqrt{2(x+8)}.$$

Hence $f_r^{-1}(x) = 2 + \sqrt{2(x+8)}$.
Also $D_{f_r^{-1}} = \{x \in \mathbb{R} \mid x \geq -8\}$ and $R_{f_r^{-1}} = \{y \in \mathbb{R} \mid y \geq 2\}$.

Check:

For $x \in D_{f_r}$

$$\begin{aligned}
\left(f_r^{-1} \circ f_r\right)(x) &= f_r^{-1}(f_r(x)) \\
&= f_r^{-1}\left(\tfrac{1}{2}(x-2)^2 - 8\right) \\
&= 2 + \sqrt{2\left(\tfrac{1}{2}(x-2)^2 - 8 + 8\right)} \\
&= 2 + \sqrt{(x-2)^2} \\
&= 2 + x - 2 \\
&= x.
\end{aligned}$$

Also for $x \in D_{f_r}^{-1}$

$$\begin{aligned}
\left(f_r \circ f_r^{-1}\right)(x) &= f_r\left(f_r^{-1}(x)\right) \\
&= f_r\left(2+\sqrt{2(x+8)}\right) \\
&= \tfrac{1}{2}\left(2+\sqrt{2(x+8)}-2\right)^2-8 \\
&= \tfrac{1}{2}\left(\sqrt{2(x+8)}\right)^2-8 \\
&= \tfrac{1}{2}(2(x+8))-8 \\
&= x+8-8 \\
&= x.
\end{aligned}$$

Notice in the check above we used $f_r(x)$ in the form $\frac{1}{2}(x-2)^2+8$. This just makes the calculations easier. You may also use $f_r(x)$ in the form $\frac{1}{2}x^2-2x-6$, and you will then obtain

$$\begin{aligned}
\left(f_r^{-1} \circ f_r\right)(x) &= f_r^{-1}(f_r(x)) \\
&= f_r^{-1}\left(\tfrac{1}{2}x^2-2x+6\right) \\
&= 2+\sqrt{2\left(\tfrac{1}{2}x^2-2x-6+8\right)} \\
&= 2+\sqrt{2\left(\tfrac{1}{2}x^2-2x+2\right)} \\
&= 2+\sqrt{x^2-4x+4} \\
&= 2+\sqrt{(x-2)^2} \\
&= 2+x-2 \\
&= x
\end{aligned}$$

and

$$\begin{aligned}
\left(f_r \circ f_r^{-1}\right)(x) &= f_r\left(f_r^{-1}(x)\right) \\
&= f_r\left(2+\sqrt{2(x+8)}\right) \\
&= \tfrac{1}{2}\left(2+\sqrt{2(x+8)}\right)^2-2\left(2+\sqrt{2(x+8)}\right)-6 \\
&= \tfrac{1}{2}\left(4+4\sqrt{2(x+8)}+2(x+8)\right)-4-2\sqrt{2(x+8)}-6 \\
&= 2+2\sqrt{2(x+8)}+x+8-4-2\sqrt{2(x+8)}-6 \\
&= x.
\end{aligned}$$

Additional Exercises 2.9.1

1. Figure 2.9.3 shows the graphs of the functions f and g, where f and g are defined by
$$y = f(x) = a|x - p| + q$$
and
$$y = g(x) = mx + c.$$

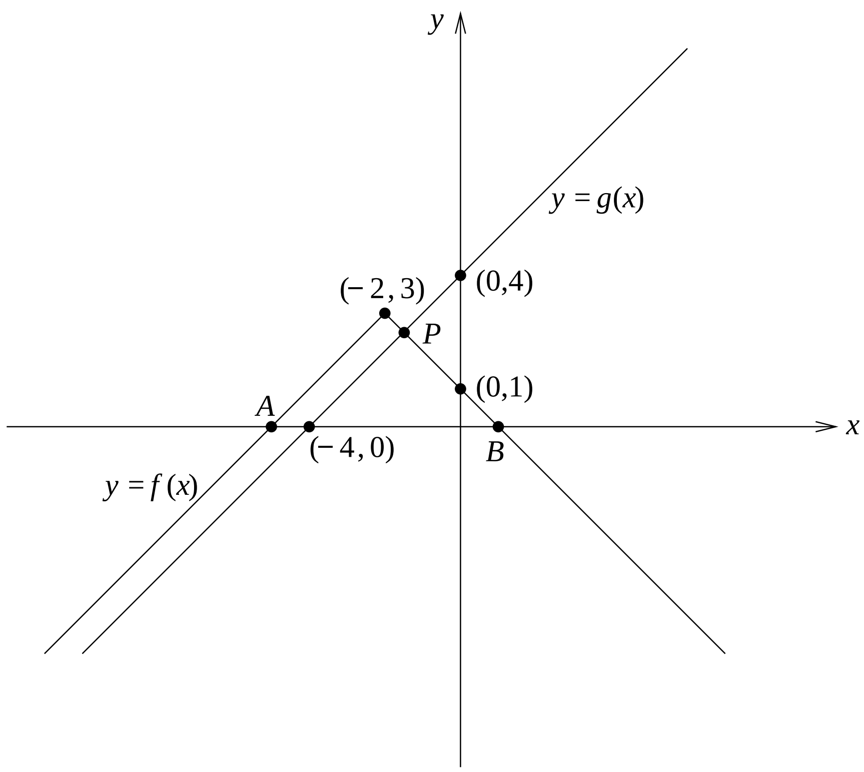


Figure 2.9.3

(a) Determine a, p, q, m and c, and write down the formula for f and for g.

(b) Find the points A, B and P which are indicated on the sketch.

(c) Why can we be sure that the graph of g and the line passing through A and $(-2, 3)$ do not intersect?

(d) For which values of x is $|x + 2| > 3$? (Use the sketch to answer this.)

2. The graphs of f and g are sketched in Figure 2.9.4. The functions f and g are defined by

$$y = f(x) = \frac{1}{2}|x+1| - 2$$

and

$$y = g(x) = -|x-2| + 3.$$

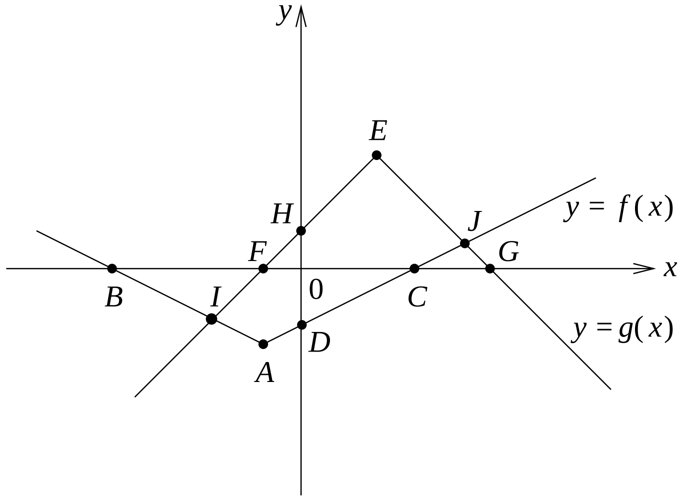


Figure 2.9.4

(a) Give the coordinates of A, B, C, D, E, F, G, H, I and J.

(b) From the graphs find the values of x for which

(i) $|x-2| < 3$

(ii) $|x+1| > 4$.

(c) Give a formula for $d(x)$ which represents the vertical distance between the graphs of f and g where x lies between the x–coordinates of I and J. Then find $d(1)$.

3. The graphs of the functions f and g are sketched in Figure 2.9.5.

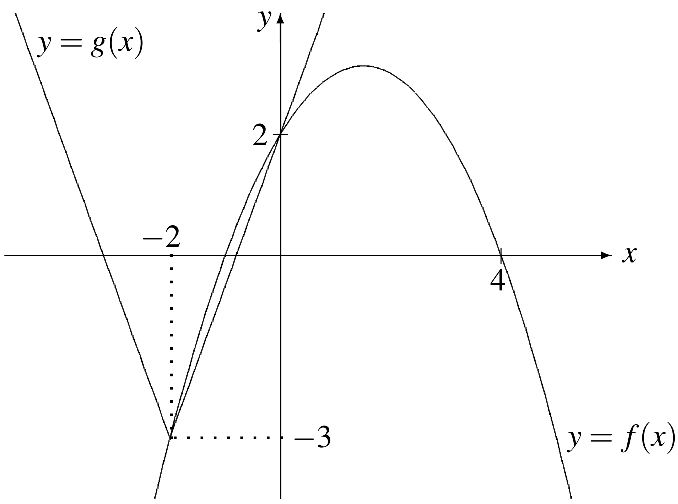


Figure 2.9.5

(a) Find the equations that define the quadratic function f and the absolute value function g.

(b) Calculate the vertical distance between the graphs of f and g for $x = -\frac{3}{2}$.

(c) Use the graphs to solve the inequality $f(x) \leq g(x)$.

(d) Restrict the domain of g so that the restricted function g_r is one–to–one. Choose the domain D_{g_r} so that it contains only negative numbers. Find the function g_r^{-1}.

4. The graphs of g and k are sketched in Figure 2.9.6 where

$$y = g(x) = c|x - p| + q$$

and k is a quadratic function such that $\left(-1, \frac{1}{3}\right)$ lies on the graph of $y = k(x)$. The salient point of the graph of $y = g(x)$ is $Q = (4, 2)$. The line segment AB is parallel to the y–axis.

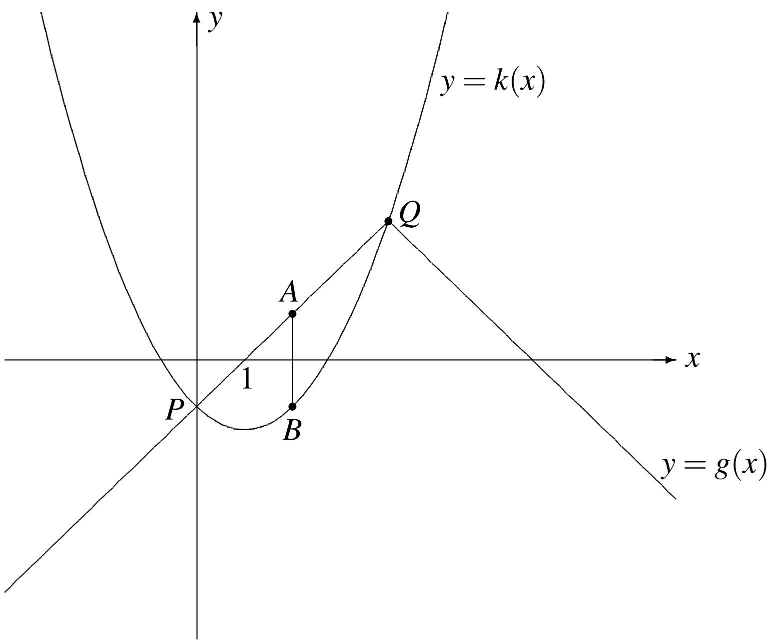


Figure 2.9.6

(a) Determine the values of c, p and q and thus write down the equation of g.

(b) If one of the x–intercepts of the graph of g is 1, use symmetry to determine the other x–intercept.

(c) Find the equation of k.

(d) Calculate the maximum length of AB if AB lies between P and Q.

(e) (i) Restrict the domain of k so that the function k_r defined by

$$k_r(x) = k(x) \text{ for all } x \in D_{k_r}$$

is a one–to–one function.

(ii) Find the equation of k_r^{-1}.

2 REVIEW AND TEST

Now assess yourself by attempting the questions in the Concept Check, trying a selection of questions from the Exercises and doing the Test at the end of Chapter 2 in SRW.

3 POLYNOMIAL AND RATIONAL FUNCTIONS

You need not study this chapter.

4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

As we have already emphasised, the concept of a function is important. You have previously studied linear and quadratic functions. In this chapter we deal with exponential and logarithmic functions, and begin to see how important they are.

4.1 EKSPONENSIëLE FUNKSIES

To study SRW, Section 4.1, and the additional notes.

Outcomes After studying this section you should be able to

- define irrational powers and hence any power in which the exponent is a real number
- manipulate powers of the form a^x, where $a \in \mathbb{R}$ and $a > 0$, for all $x \in \mathbb{R}$
- calculate the values of expressions containing powers for given values of the bases and exponents (with and without a calculator)
- define the exponential function f with base a where $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$
- identify the domain and range of $f(x) = a^x$
- sketch the graph of $f(x) = a^x$
- recognise the position of the graph of $f(x) = a^x$ relative to the axes as a varies, i.e. for $0 < a < 1$ and for $a > 1$
- recognise whether $f(x) = a^x$ is an increasing $(a > 1)$ or a decreasing $(0 < a < 1)$ function

	▶ apply principles of transformation (horizontal and verti– cal shrinking, stretching and shifting, as well as reflec– tion in the x– and y– axes) to the graph of $f(x) = a^x$, to obtain new exponential graphs; to find the equations of these graphs
	▶ define the **natural** exponential function (also called **the** exponential function)
	▶ sketch the graph of the natural exponential function
	▶ determine the position of the graph of the natural expo– nential function relative to the positions of the graphs of other exponential functions
	▶ apply the principles of transformation (horizontal and vertical shrinking, stretching and shifting, as well as re– flection in the x– and y–axes) to the graph of the natural exponential function to obtain new graphs; to find the equations of such graphs
	▶ find the values of expressions containing powers of e, using a calculator
	▶ apply knowledge of the natural exponential function, and of exponential functions in general, to solve problems related to compound interest, appreciation and depreciation, and other applied problems.
Exercises	Exercises 4.1: questions 1–41 (odd numbers) 65, 67, 69, 73-81 (odd numbers) Additional Exercises 4.1.1

Read the first two subsections carefully. We define the exponential function f with base a for all real numbers x as follows. For $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$, the exponential function f with base a is defined by $f(x) = a^x$. In Example 2 the graphs are drawn by plotting points. Figure 2 shows the pattern that develops as the base changes. Note that all the graphs in the figure are defined by $y = a^x$, where $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$, and $x \in \mathbb{R}$. We assume $a \neq 1$ since the function $f(x) = 1^x = 1$ is just a **constant** function. We summarise the results as follows.

Characteristics of $y = f(x) = a^x$, where $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$
1. $D_f = \mathbb{R}$.
2. $R_f = \{y \in \mathbb{R} \mid y > 0\}$.
3. f **increases** when $a > 1$. f is **constant** when $a = 1$. f **decreases** when $0 < a < 1$.
4. f is a one-to-one function.
5. The graph of f cuts the y–axis at $(0, 1)$. There are no x-intercepts.
6. The **horizontal asymptote** of the graph is the **negative** x**–axis** when $a > 1$ the **positive** x**–axis** when $0 < a < 1$.
7. ► When $a > 1$, the **bigger** a becomes, the **closer** the graph moves to the **positive** y**–axis** when $x > 0$, the **closer** the graph moves to the **negative** x**–axis** when $x < 0$. ► When $0 < a < 1$, the **smaller** a becomes, the **closer** the graph moves to the **positive** y**–axis** when $x < 0$, the **closer** the graph moves to the **positive** x**–axis** when $x > 0$.

Table 4.1.1

Note: if $a > 1$ then $\frac{1}{a} < 1$; if $0 < a < 1$ (i.e. if a is a fraction smaller than 1) then $\frac{1}{a} > 1$.

The graph of $y = f(-x) = a^{-x}$ is the mirror image, in the y–axis, of the graph of $y = f(x) = a^x$. For example, compare the graphs of $y = 2^x$ and $y = 2^{-x}$, i.e. $y = (\frac{1}{2})^x$.

Example 3 deals with identifying the graphs of exponential functions. “Identifying the graph” in this case means finding the equation of the graph. In Example 3 we use the fact that

- ► if a point $A(x, y)$ lies on the graph of a function k, defined by the equation $y = k(x)$, then the coordinates (x, y) of the point satisfy the equation
- ► if the coordinates x and y of a point $A(x, y)$ satisfy the equation $y = k(x)$ then the point A lies on the graph of k.

When we have a function defined by $y = a^x$, and we are given one point on the graph of the function, we can find the equation that defines the function.

In Example 3(a), the graph is defined by $y = a^x$ and the point $(2, 25)$ lies on the graph. We thus have

$$25 = a^2$$

and since $a > 0$ we conclude that $a = 5$, i.e. that the equation is

$$y = 5^x.$$

In Example 3(b) the graph is again defined by $y = a^x$, and the point $\left(3, \frac{1}{8}\right)$ lies on the graph. Thus, by substitution, we have

$$\tfrac{1}{8} = a^3$$

i.e. $$\left(\tfrac{1}{2}\right)^3 = a^3$$

i.e. $$a = \tfrac{1}{2}$$

and hence the function is defined by

$$y = \left(\tfrac{1}{2}\right)^x.$$

If we have a function defined by $y = a^x$, and we know that $(2, 20)$ lies on the graph of this function, then

$$20 = a^2$$

and since $a > 0$

$$\begin{aligned} a &= \sqrt{20} \\ &= \sqrt{4 \times 5} \\ &= 2\sqrt{5} \end{aligned}$$

which is an irrational number. The function is defined by $y = (2\sqrt{5})^x$.

In Example 4 we apply the concepts dealt with in general in Section 2.4. New graphs can be obtained from existing graphs by means of a succession of different steps. In Example 4(a) the transformation takes place by means of a vertical shift, upwards, of one unit (illustrated in Figure 3(a)); in Example 4(b) the transformation occurs through a reflection in the x–axis (illustrated in Figure 3(b)). In Example 4(c) the transformation occurs through a horizontal shift, one unit to the right (see Figure 3(c)). Example 4(a) shows (see Figure 3(a)) that any vertical shifting of a graph results in a new graph whose horizontal asymptote is found by shifting the horizontal asymptote of the original graph the same amount, in the same direction.

You may omit Example 5. It relates to power functions, which we do not deal with in this module; it also depends on the use of a graphing calculator, which we exclude.

■ The Natural Exponential Function

Read this subsection up to Compound Interest. It is interesting to note that

$$n \to \infty \Rightarrow \left(1+\frac{1}{n}\right)^n \to e.$$

This is mathematical "shorthand" for: "as n becomes infinitely big (i.e. as n increases indefinitely), the expression $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to the irrational number e". This is illustrated in the table in the margin next to the paragraph where we see that $e \approx 2,71828$, correct to five decimal places.

Figure 5 shows the graph of the natural exponential function; it also shows the position of the graph of the natural exponential function relative to the graphs of two other exponential functions, namely $y = 2^x$ (note that $2 < e$) and $y = 3^x$ (note that $3 > e$).

You can practise using your calculator by trying to do Example 6. Note that in 6(b) you need to calculate $2e^{-0.53}$, and you must remember that

$$2e^{-0.53} \neq (2e)^{-0.53}.$$

To convince yourself, calculate both of these using a calculator. You will see that

$$2e^{-0.53} \approx 1.17721 \text{ (correct to five decimal places)}$$

whereas

$$(2e)^{-0.53} \approx 0.40764 \text{ (correct to five decimal places).}$$

In Example 7(a) (see Figure 6) we have the graphs of

$$y = g(x) = e^x$$

and

$$y = f(x) = e^{-x}.$$

We note that $g(x) = f(-x)$, and we would thus expect that the graph of g is the mirror image, in the y–axis, of the graph of f.

In Example 7(b) (see Figure 7) several points have been found using a calculator. However, a rough sketch of this graph can also be made by starting with the graph of $y = f(x) = e^x$ and applying various transformations.

► Note that the graph of the function k, defined by

$$y = k(x) = f\left(\frac{x}{2}\right) = e^{0.5x}$$

is obtained by stretching the graph of f horizontally, by a factor of 2.

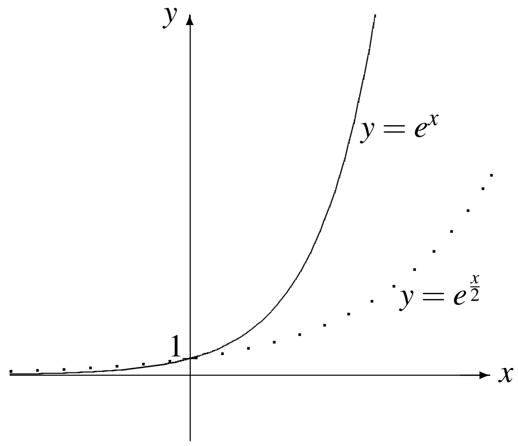


Figure 4.1.1

► The graph of the function ℓ, defined by

$$y = \ell(x) = 3k(x) = 3e^{0.5x}$$

is obtained by vertically stretching the graph of k, by a factor of 3.

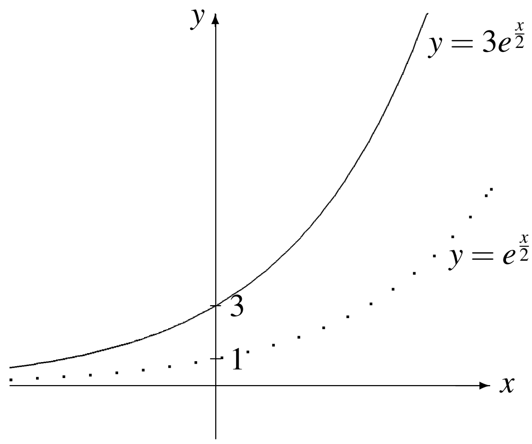


Figure 4.1.2

We can also find the equation of a graph, without necessarily drawing the graph, if we know the transformation processes that are applied.

Example 4.1.1

Suppose the graph of the natural exponential function is

- horizontally compressed by a factor of $\frac{1}{3}$;
- the resulting graph is then shifted horizontally, one unit to the left, and then vertically, one unit downwards; and finally
- the resulting graph is reflected in the x–axis.

What is the equation of the final graph?

Solution

We begin with the graph of the natural exponential function f, where

$$f(x) = e^x.$$

If the graph is horizontally compressed by a factor of $\frac{1}{3}$ we obtain the graph of a function g, where

$$g(x) = f(3x)$$

i.e.

$$g(x) = e^{3x}.$$

If the graph of g is shifted horizontally to the left by one unit, we obtain the graph of a function h, where

$$h(x) = g(x+1)$$

i.e.

$$h(x) = e^{3(x+1)}.$$

If the graph of h is shifted vertically downwards by one unit we have the graph of a function k, where

$$k(x) = h(x) - 1$$

i.e.

$$k(x) = e^{3(x+1)} - 1.$$

The final transformation process is a reflection in the x–axis. If the graph of k is reflected in the x–axis we obtain the graph of the function s, where

$$s(x) = -k(x)$$

and thus the required equation is

$$\begin{aligned} s(x) &= -\left[e^{3(x+1)} - 1\right] \\ &= -e^{3x+3} + 1. \end{aligned}$$

■ Compound Interest

Study the introduction to compound interest carefully in the first paragraph of the subsection. In particular take note of words that may be unfamiliar, such as **principal**. The interest rate r is the **annual interest rate**, i.e. the rate at which interest is calculated during one full year. Note that the interest rate r is **expressed as a percentage**. In the formula, in which interest is **compounded annually** , we have

Make sure you understand what it means when we say that interest is **compounded** annually.

$$A = P(1+r)^k.$$

Suppose we have actual values for P, r and k, i.e. suppose that an amount of R1 000 is invested for 3 years at an annual interest rate of 9%, and interest is compounded annually (i.e. once a year). Then the amount of money available at the end of the period (i.e. A) is found by calculating

$$A = 1\,000\left(1+\frac{9}{100}\right)^3.$$

We see that $r = 9\% = \frac{9}{100}$, hence the formula uses $\frac{9}{100}$ and not 9.

The general compound interest formula appears in the blue block above Example 9. Note that r is also referred to as the nominal annual interest rate. Example 9 shows how the formula is used. Make sure that you do not confuse

t : the **number of years** that the money is invested; and
n : the **number of times per year** that interest is compounded.

Comment

You may wonder why the formula contains the term $\frac{r}{n}$. An example may be useful. Suppose the annual interest rate is 12%, and interest is compounded every three months. There are four three month periods in one year, and in each period of three months the interest rate is $\frac{12}{4}\%$, i.e. 3%. Now

$$\text{interest} = 12\% \times P \ \text{(in one year)}.$$

Thus, if there are four time periods,

$$\begin{aligned}\text{interest} &= \frac{12\% \times P}{4} + \frac{12\% \times P}{4} + \frac{12\% \times P}{4} + \frac{12\% \times P}{4} \\ &= 4\left(\frac{12\% \times P}{4}\right) \\ &= 12\% \times P.\end{aligned}$$

In each time period we thus calculate interest at the rate of 3%. In the formula we have

$$
\begin{aligned}
A(t) &= P\left(1+\frac{r}{n}\right)^{nt} \\
&= P\left(1+\frac{12\%}{4}\right)^{4t} && \text{Remember to write } r \text{ as a percentage.} \\
&= P\left(1+\frac{0.12}{4}\right)^{4t} \\
&= P(1+0.03)^{4t} \\
&= P(1.03)^{4t}.
\end{aligned}
$$

Although in practical terms it is not possible to increase the number of compounding periods indefinitely (i.e. even more than every second of every minute of every day of every year) in theory this is mathematically possible. The formula in the blue block above Example 10 gives the value of an initial investment of P, after t years, when the annual interest rate is r and interest is **compounded continuously**. This is illustrated in Example 10.

In the examples you have considered so far it has either been possible to do the calculation directly, or by using a calculator. Although you would usually use calculators or computers to perform these calculations, we can also use compound interest tables to find the approximate values of certain expressions. An example of such a table is given below.

COMPOUND INTEREST TABLE

n	$(1.01)^n$	$(1.02)^n$	$(1.04)^n$	$(1.08)^n$	$(1.12)^n$
1	1.01000000	1.02000000	1.04000000	1.08000000	1.12000000
2	1.02010000	1.04040000	1.04040000	1.06640000	1.25440000
3	1.03030100	1.06120800	1.12486400	1.25971200	1.40492800
4	1.04060401	1.08243216	1.16985856	1.36048896	1.57351936
5	1.05101005	1.10408080	1.21665290	1.46932808	1.76234168
6	1.06152015	1.12616242	1.26531902	1.58687432	1.97382269
7	1.07213535	1.14868567	1.31593178	1.71382427	2.21068141
8	1.08285671	1.17165938	1.36856905	1.85093021	2.47596318
9	1.09368527	1.19509257	1.42331181	1.99900463	2.77307876
10	1.10462213	1.21899442	1.48024428	2.15892500	3.10484821
11	1.11566835	1.24337431	1.53945406	2.33163900	3.47854999
12	1.12682503	1.26824179	1.60103222	2.51817012	3.89597599
15	1.16096896	1.34586834	1.80094351	3.17216911	5.47356576
20	1.22019004	1.48594740	2.19112314	4.66095714	9.64629309
25	1.28243200	1.64060599	2.65583633	6.84847520	17.00006441
30	1.34784892	1.81136158	3.24339751	10.06265689	29.95992212
35	1.41660276	1.99988955	3.94608899	14.78534429	52.79961958
40	1.48886373	2.20803966	4.80102063	21.72452150	93.05097044
45	1.56481075	2.43785421	5.84117568	31.92044939	163.98760387
50	1.64463182	2.69158803	7.10668335	46.90161251	289.00218983
55	1.72852457	2.97173067	8.64636692	68.91385611	509.32060567
60	1.81669670	3.28103079	10.51962741	101.25706367	897.59693349
65	1.90936649	3.62252311	12.79873522	148.77984662	1581.87249060
70	2.00676337	3.99955822	15.57161835	218.60640590	2787.79982770
75	2.10912847	4.41583546	18.94525466	321.20452996	4913.05584077
80	2.21671522	4.87543916	23.04979907	471.95483426	8656.48310008
85	2.32978997	5.38287878	28.04360494	693.45648897	15259.20568055
90	2.44863267	5.94313313	34.11933334	1018.91508928	26891.93422336
95	2.57353755	6.56169920	41.51138594	1497.12054855	47392.77662369
100	2.70481388	7.24464612	50.50494818	2199.76125634	83522.26572652

The table gives only selected rates. For example we cannot use it in the case where the interest rate is 7% per annum, with interest compounded annually for 14 years. You would then have to use either a calculator or a computer.

The table is easy to use. Suppose you have an interest rate of 2%, and the formula you are using becomes

$$A = P(1.02)^{15}.$$

In this case $n = 15$. To find $(1.02)^{15}$ place a ruler horizontally across the table at the place where $n = 15$. Read across the row until you have the number in the column with $(1.02)^n$ at the top. The answer is 1.345 86834.

■ Appreciation and Depreciation

We include this subsection as it is related to the principles of compound interest.

We do not only measure the increase in value in the case of money. For example, items of value such as property, art works and diamonds may also **appreciate**. A house which was bought ten years ago for R50 000 may be worth R200 000 today. On the other hand, many items which are expensive to buy, such as motor cars, may **depreciate**, i.e. decrease in value with time.

The formulas for appreciation and depreciation are similar to those used for compounded interest.

Formula for appreciation

For **appreciation** we have

$$A = P\,(1+r)^t\,.$$

Formula for depreciation

For **depreciation** we have

$$A = P\,(1-r)^t$$

where

P is the initial value of the asset

r is the rate of appreciation or depreciation, expressed as a percentage

t is the time, in years, over which the asset appreciates or depreciates

A is the value of the asset after t years.

Consider the following example.

Example 4.1.2

Suppose factory machinery depreciates at 10% per year. If the machinery is valued at R25 000 today, what will it be worth in 5 years time?

Solution

We use

$$A = P\,(1-r)^t$$

where $P = 25\ 000,\ r = 10\%$ and $t = 5$. Thus

$$\begin{aligned} A &= 25\,000\left(1-\frac{10}{100}\right)^5 \\ &= 25\,000\,(1-0.1)^5 \\ &= 25\,000\,(0.9)^5 \\ &= 14\,762.25. \end{aligned}$$

Thus, in 5 years time, the machinery will be worth approximately R14 762.

Additional Exercises 4.1.1

1. Draw four sets of axes. On the four different systems of axes sketch the pairs of graphs defined by the following equations.

   (a) $y = 2^x; \quad y = \left(\frac{1}{2}\right)^x$

   (b) $y = 2^x; \quad y = 3^x$

   (c) $y = 2^x; \quad y = -2^x$

   (d) $y = 2^x; \quad y = 2^{x-1}$

2. Explain how you would obtain the graph of each of the following functions from the graph of the function g defined by $y = g(x) = 3^{-x}$.

   (a) The function k, defined by $y = k(x) = -3^x$.

   (b) The function f, defined by $y = f(x) = 3^{2x-1}$.

   (c) The function ℓ, defined by $y = \ell(x) = 3^{-2(x-1)}$.

3. The graph of the natural exponential function is shifted two units to the right; the resulting graph is stretched vertically by a factor of three and then shifted vertically downwards by one unit.

   (a) What is the equation of the new graph?

   (b) What is its horizontal asymptote?

   (c) What are the domain and range of the function represented by the new graph?

4. The graph of a function g is found in the following way.

   The graph of the function f, defined by $y = f(x) = e^{\frac{x}{3}} - 3$ is shifted vertically three units upwards; the resulting graph is horizontally shrunk by a factor of 3 and then reflected in the y–axis.

   Write down the equation of g.

5. Suppose a function k defined by

   $$y = k(t) = pe^{ct}$$

   describes the growth of a given population, where c represents the growth rate.

   (a) Write down the equation of the function ℓ which describes population growth at double the rate of the function k.

   (b) On the same system of axes, without doing any calculations, sketch the graphs of k and ℓ. Since p, c and t are unknown, just give a rough sketch to show the positions of the graphs in relation to each other.

4.2 LOGARITHMIC FUNCTIONS

To study	SRW, Section 4.2.
Outcomes	After studying this section you should be able to ▶ define the logarithmic function with base a, $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$; extend this definition to **common** and **natural** logarithms ▶ change an equation given in exponential form into an equation in logarithmic form, and vice versa ▶ identify the domain and range of any logarithmic function ▶ calculate the value of a logarithmic expression (with and without using a calculator) ▶ sketch the graphs of logarithmic functions by plotting points ▶ apply principles of transformation (horizontal and vertical shrinking, stretching and shifting, as well as reflection in the x– and y–axes) to obtain new logarithmic graphs; to find the equations of such graphs ▶ know and apply the properties of logarithms ▶ interpret combinations of graphs of logarithmic or exponential functions and other functions.
Exercises	Exercises 4.2: questions 1–63 (odd numbers, a selection) Additional Exercises 4.2.1

Read carefully Section 4.2, SRW up to the end of Figure 4. This section illustrates the importance of having understood (from Section 2.8) how to:

– define an inverse function f^{-1} for any one–to–one function f

– draw the graph of the function f^{-1} by reflecting the graph of a one–to–one function f in the line defined by $y = x$

– identify the domain and range of the inverse function f^{-1} from the range and domain of the one–to–one function f.

If you had difficulty with the concepts explained in Section 2.8 you may want to go over them again now.

It is clear from the definition of a logarithmic function that **a logarithm is an exponent**. If you remember this it will be easier to remember the properties of logarithms (given in the blue block above Example 3).

The graphs of logarithms given in Figures 2, 3 and 4 are defined by $y = \log_a x$ where $a \in \mathbb{R}$ and $a > 1$. What happens when $0 < a < 1$?

We remember that for any $a \in \mathbb{R}$, $a > 0$ and $a \neq 1$,

$$\log_a x = y \Leftrightarrow a^y = x$$

which means that for the exponential function f defined by $y = a^x$ there exists an inverse function f^{-1} which is defined by

$$y = \log_a x.$$

You may find the following diagram helpful.

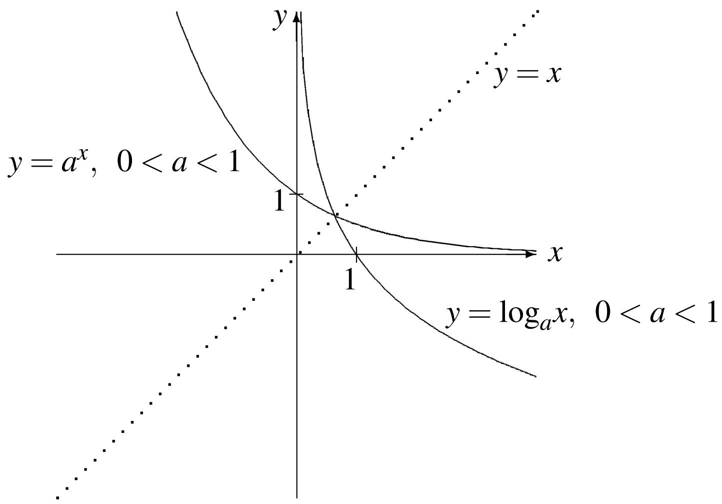


Figure 4.2.1

Recall that the procedure for finding f^{-1} for a one–to–one function f requires that we carry out the following steps.

- ► Write the equation $y =$ .... so that we have $x =$ ...., i.e. we express x in terms of y.
- ► This gives an equation of the form

$$x = f^{-1}(y) = \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots}_{\text{an expression containing } y}.$$

► Interchange x and y. Then we have

$$y = f^{-1}(x) = \underbrace{\dots\dots\dots\dots\dots\dots}_{\text{an expression containing } x}.$$

Thus, for the one–to–one function f defined by

$$y = a^x$$

we have $x = \log_a y$ Changing from exponential form to logarithmic form.

i.e. $x = f^{-1}(y) = \log_a y.$

Then we interchange x and y to get

$$y = f^{-1}(x) = \log_a x.$$

The graph of the inverse function f^{-1} is obtained by reflecting the graph of f in the line defined by $y = x$. Thus, if $0 < a < 1$, we have the following situation.

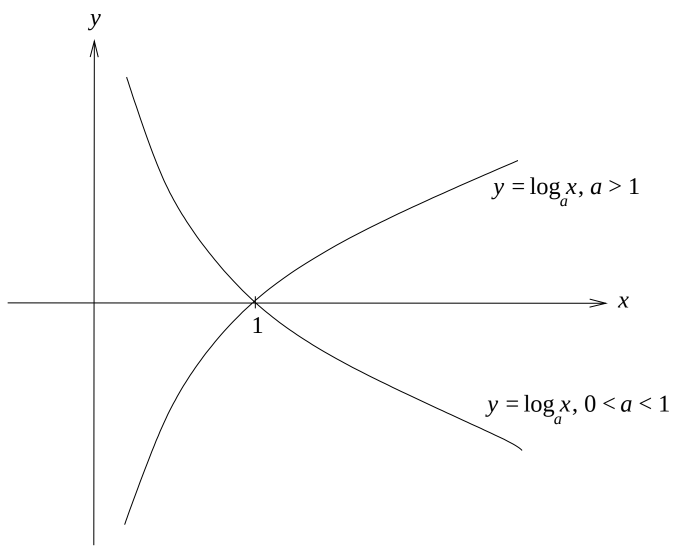


Figure 4.2.2

The graph of $y = \log_a x$, $0 < a < 1$, is the reflection of the graph of $y = a^x$ in the line defined by $y = x$.

Comparing the graphs of $y = \log_a x$ in Figure 2 (where $a > 1$) and Figure 4.2.2 (where $0 < a < 1$) we see that

- ► if $0 < a < 1$ the function is **decreasing**
  if $a > 1$ the function is **increasing**
- ► if $0 < a < 1$ the vertical asymptote of the graph is the positive y–axis
  if $a > 1$ the vertical asymptote of the graph is the negative y–axis.

In Section 4.1 we summarised the characteristics of exponential functions. The discussion in SRW, Section 4.2, up to end of Example 4, and Figures 2, 3 and 4, now enable us to summarise the characteristics of logarithmic functions.

Characteristics of $y = f(x) = \log_a x$, **where** $a \in \mathbb{R}$, $a > 0$ **and** $a \neq 1$
1. $D_f = \{x \in \mathbb{R} \mid x > 0\}$.
2. $R_f = \mathbb{R}$.
3. f **increases** when $a > 1$. f **decreases** when $0 < a < 1$.
4. f is a one–to–one function.
5. The graph of f cuts the x–axis at $(1,0)$. There is no y–intercept.
6. The **vertical asymptote** of the graph is the **negative** y**–axis** when $a > 1$ the **postive** y**–axis** when $0 < a < 1$.
7. ► When $a > 1$, the **bigger** a becomes, the **closer** the graph moves to the **positive** x**–axis** when $x > 1$, the **closer** the graph moves to the **negative** y**–axis** when $0 < x < 1$. ► When $0 < a < 1$, the **smaller** a becomes, the **closer** the graph moves to the **positive** x**–axis** when $x > 1$, the **closer** the graph moves to the **positive** y**–axis** when $0 < x < 1$.

Table 4.2.1

Now study Examples 5 and 6 in Section 4.2, SRW. In Examples 5 and 6 we apply the properties of transformation (reflection in the x– and y–axes, in Example 5; vertical and horizontal shifting, in Example 6) to the basic graphs sketched in Figure 4.

Compare (a) in Figure 5 with the graphs of $y = \log_a x$ for $a > 1$ (given in Figure 2) and $y = \log_a x$ for $0 < a < 1$ (given in Figure 4.2.2). Notice that if we reflect the graph of $y = 2^x$ in the line defined by $y = x$ we obtain the graph of $y = \log_2 x$ shown in Figure 5(a).

If we were to start with the graph of $y = \left(\frac{1}{2}\right)^x$ and reflect it in the line defined by $y = x$ we would obtain the red graph given in Figure 5(a). We see that the graph of $y = \log_{\frac{1}{2}} x$ is the same as the graph of $y = -\log_2 x$ (obtained by reflecting the graph of $y = \log_2 x$ in the x–axis). When we study the change of base formula (in Section 4.3) we will see that

$$\log_{\frac{1}{2}} x = -\log_2 x$$

is true for all $x \in \mathbb{R}$, $x > 0$. In calculations involving logarithms, or in solving logarithmic equations and inequalities, we often need to apply the four properties of logarithms given in the table just above Example 3.

It may be helpful to try to prove these properties, if we want to understand them better. Suppose we want to prove that

$$\log_a 1 = 0, \ a \in \mathbb{R}, \ 0 < a < 1 \text{ or } a > 1.$$

Let

$$\log_a 1 = p, \quad \text{where } p \in \mathbb{R}.$$

Then

$$a^p = 1$$

We change from logarithmic form to exponential form.

and hence

$$p = 0.$$

$a^0 = 1$;
$a^p \neq 1$ for any other value of $p \in \mathbb{R}$.

Thus

$$\log_a 1 = 0.$$

We leave it as an exercise for you to prove the remaining three properties. Example 3 illustrates how these properties are applied.

■ Common and Natural Logarithms

The logarithm with base 10 is called the common logarithm, and is denoted by omitting the base:

$$\log x = \log_{10} x.$$

The natural logarithm is the logarithm with base e.

We have discussed exponential functions in general, and the natural exponential function which has e as its base. We have a similar distinction in the case of logarithmic functions, except that we now have two special logarithms, called common logarithms and natural logarithms, respectively. The definitions are given in the subsections Common Logarithms and Natural Logarithms in Section 4.2 of SRW.

Example 7 deals with calculating values of the common logarithm. Note that the numbers in the right hand column of the table next to Figure 8 are not all exact. For example:

$$\begin{aligned} x = 0.01 \Rightarrow \log x &= \log 0.01 \\ &= \log \frac{1}{100} \\ &= \log 10^{-2} \\ &= -2 \quad \text{Property 3 of Properties of Logarithms.} \end{aligned}$$

and

$$\begin{aligned} x = 0.1 \Rightarrow \log x &= \log 0.1 \\ &= \log \frac{1}{10} \\ &= \log 10^{-1} \\ &= -1. \quad \text{Property 3 of Properties of Logarithms.} \end{aligned}$$

However

$$x = 0.5 \Rightarrow \log x = \log 0.5$$

and since we cannot write 0.5 as a power of 10 we need to use a calculator. The answer is $-0,301029...$ which has been approximated to three decimal places in the table. Similarly, the values of $\log x$ given for $x = 4$ and $x = 5$ are approximations.

The notation ln, which abbreviates *logarithmus naturalis*, and which denotes the natural logarithm, is pronounced as "lin".

Calculators are also required to find the value of any natural logarithm (we use the ln key), and most answers must then be approximated to a particular number of decimals, as in (c) of Example 9, where the answer is given correct to three decimal places.

Example 10 is important, as we will need to apply this whenever we solve logarithmic equations (i.e. equations in which the logarithm of a variable occurs). We return to this in Section 4.4.

Additional Exercises 4.2.1

For each of questions 1–7 in this exercise sketch the following pairs of graphs on the same set of axes. Refer to questions 49–56 (except question 55) in Exercises 4.2 of SRW. Follow the instructions given for these questions.

1. (a) Do question 49 (b) Sketch the graph of $k(x) = \log_2 x - 4$.
2. (a) Do question 50 (b) Sketch the graph of $k(x) = \log(-x)$.
3. (a) Do question 51 (b) Sketch the graph of $k(x) = -\log_5 x$.
4. (a) Do question 52 (b) Sketch the graph of $k(x) = \ln x + 2$.

5. (a) Do question 53 (b) Sketch the graph of $k(x) = \log_3(x+2)$.

6. (a) Do question 54 (b) Sketch the graph of $k(x) = \log_3(x-3)$.

7. (a) Do question 56 (b) Sketch the graph of $k(x) = 1 - \ln x$.

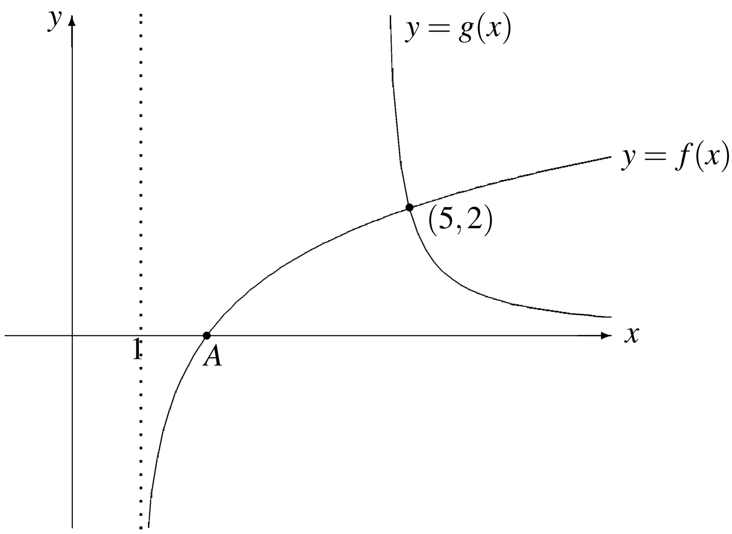
8. Prove Properties 2, 3 and 4 in the blue block above Example 9 (Properties of Natural Logarithms).

9. Sketch the graph of $y = |\log_3 x|$.

10. Draw the graph of $y = f(x) = \log_2(x-1)$. By reflecting this graph in the line defined by $y = x$ sketch the graph of the inverse function f^{-1}. Obtain a formula which defines the inverse function f^{-1}. Show that

$$\left(f^{-1} \circ f\right)(x) = x \text{ for all } x > 1.$$

11.



The graphs of the functions f and g, where

$$y = f(x) = \log_b(x-a), \ \ b > 0 \text{ and } b \neq 1,$$

and

$$y = g(x) = \frac{1}{x+k}, \ \ x \neq -k$$

are sketched above.

The line defined by $x = 1$ is a vertical asymptote of the graph of f.

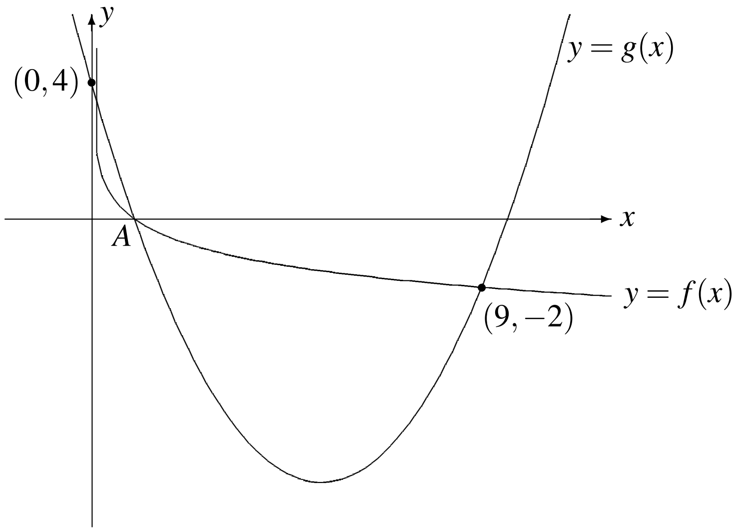
(a) Determine the value of a.

(b) Determine the coordinates of A.

(c) Determine the values of b and k.

(d) If the graph of the function h is the reflection of the graph of f in the y–axis, give the equation of h in the form $y = h(x) = \ldots$.

12.



The graphs of

$$y = f(x) = \log_q(x)\,, \quad q > 0 \ \text{ and } \ q \neq 1,$$

and

$$y = g(x) = ax^2 + bx + c$$

are sketched above.

(a) Give the coordinates of A.

(b) Determine the values of q, a, b and c.

(d) If the graph of the function k is the reflection of the graph of f in the line defined by $y = x$, give the equation of k in the form

$$y = k(x) = \ldots \ .$$

4.3 LAWS OF LOGARITHMS

Assumed knowledge	SRW, Section 4.3: Laws of logarithms Change of base formula

Diagnostic Test 4.3

Exercises 4.3: questions 1–61 (odd numbers) a selection, excluding 57.

4.4 EXPONENTIAL AND LOGARITHMIC EQUATIONS AND INEQUALITIES

To study	SRW, all of Section 4.4, and additional notes dealing with expo–nential and logarithmic inequalities.
Outcomes	After studying this section you should be able to ► use the laws of exponents, and the laws and properties of logarithms, to solve exponential and logarithmic equations and inequalities ► apply the characteristics of exponential and logarithmic functions to solve exponential and logarithmic equations and inequalities ► solve applied problems .
Exercises	Exercises 4.4: questions 1–49, 63–71, 75–79 (odd numbers only) Additional Exercises 4.4.1

■ Exponential Equations

Equations containing terms of the form a^x, $a > 0$ and $a \neq 1$, are called **exponential equations**. In Section 4.5 we consider applications of these equations.

The simplest types of exponential equations can be solved directly and do not require the use of a calculator or additional algebraic manipulation. For example, if

$$2^x = 16$$

then, by writing the term on the right hand side as a power with base 2, we have

$$2^x = 2^4$$

and thus

$$x = 4,$$

since the exponential function with base 2 is a one–to–one function. The one–to–one property of exponential functions helps us solve exponential equations. This property states that for a one–to–one function f

$$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2.$$

We may rewrite the one–to–one property in terms of an exponential function f, defined by $y = a^x$, as

$$a^{x_1} = a^{x_2} \Leftrightarrow x_1 = x_2.$$

This means that we can solve exponential equations directly, by

Because $y = 2^x$ defines a one–to–one function, we know that if $2^x = 2^4$ then 4 is the **only** value of x for which this statement is true.

- simplifying algebraically, if necessary
- writing the expressions on both sides of the equation as powers with the same base
- equating the exponents.

Consider Examples 4.4.1 and 4.4.2.

Example 4.4.1

Solve the equation $\dfrac{1}{2^{1-x}} = 32$.

Solution

$$\begin{array}{llll} & \dfrac{1}{2^{1-x}} = 32 & \\ \Leftrightarrow & 2^{x-1} = 2^5 & \text{Rule for negative exponents.} \\ \Leftrightarrow & x-1 = 5 & \text{One–to–one property.} \\ \Leftrightarrow & x = 6. & \end{array}$$

In Example 4.4.2 we first need to reduce the equation to a form in which we can equate the exponents.

Example 4.4.2

$$5^{x+1} + 5^{x-1} = 26.$$

Solution

Caution: Although we note that the equation can be written as

$$\begin{aligned} 5^{x+1} + 5^{x-1} &= 25 + 1 \\ &= 5^2 + 5^0 \end{aligned}$$

this does **not** mean that we can deduce that

$$x + 1 + x - 1 = 2 + 0$$

even though *in this case* we will obtain the correct answer.

As a counter–example consider

$$3^{x+1} + 3^{x-1} = 10.$$

If we use the **incorrect** procedure above, we may deduce that

$$\begin{array}{llll} & 3^{x+1} + 3^{x-1} & = & 3^3 + 3^0 \\ \Rightarrow & x + 1 + x - 1 & = & 3 \\ \Rightarrow & 2x & = & 3 \\ \Rightarrow & x & = & \dfrac{3}{2}. \end{array}$$

However, if we substitute $x = \frac{3}{2}$ into the left side of the equation, we have

$$\begin{aligned} 3^{\frac{3}{2}+1} + 3^{\frac{3}{2}-1} &= 3^{\frac{5}{2}} + 3^{\frac{1}{2}} \\ &= 3^{\frac{1}{2}}\left(3^2+1\right) \\ &= \sqrt{3}(10) \\ &\neq 10. \end{aligned}$$

The mistake arises when we misinterpret the one–to–one property. We must have $f(x_1) = f(x_2)$ in order to deduce that $x_1 = x_2$. We may not have a **combination** of $f(x_1)$, $f(x_2)$, etc. on the one side and another combination on the other side.

The correct way to solve the equation is now given.

$$5^{x+1} + 5^{x-1} = 26$$

$$\begin{aligned} \Leftrightarrow \quad 5^x \cdot 5 + 5^x \cdot 5^{-1} &= 26 && \text{Law 1 of exponents.} \\ \Leftrightarrow \quad 5^x\left(5+\frac{1}{5}\right) &= 26 && \text{Rule for negative exponents.} \\ \Leftrightarrow \quad 5^x\left(\frac{25+1}{5}\right) &= 26 \\ \Leftrightarrow \quad 5^x\left(\frac{26}{5}\right) &= 26 \\ \Leftrightarrow \quad 5^x &= \frac{26\times 5}{26} \\ \Leftrightarrow \quad 5^x &= 5 \\ \Leftrightarrow \quad x &= 1 && \text{One–to–one property.} \end{aligned}$$

In certain cases we may first need to reduce the equation to a quadratic form, as in the following example.

Example 4.4.3

Solve the equation $3^{2x} + 3^{x+1} = 4$.

Solution

Again, note that we may **not** write $3^{2x} + 3^{x-1} = 3^1 + 3^0$ $\Rightarrow 2x + x - 1 = 1$.

$$\begin{aligned} 3^{2x} + 3^{x+1} &= 4 \\ \Leftrightarrow \quad 3^{2x} + 3^x \cdot 3 - 4 &= 0 \end{aligned}$$

Now let $3^x = k$. We then have

$$\begin{aligned} & k^2+3k-4 &=& 0 \\ \Leftrightarrow\quad & (k-1)(k+4) &=& 0 \qquad \text{Zero–product property.} \\ \Leftrightarrow\quad & k=1 \quad \text{or} \quad k=-4. \end{aligned}$$

Thus $3^x = 1$ or $3^x = -4$. Since the range of the function defined by $y = 3^x$ is the set of positive real numbers we know that $3^x \neq -4$. The only solution is thus

$$\begin{aligned} & 3^x &=& 1 \\ \text{i.e.}\quad & 3^x &=& 3^0 \\ \text{i.e.}\quad & x &=& 0. \qquad \text{One–to–one property.} \end{aligned}$$

Not many exponential equations are as simple as these, and we need to apply the Laws of Logarithms (see the box at the start of Section 4.3 in SRW) to solve them.

Read the first example in Section 4.4 of SRW. We need to solve $2^x = 7$. In the second step we read the comment "Take ln of each side". This is not the only way of solving such an equation. We can also take the common logarithm of each side. In that case we reason as follows.

We may take the logarithm with **any** base, on both sides of the equation. However, when a calculator is involved we must use bases 10 or e, since log and ln are the only two functions available.

$$\begin{aligned} & 2^x &=& 7 \\ \Leftrightarrow\quad & \log 2^x &=& \log 7 \qquad \text{Take log of each side.} \\ \Leftrightarrow\quad & x\log 2 &=& \log 7 \qquad \text{By log law 3.} \\ \Leftrightarrow\quad & x &=& \frac{\log 7}{\log 2} \\ \Leftrightarrow\quad & x &\approx& 2.807 \qquad \text{Use a calculator; give the answer correct to three decimal places.} \end{aligned}$$

The technique for solving exponential equations is summarised in SRW in the block following the example that we have just discussed. These guidelines are based on the fact that the exponential and logarithmic functions are inverses of each other.

SRW seems to approximate answers to an arbitrary number of decimal places. To avoid confusion, when we set exercise, assignment or exam questions, we will specify the required number of decimal places.

In Example 1, Section 4.4, SRW, take note of the comment in the margin which again points out that there is more than one way of solving the equation. Yet another way of approaching Example 1 makes use of the laws of exponents (see Section 1.2).

$$
\begin{array}{rrcll}
 & 3^{x+2} & = & 7 & \\
\Leftrightarrow & 3^2.3^x & = & 7 & \text{Law 1 of exponents.} \\
\Leftrightarrow & 9.3^x & = & 7 & \\
\Leftrightarrow & 3^x & = & \dfrac{7}{9} & \\
\Leftrightarrow & x\log 3 & = & \log\dfrac{7}{9} & \text{Take log of both sides.} \\
\Leftrightarrow & x & = & \dfrac{\log\frac{7}{9}}{\log 3} & \text{Solve for } x. \\
\Leftrightarrow & x & = & \dfrac{\log 7 - \log 9}{\log 3} & \\
 & & \approx & -0.228756 & \text{Use a calculator; answer correct to six decimal places.}
\end{array}
$$

At the end of Example 1 there is a "Check your answer" block. It is always a good idea to carry out this process, in case you made a careless mistake in the calculation. In the case of exponential equations this is the only purpose of checking. However, as we shall see, in logarithmic equations we must carry out the check since the domain is not the set of all real numbers, and answers obtained in the solution may not be valid.

Study Examples 2–5 in Section 4.4 of SRW. Example 3 shows how the equation may also be solved graphically. Even though you are not expected to have or use a graphing calculator or a computer at this stage, it is still important to remember that the solution of

$$e^{3-2x} = 4$$

is the simultaneous solution of the system

$$\left.\begin{array}{l} y = e^{3-2x} \\ y = 4 \end{array}\right\}$$

which is found at the point of intersection of the two graphs $y = e^{3-2x}$ and $y = 4$.

Note that this equation also reduces to quadratic form.

When you solve equations such as the one given in Example 4 we prefer that you include the word "or" in the last step as well. After the first three lines of the Solution we thus have

$$
\begin{array}{rcl}
(e^x - 3)(e^x + 2) & = & 0 \\
e^x - 3 = 0 & \text{or} & e^x + 2 = 0 \\
e^x = 3 & \text{or} & e^x = -2,
\end{array}
$$

and then the final paragraph follows.

Comments

In Example 5 it is important to recognise that the terms on the left side have a common factor. By factorising we reduce a **sum** of terms to a **product** of terms, and we can then apply the zero–product property.

Do you find it confusing that we have "$x = 0$ **or** $x = -3$", followed by "the solutions are $x = 0$ **and** $x = -3$"? Recall the zero–product property (given in Section 1.5, SRW). To satisfy $x(3+x) = 0$ we require

$$x = 0 \quad \textbf{or} \quad 3 + x = 0.$$

If **any one** of the three factors in the equation is zero, the product will be zero. But **all** the values of x for which the factors are zero are solutions of the equation. We have seen that $e^x = 0$ has been excluded since $e^x \neq 0$, and the solutions of $x = 0$ **and** $3 + x = 0$ are both solutions of the original equation.

Mathematics develops in a hierarchical way, in that new knowledge is built on and depends on previous knowledge. We cannot compartmentalise any aspect of mathematics, and expect that at any given time we will only consider one particular concept.

Although we are now dealing with exponential equations, in order to solve them we need to have understood

simplification of algebraic expressions
factorisation, which may include
  taking out a common factor
  difference of squares; sum or difference of cubes
  grouping
  factorising a quadratic trinomial
zero–product property
exponential and logarithmic laws and properties
characteristics of exponential and logarithmic functions
calculator functions
rules that apply to solving equations.

In Example 4 you cannot solve the equation if you do not recognise that it is a **quadratic equation**, if you cannot solve quadratic equations using the **zero–product property**, if you do not know that negative numbers are not in the **range of the natural exponential function**, or if you cannot **use a calculator**. In Example 5 you must be able to recognise that xe^x is the **highest common factor**, apply the **zero–product property**, know the **range** of the natural exponential function. In the exercises at the end of this section there are several questions which require an understanding of many other concepts as well, for example

decimals
negative exponents
properties of ln
exponential laws and
operations with fractions.

We consider, as an example, a question from Exercises 4.4, SRW.

Example 4.4.4

Solve the equation for x.

$$e^x - 12e^{-x} - 1 = 0$$

Solution

It is not immediately obvious that this is a quadratic equation in e^x. However, it can be converted to a quadratic equation, and then solved.

$$\begin{array}{rrcll}
 & e^x - 12e^{-x} - 1 & = & 0 & \\
\Leftrightarrow & e^x \cdot e^x - 12e^{-x} \cdot e^x - 1 \cdot e^x & = & 0 & \text{Multiply each term by } e^x. \\
\Leftrightarrow & e^{2x} - 12 - e^x & = & 0 & \\
\Leftrightarrow & (e^x)^2 - e^x - 12 & = & 0 & \text{A quadratic equation in } e^x. \\
\Leftrightarrow & (e^x - 4)(e^x + 3) & = & 0 & \text{Factorise.} \\
\Leftrightarrow & e^x = 4 \quad \text{or} \quad e^x & = & -3 & \text{Zero–product property.}
\end{array}$$

Now $e^x = -3$ has no solution, since $e^x > 0$ for all $x \in \mathbb{R}$. Thus, if

$$e^x = 4$$

then

$$\ln e^x = \ln 4 \qquad \text{Take ln on both sides.}$$

i.e.

$$x = \ln 4 \qquad \text{Property 3 of ln.}$$

i.e.

$$x \approx 1{,}386. \qquad \text{Correct to 3 decimal places.}$$

■ Logarithmic Equations

The example at the start of this subsection in SRW leads to the guidelines for solving logarithmic equations. Note that exponential equations are often solved by applying log laws; log equations are often solved by applying exponential laws. This is hardly surprising, since the logarithmic function defined by $y = \log_a x$ is the inverse of the exponential function defined by $y = a^x$ (recall Figure 2, Section 4.2).

The method of solving the simplest forms of logarithmic equations is as follows.

Suppose

$$\log_2 x = 3.$$

Then $$x = 2^3$$

i.e. $$x = 8,$$

changing from logarithmic form to exponential form. Thus the solution is $x = 8$.

Examples 6, 7 and 8 (Section 4.4, SRW) illustrate the techniques generally used to solve logarithmic equations. In these examples we are solving equations which involve functions where the domain is not the set of all real numbers, and it is important to check the answer. Consider Example 4.4.5.

Example 4.4.5

Solve the following equation for x.

$$\log_2 x + \log_2(x+1) = \log_2 20$$

Solution

Note that the equation is only defined for $x > 0$.
For $x > 0$,

$$\log_2 x + \log_2(x+1) = \log_2 20$$

$\Rightarrow$ $$\log_2 x(x+1) = \log_2 20$$ By log law 1.

$\Rightarrow$ $$x(x+1) = 20$$ The function $\log_2$ is one–to–one.

$\Rightarrow$ $$x^2 + x - 20 = 0$$

$\Rightarrow$ $$(x+5)(x-4) = 0$$ Factorise.

$\Rightarrow$ $$x+5 = 0 \text{ or } x-4 = 0$$ Zero–product property.

$\Rightarrow$ $$x = -5 \text{ or } x = 4.$$

Since $\log_2 x$ is only defined for $x > 0$, $x = -5$ is not a solution. Hence the only solution is $x = 4$.

The importance of checking the answer.

Caution: By the one–to–one property of logarithmic functions we may deduce that if

$$\log_a p = \log_a q$$

then

$$p = q.$$

In the first step of the example above do not make the mistake of thinking that

$$\begin{aligned} \log_2 x + \log_2 (x+1) &= \log_2 20 \\ \Rightarrow \quad x + x + 1 &= 20. \end{aligned}$$

It is easy to see that this is **not true**. Suppose we consider a similar equation with base 10, and reasoning incorrectly, we find that

$$\begin{aligned} \log x + \log (x+1) &= \log 20 \\ \Rightarrow \quad x + x + 1 &= 20 \\ \Rightarrow \quad 2x &= 19 \\ \Rightarrow \quad x &= 8\tfrac{1}{2}. \end{aligned}$$

If we substitute $x = 8\frac{1}{2}$ into the left side of the equation, we find that

$$\log 8.5 + \log 9.5 \approx 1.907 \neq 20.$$

Another example showing the importance of checking the answer is given below. In this example we have the unknown variable in the base of the logarithm as well.

Example 4.4.6

Solve the following equation.

$$\log_x 2 - \log_x (x+1) = \log_x 3$$

Solution

Note that the equation is only defined for $x > 0$ and $x \neq 1$.

$$\begin{aligned} \log_x 2 - \log_x (x+1) &= \log_x 3 & \\ \Rightarrow \quad \log_x \frac{2}{x+1} &= \log_x 3 & \text{By log law 2.} \\ \Rightarrow \quad \frac{2}{x+1} &= 3 & \text{One–to–one property of log functions.} \\ \Rightarrow \quad x+1 &= \frac{2}{3} & \\ \Rightarrow \quad x &= -\frac{1}{3} & \end{aligned}$$

But, by definition of logarithms, the base must be positive, and hence the equation has no solution.

■ Logarithmic Equations where the Change of Base Formula is used

There are no examples in SRW of logarithmic equations in which the variable for which we need to solve appears as the base of the logarithm. In Example 4.4.6 the variable occurs as the base of the logarithm, but the equation is easily simplified. We now consider an example in which we first need to apply the change of base formula.

Example 4.4.7

Solve for x.

$$\log_2 x + 1 = 6 \log_x 2$$

Solution

$$\log_2 x + 1 = 6 \log_x 2$$

$$\Rightarrow \quad \log_2 x + 1 = 6\frac{\log_2 2}{\log_2 x}$$ Change base x to base 2.

$$\Rightarrow \quad \log_2 x + 1 = \frac{6}{\log_2 x}$$ Property 2 of logarithms.

$$\Rightarrow \quad (\log_2 x)^2 + \log_2 x = 6$$ Multiply by $\log_2 x$.

$$\Rightarrow \quad (\log_2 x)^2 + \log_2 x - 6 = 0$$ Quadratic equation.

$$\Rightarrow \quad (\log_2 x - 2)(\log_2 x + 3) = 0$$

$$\Rightarrow \quad \log_2 x - 2 = 0 \quad \text{or} \quad \log_2 x + 3 = 0$$ Zero–product property.

$$\Rightarrow \quad \log_2 x = 2 \quad \text{or} \quad \log_2 x = -3$$

$$\Rightarrow \quad x = 2^2 \quad \text{or} \quad x = 2^{-3}$$ Change from log form to exponential form.

$$\Rightarrow \quad x = 4 \quad \text{or} \quad x = \frac{1}{8}$$

Check

Substitute $x = 4$ into the left side of the equation.

$$\begin{aligned} x = 4 \ \Rightarrow \ \log_2 x + 1 &= \log_2 4 + 1 \\ &= \log_2 2^2 + 1 \\ &= 2\log_2 2 + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

We need not write 2 as $4^{\frac{1}{2}}$. We can also use the change of base formula:

$$\log_4 2 = \frac{\log_2 2}{\log_2 4} = \frac{\log_2 2}{2\log_2 2} = \frac{1}{2}.$$

Substitute $x = 4$ into the right side of the equation.

$$\begin{aligned} x = 4 \ \Rightarrow \ 6\log_4 2 &= 6\log_4 4^{\frac{1}{2}} && \text{Write 2 as } \sqrt{4} = 4^{\frac{1}{2}}. \\ &= 6.\tfrac{1}{2}.\log_4 4 \\ &= 6.\tfrac{1}{2} && \text{Property 2 of logs.} \\ &= 3 \end{aligned}$$

Since LHS $=$RHS we know that 4 is a valid solution.

We now substitute $x = \frac{1}{8}$ into both sides of the equation.

$$\begin{aligned} \text{LHS} &= \log_2 \tfrac{1}{8} + 1 & \text{RHS} &= 6\log_{\frac{1}{8}} 2 \\ &= \log_2 \left(\tfrac{1}{2}\right)^3 + 1 & &= 6\frac{\log_2 2}{\log_2 \frac{1}{8}} \\ &= \log_2 (2)^{-3} + 1 & &= \frac{6}{\log_2 (2)^{-3}} \\ &= -3\log_2 2 + 1 & &= \frac{6}{-3\log_2 2} \\ &= -3 + 1 & &= \frac{6}{-3} \\ &= -2 & &= -2 \end{aligned}$$

Since LHS $=$RHS, $x = \frac{1}{8}$ is a valid solution. Thus the solutions of the equation are $x = 4$ and $x = \frac{1}{8}$.

SRW does not discuss exponential or logarithmic inequalities, but they are also important, and we deal with them below.

■ Exponential Inequalities

All the principles that apply to solving algebraic inequalities also apply to solving exponential inequalities. You may want to read through Chapter 1, Section 1.7, SRW, before carrying on, especially the rules for inequalities, and the guidelines for solving non–linear inequalities.

Apart from the general principles governing the solution of inequalities we will again apply the one–to–one property of exponential functions, but now we also need to take note of whether the exponential function we are dealing with is increasing or decreasing. Why is this important? Have a look at two familiar graphs.

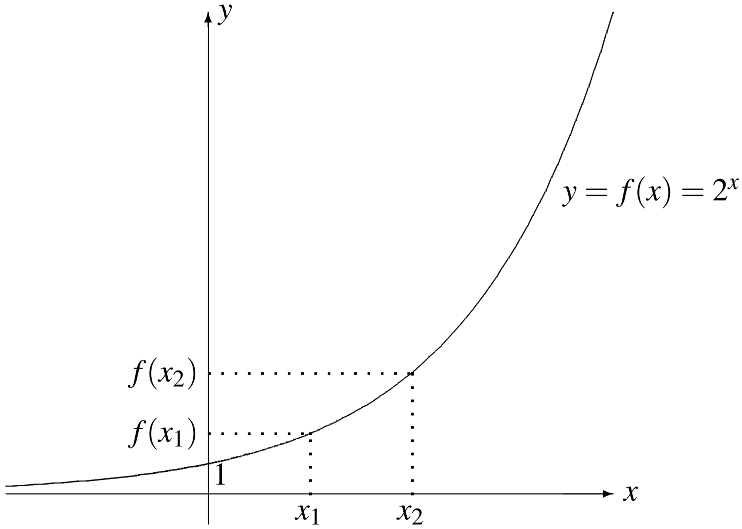


Figure 4.4.1(a)

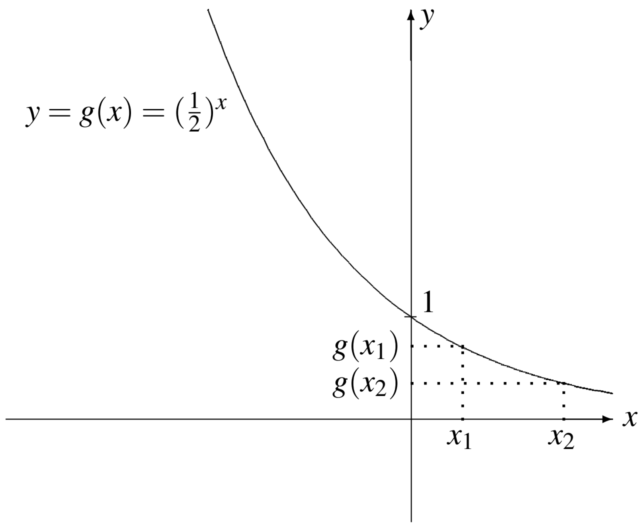


Figure 4.4.1(b)

For the function f defined by $y = 2^x$,

$$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2),$$

and f is an **increasing** function.

For the function g defined by $y = \left(\frac{1}{2}\right)^x$

$$x_1 < x_2 \Leftrightarrow g(x_1) > g(x_2),$$

and g is a **decreasing** function.

The following example shows how the fact that a function is increasing or decreasing affects the solution of an inequality.

Example 4.4.8

Solve for x.

(a) $5^x \le \frac{1}{125}$

(b) $\left(\frac{1}{49}\right)^x \le 7$

Solution

(a)
$$\begin{aligned} & 5^x \le \tfrac{1}{125} \\ \Leftrightarrow \quad & 5^x \le \left(\tfrac{1}{5}\right)^3 \\ \Leftrightarrow \quad & 5^x \le 5^{-3} \\ \Leftrightarrow \quad & x \le -3 \end{aligned}$$

by the one–to–one property and because the exponential function defined by $y = 5^x$ has base $a = 5$, i.e. $a > 1$, so the function is increasing.

(b)
$$\begin{aligned} & \left(\tfrac{1}{49}\right)^x \le 7 \\ \Leftrightarrow \quad & \left(\frac{1}{7^2}\right)^x \le 7 \\ \Leftrightarrow \quad & \left(\tfrac{1}{7}\right)^{2x} \le 7 \end{aligned}$$

We have

$$\left(\tfrac{1}{7}\right)^{2x} \le \left(\tfrac{1}{7}\right)^{-1}$$

i.e.

$$2x \ge -1$$

by the one–to–one property and because the exponential function defined by $y = \left(\frac{1}{7}\right)^x$ has base $a = \frac{1}{7}$, i.e. $a < 1$, so the function is decreasing,

i.e.

$$x \ge -\tfrac{1}{2}.$$

Note that we could also have reasoned this way.

$$\begin{array}{rrcl} & \left(\frac{1}{7^2}\right)^x & \leq & 7 \\ \Leftrightarrow & \left(\frac{1}{7}\right)^{2x} & \leq & 7 \\ \Leftrightarrow & 7^{-2x} & \leq & 7^1 \\ \Leftrightarrow & -2x & \leq & 1 \end{array}$$

by the one–to–one property, and since the exponential function defined by $y = 7^x$ is an **increasing** function (because we have base $a = 7$, i.e. $a > 1$).

We thus have

$$x \geq -\tfrac{1}{2}.$$

We give one final example.

Example 4.4.9

Sketch the graphs of the functions g and k defined by $y = g(x) = \left(\frac{3}{5}\right)^x$ and $y = k(x) = \left(\frac{25}{9}\right)^x$ on the same system of axes. From the graphs determine $\left\{x \in \mathbb{R} \mid \left(\frac{25}{9}\right)^x < \left(\frac{3}{5}\right)^x\right\}$ and check the answer algebraically.

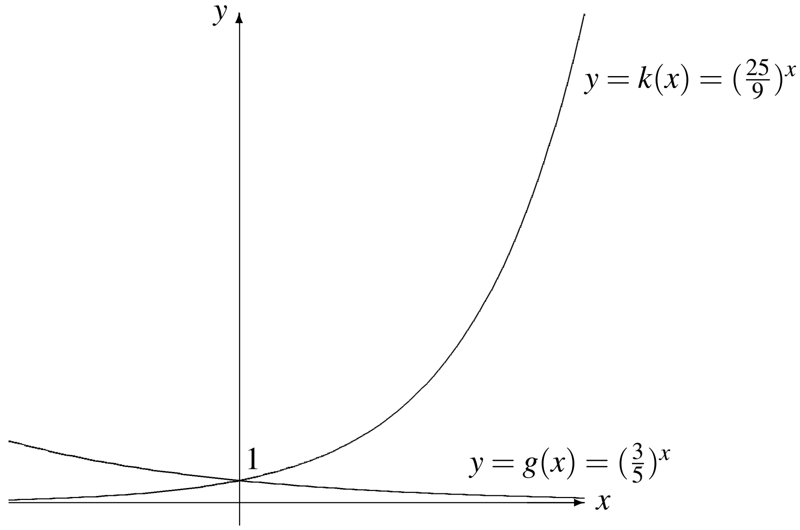


Figure 4.4.2

In Figure 4.4.2 we see that the graphs of k and g intersect at the point $(0, 1)$, and that the graph of k lies **below** the graph of g for all $x < 0$. We thus have

$$\left\{x \in \mathbb{R} \mid \left(\frac{25}{9}\right)^x < \left(\frac{3}{5}\right)^x\right\} = \{x \in \mathbb{R} \mid x < 0\}.$$

We check this algebraically:

$$
\begin{aligned}
& \left(\tfrac{25}{9}\right)^x < \left(\tfrac{3}{5}\right)^x \\
\Leftrightarrow\quad & \left(\left(\tfrac{5}{3}\right)^2\right)^x < \left(\tfrac{5}{3}\right)^{-x} \\
\Leftrightarrow\quad & \left(\tfrac{5}{3}\right)^{2x} < \left(\tfrac{5}{3}\right)^{-x} \\
\Leftrightarrow\quad & 2x < -x,
\end{aligned}
$$

since the function defined by $y = \left(\frac{5}{3}\right)^x$ is increasing (because $y = a^x$ has base $a = \frac{5}{3}$, i.e. $a > 1$.) We thus have

$$
\begin{aligned}
& 3x < 0 \\
\text{i.e.}\quad & x < 0
\end{aligned}
$$

and thus

$$\left\{x \in \mathbb{R} \mid \left(\tfrac{25}{9}\right)^x < \left(\tfrac{3}{5}\right)^x\right\} = \{x \in \mathbb{R} \mid x < 0\}.$$

■ Logarithmic Inequalities

All the principles that apply to solving algebraic inequalities also apply to solving logarithmic inequalities. In addition we need to

- ► take into account the values of x for which the inequality is defined (since if $f(x) = \log_a x$ then $D_f = \{x \in \mathbb{R} \mid x > 0\}$)
- ► take note of whether the logarithmic functions involved in the inequality are increasing or decreasing.

The following examples illustrate the need for these requirements.

Example 4.4.10

Solve for x.

(a) $\log_{\frac{1}{2}} x \geq -4$

(b) $\log_5 (x-2) + \log_5 (x+2) < 1$

Solution

(a) The inequality is defined for $x > 0$.
For $x > 0$,

	$\log_{\frac{1}{2}} x \geq -4$	
$\Leftrightarrow$	$\log_{\frac{1}{2}} x \geq -4\log_{\frac{1}{2}} \left(\frac{1}{2}\right)$	By property 2 of logs.
$\Leftrightarrow$	$\log_{\frac{1}{2}} x \geq \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-4}$	By log law 3.
$\Leftrightarrow$	$\log_{\frac{1}{2}} x \geq \log_{\frac{1}{2}} (2)^4$	See negative exponents, Section 1.2.
$\Leftrightarrow$	$x \leq 2^4$	Since $\log_{\frac{1}{2}}$ is a decreasing function, the direction of the inequality sign changes.
$\Leftrightarrow$	$x \leq 16.$	

Thus the solution is $0 < x \leq 16$.

(b) The inequality is defined for $(x-2) > 0$ and $(x+2) > 0$, i.e. for $x > 2$ **and** $x > -2$. Thus it is defined for $x > 2$.
For $x > 2$,

	$\log_5 (x-2) + \log_5 (x+2) < 1$	
$\Leftrightarrow$	$\log_5 (x-2)(x+2) < \log_5 5$	
$\Leftrightarrow$	$(x-2)(x+2) < 5$	Because $\log_5$ is an increasing function the direction of the inequality sign doesn't change.
$\Leftrightarrow$	$x^2 - 4 < 5$	
$\Leftrightarrow$	$x^2 < 9$	
$\Leftrightarrow$	$-3 < x < 3.$	

If this solution is not immediately clear you can use a table of signs.

$$x^2 < 9$$
$$\Leftrightarrow \quad x^2 - 9 < 0$$
$$\Leftrightarrow \quad (x-3)(x+3) < 0$$

We use $x = -3$ and $x = 3$ as the split points and obtain the following table of signs. The product is negative in the middle column, i.e. for $-3 < x < 3$.

	$x < -3$	$-3 < x < 3$	$x > 3$
$x+3$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
x^2-9	$+$	$-$	$+$

As a final example we solve an inequality in which we also need to change the base of the logarithm.

Example 4.4.11

Solve for x.

$$\log_{\frac{1}{3}}\left(\frac{x}{x+2}\right) \geq \log_3 x$$

Solution

We first note the values of x for which the inequality is defined. Since $\log_a A$ is only defined for $A > 0$ (where A is an algebraic expression) we must have $x > 0$ (from the right hand side) and $\frac{x}{x+2} > 0$, $x \neq -2$ (from the left hand side). Now if $x > 0$ **and** $\frac{x}{x+2} > 0$ then $x+2 > 0$ (if the numerator of a positive fraction is positive then the denominator must also be positive). The inequality is thus defined for $x > -2$ and $x > 0$, i.e. for $x > 0$.

We can change the base from $\frac{1}{3}$ to 3, or from 3 to $\frac{1}{3}$. Since changing the base introduces additional terms we choose to apply the change of base formula to the simplest term, i.e. to the term on the right side of the inequality.

For $x > 0$,

$$\log_{\frac{1}{3}} \frac{x}{x+2} \geq \frac{\log_{\frac{1}{3}} x}{\log_{\frac{1}{3}} 3}$$

$$\Rightarrow \quad \log_{\frac{1}{3}} \frac{x}{x+2} \geq \frac{\log_{\frac{1}{3}} x}{\log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{-1}}$$

$$\Rightarrow \quad \log_{\frac{1}{3}} \frac{x}{x+2} \geq \frac{\log_{\frac{1}{3}} x}{-1} \qquad \text{By log law 3 and log property 2.}$$

$$\Rightarrow \quad \log_{\frac{1}{3}} \frac{x}{x+2} \geq -\log_{\frac{1}{3}} x$$

$$\Rightarrow \quad \log_{\frac{1}{3}} \frac{x}{x+2} \geq \log_{\frac{1}{3}} x^{-1} \qquad \text{By log law 3.}$$

$$\Rightarrow \quad \log_{\frac{1}{3}} \frac{x}{x+2} \geq \log_{\frac{1}{3}} \frac{1}{x}$$

$$\Rightarrow \quad \frac{x}{x+2} \leq \frac{1}{x}. \qquad \text{Since } \log_{\frac{1}{3}} \text{ is a decreasing function.}$$

Since $x > 0$ and $x+2 > 0$ we may multiply both sides of the inequality by $x(x+2)$.

We then obtain

$$x^2 \leq x+2$$

i.e.

$$x^2 - x - 2 \leq 0$$

i.e.

$$(x-2)(x+1) \leq 0. \tag{1}$$

We may either use a table of signs, or a sketch of the parabola defined by $y = x^2 - x - 2$ to solve (1). We find that $(x-2)(x+1) \leq 0 \Leftrightarrow -1 \leq x \leq 2$. But since we already have $x > 0$, the solution of the inequality is

$$0 < x \leq 2.$$

■ Compound Interest

Study Examples 11, 12 and 13 in SRW, Section 4.4. Example 13 is interesting, but does not require knowledge of exponential or logarithmic equations.

Example 4.4.12

At what annual interest rate (to the nearest whole number) must R1 000 be invested so that the investment is worth R1 500 after 2 years, if

(a) interest is compounded quarterly

(b) interest is compounded continuously?

Solution

(a)
$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

We have $A(t) = 1\,500$, $P = 1\,000$, $n = 4$ and $t = 2$. We thus need to find r.

Now

$$1\,500 = 1\,000\left(1+\frac{r}{4}\right)^{4\times 2}$$

i.e.

$$1.5 = \left(1+\frac{r}{4}\right)^{8}$$

i.e.

$$\sqrt[8]{1.5} = 1+\frac{r}{4}$$

i.e.

$$\frac{r}{4} = \sqrt[8]{1.5}-1$$

i.e.

$$r = 4\left(\sqrt[8]{1.5}-1\right)$$

$$\approx 0.208.$$

Thus the interest rate must be approximately 21%.

(b)
$$A(t) = Pe^{rt}$$
where $A(t) = \text{R}1\,500$, $P = \text{R}1\,000$ and $t = 2$ years.

Thus
$$1\,500 = 1\,000e^{2r}$$
i.e.
$$1.5 = e^{2r}$$
i.e.
$$\ln 1.5 = 2r$$
i.e.
$$r = \frac{\ln 1.5}{2}$$
i.e.
$$r \approx 0.203.$$
Thus the interest rate must be approximately 20%.

Additional Exercises 4.4.1

1. Solve the following equations.

   (a) $4^{4x} - 3^{4x} = 2 \cdot 3^{4x} - 3 \cdot 4^{4x}$

   (b) $\dfrac{2^x \sqrt[x]{8}}{2^{7-x}} = 1$

   (c) $2^{2x} = 2 - 7 \cdot 2^{x-1}$

   (d) $3^{2x} + 3^{x+1} - 3 = 1$ (Note the "caution" comment that follows Example 4.4.2)

   (e) $2\log x + \log 3 - \log 4 = \log\left(2x + \frac{3}{4}\right)$

   (f) $\log_2(x-3) + \log_2(x-4) = 1$

   (g) $\log_3 x - 2\log_x 3 = 1$

   (h) $10\log_x e - \ln x = 3$

   (i) $2\log_3 x - 9\log_x 3 + 3 = 0$
   Give the answer correct to two decimal places.

2. Show that the solution set for
$$\frac{3^{2x+1} + 3^{2x-1}}{4 \cdot 9^{x+1} + 6 \cdot 9^{x-1}} = \frac{1}{11}$$
is $\mathbb{R}$.

3. Solve the following inequalities.

   (a) $2^{2x} + 2 \cdot 2^x - 8 < 0$

   (b) $\log_{\frac{1}{3}} x < 2$

   (c) $\log_2 (x-3) + \log_2 (x-4) - 1 < 0$

   (d) $\ln (x+5) + \ln (x-2) > \ln 8$

4.5 MODELING WITH EXPONENTIAL AND LOGARITHMIC FUNCTIONS

To study SRW, Section 4.5.

Outcomes After studying this section you should be able to

- ► solve real–life problems in which knowledge of expo– nential and/or logarithmic equations is required, namely problems dealing with exponential growth of populations; radioactive decay of substances; Newton's Law of Cooling; logarithmic scales.

Exercises Exercises 4.5: 1–41 (odd numbers only)
Additional Exercises 4.5.1

In Section 4.1 we were able to solve problems provided the unknown variable did not appear in the exponent of the formula. Now the techniques of solving exponential and logarithmic equations enable us to solve a wider variety of problems. We consider several types of problems in this section.

■ Exponential Models of Population Growth

We now consider the processes of exponential growth and decay which are prevalent in the biological sciences. SRW begins by considering the doubling time of a population of bacteria, and shows how this leads to a formula for the growth of that particular type of bacteria. In general,

$$n(t) = n_0 2^{ct} \tag{1}$$

where $c = \frac{1}{a}$. In this case n_0 is the initial size of the population; $n(t)$ is the size after time period t has elapsed and the doubling period is a.

After this formula there is a statement "If we knew the tripling time b, then the formula would be

$$n(t) = n_0 3^{ct} \tag{2}$$

where $c = \frac{1}{b}$."

Look at the way formula (1) is derived from the pattern given in SRW, and see whether you can derive formula (2). Assume that the culture begins with 1 000 bacteria and that it **triples** in size every 5 hours.

The formula we use to determine exponential growth is

$$n(t) = n_0 e^{rt}$$

where
- n_0 is the initial population size
- $n(t)$ is the size of the population after time period t
- t is the period of time (expressed in any unit, such as hours, days, years) over which the population increases
- r is a constant (which is given or which needs to be calculated) relative to the particular population; it expresses the rate of that population's growth.

If we can find the relative growth rate of a population over a specific time period, we can use this information to predict the size of the population at some future stage. Note that such a prediction is dependent on the fact that all the conditions under which the population initially increased remain unchanged. For example, we realise that if the growth rate of a particular culture of bacteria is 2%, and the temperature of the environment in which the bacteria are kept changes, the growth rate of 2% would no longer be accurate. It may be mathematically possible to maintain all the conditions, however, in practice growth does not continue to increase exponentially, at the same rate. Read the comment at the end of Example 3, SRW.

In SRW, Example 1 shows how the size of a population can be predicted. Example 2 requires the use of a graphing calculator, but it is interesting to read it anyway, to see how a small change in the relative rate of growth makes a significant difference in the long term. Example 3 reverses the way in which the formula is applied: if we know the population at the end of a certain period and the relative growth rate then we can find out what the population was at the beginning. In this example we know $n(t)$, t and r, and we need to find n_0.

What happens when we do not know the relative growth rate r, or the time period t? Provided we have three out of the four unknowns in the equation

$$n(t) = n_0 e^{rt}$$

we can find the value of the remaining unknown. However, if we need to solve for r or t, we need to be able to solve **exponential equations** (an equation in which the variable for which we must solve appears as the exponent).

If, for example, r is the only unknown in the formula $n(t) = n_0 e^{rt}$, we can calculate r provided we know the values of $n(t)$, n_0 and t. We can then use this value of r to predict $n(t)$ for different values of t.

Example 4.5.1

Suppose a culture of 20 000 bacteria grows to 25 000 in one hour and forty minutes. How many bacteria will there be after 12 hours?

Solution

We have

$$n(t) = n_0 e^{rt}$$

where $n_0 = 20000$, $n(t) = 25\,000$ and $t = 1\frac{2}{3}$ (remember that 40 minutes is $\frac{40}{60}$ hour, i.e. $\frac{2}{3}$ hour). If we substitute into the equation we have

$$n(t) = 25\,000 = 20000 e^{r \cdot 1\frac{2}{3}}$$

i.e. $$e^{\frac{5}{3}r} = \frac{25\,000}{20\,000}$$

i.e. $$e^{\frac{5r}{3}} = \frac{5}{4}.$$

This is an exponential equation: it is an equation in which the variable for which we must solve appears in the exponent. In this case we need to solve for r.

Now

$$\ln\left(e^{\frac{5r}{3}}\right) = \ln\left(\frac{5}{4}\right)$$

i.e. $$\tfrac{5r}{3} = \ln\frac{5}{4}$$

i.e. $$r = \frac{3}{5}\ln\frac{5}{4} \approx 0.13389.$$

We substitute this value of r, $n_0 = 20\ 000$ and $t = 12$ into the equation

$$n(t) = n_0 e^{rt}$$

to find how many bacteria there will be after 12 hours, that is,

$$n(12) = 20\ 000 e^{(0.13389)(12)} \approx 99\ 720.$$

Comment

If we consider the equation

$$n(t) = n_0 e^{rt}$$

as the definition of an exponential function n of t , then n_0 *is* the **value of the function** when $t = 0$, that is

$$n(0) = n_0 e^{r(0)} = n_0 e^0 = n_0.$$

Obviously, if $r = 0$ then $e^{rt} = e^0 = 1$, and thus in the formula

$$n(t) = n_0 e^{rt}$$

we have

$$n(t) = n_0$$

for **any** $t > 0$ when $r = 0$. We interpret this as follows: when $r = 0$ (i.e. the relative rate of growth is zero) then **no** growth takes place.

In SRW all the examples given in this subsection require the use of a calculator. In the following example you do not need to use a calculator.

Example 4.5.2

The population of a country was 40 million in 1980 and 50 million in 1990. Without using a calculator answer the following questions.

(a) Express the growth rate in terms of ln.

(b) What is the projected population for 2010?

Solution

(a) We have

$$n(t) = n_0 e^{rt}$$

where $n(t) = 50$, $n_0 = 40$ and $t = 10$.

Now

$$50 = 40e^{10r}$$

$$\Leftrightarrow \quad \frac{5}{4} = e^{10r}$$

$$\Leftrightarrow \quad \ln\frac{5}{4} = 10r$$

$$\Leftrightarrow \quad r = \frac{1}{10}\ln\frac{5}{4}.$$

Thus the growth rate is given by $\frac{1}{10}\ln\frac{5}{4}$.

(b) In 2010 we have $t = 30$, $n_0 = 40$ and r as expressed in (a). Thus

$$\begin{aligned} n(t) &= 40e^{\left(\frac{1}{10}\ln\frac{5}{4}\right)30} \\ &= 40e^{\frac{30}{10}\ln\frac{5}{4}} \\ &= 40e^{3\ln\frac{5}{4}} \\ &= 40e^{\ln\left(\frac{5}{4}\right)^3} \\ &= 40\left(\frac{5}{4}\right)^3 \\ &= \frac{40\times 125}{64} \\ &= 78\tfrac{1}{8}. \end{aligned}$$

i.e. in 2010 the projected population will be $78\frac{1}{8}$ million people, i.e. 78 125 000 people.

■ Radioactive Decay

If you have never studied chemistry the expression "radioactive decay" may be unfamiliar. Radioactive chemicals decay at various rates. This means that an initial amount of a chemical substance, say 50 g, decays so that after a certain period of time there is less than 50 g of the substance left.

Example 4.5.3

We use Question 66, Exercises 4.1, SRW as an example.

Solution

In Question 66 we begin with an unspecified amount (mass, in grams) of the substance (we can call it m_0); the rate of decay is 8.7%, and at time t the amount is given by

$8.7\% = \frac{8.7}{100} = 0.087$

$$m(t) = 6e^{-0.087t}.$$

(a) We need to find m_0, i.e. the mass, in grams, at the beginning, before the decay process has started (we call this moment $t = 0$).

If $m(t) = 6e^{-0.087t}$ then $t = 0 \Rightarrow m(0) = 6e^0 = 6$, i.e. at time $t = 0$ the mass of the substance is 6 grams.

(b) After 20 days the amount (mass, in grams) of substance that remains is given by $m(20)$, where

$$\begin{aligned} m(20) &= 6e^{(-0.087)\times 20} \\ &= 6e^{-1.74} \\ &\approx 1.053 \text{ (correct to three decimal places)} \end{aligned}$$

i.e. after 20 days there will be approximately 1 gram of substance left.

Read the comment in the margin of this subsection, and the discussion leading up to the formula above Example 6. Study Example 6, noting (for interest) the graph drawn for part (d).

Comment

We do not always need to relate radioactive decay to the half–life of a substance. Consider the following example.

Example 4.5.4

A radioactive substance decays in such a way that the initial amount of substance is 20 grams, and after 5 years an amount of 16 grams is left.

(a) Find an expression (in terms of ln) for the rate of decay.

(b) Find the amount of substance that will be left after 10 years, without using a calculator.

Solution

We use the formula

$$m(t) = m_0 e^{-rt}$$

where r is the relative rate of decay.

(a) We have $m(t) = m_0 e^{-rt}$, where $m(t) = 16$, $m_0 = 20$ and $t = 5$.

Now

$$
\begin{aligned}
& & 16 &= 20e^{-5r} \\
&\Leftrightarrow & \frac{16}{20} &= e^{-5r} \\
&\Leftrightarrow & \ln\frac{4}{5} &= \ln e^{-5r} \\
&\Leftrightarrow & \ln\frac{4}{5} &= -5r\ln e \\
&\Leftrightarrow & r &= -\frac{1}{5}\ln\frac{4}{5}.
\end{aligned}
$$

(b) We need to calculate $m(t)$, where $t = 10$. Thus

$$
\begin{aligned}
m(10) &= 20e^{-10\left(-\frac{1}{5}\ln\frac{4}{5}\right)} \\
&= 20e^{2\ln\frac{4}{5}} \\
&= 20e^{\ln\left(\frac{4}{5}\right)^2} \\
&= 20\left(\frac{4}{5}\right)^2 \\
&= 20 \times \frac{16}{25} \\
&= 12.8.
\end{aligned}
$$

Thus, after 10 years there will be 12.8 grams of the substance left.

■ Newton's Law of Cooling

This may also be unfamiliar if you have not studied Physics, but the principle is clearly explained in the subsection.

Study Example 7.

■ Logarithmic Scales

Study the comment at the start of the subsection. Logarithms serve a very useful purpose. Common logarithms can be used to scale down very large numbers so that we have a more manageable set of numbers. For example, if a certain quantity can take on values from 10^{-15} to 10^{10}, the common logarithms of these numbers will vary from -15 to 10. We represent this diagrammatically on the next page

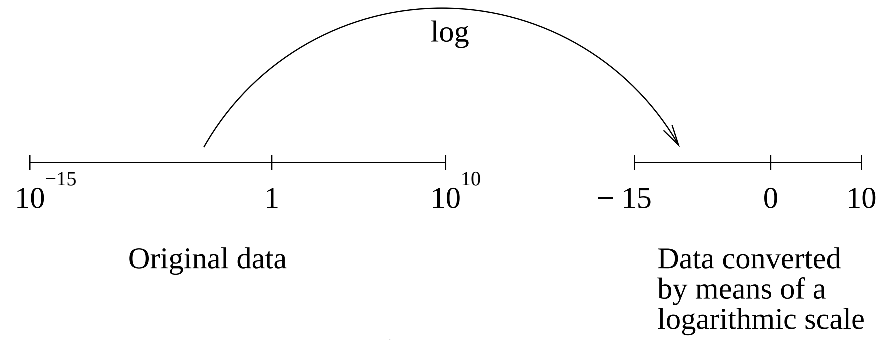


Figure 4.5.1

Now work through Examples 8, 9, 10 and 11.

Additional Exercises 4.5.1

1. An amount of R1 000 is invested at an annual interest rate of 8% and interest is compounded quarterly.

   (a) When will the investment be worth R1 200?

   (b) When will the investment have doubled?

   In both cases give your answer correct to the nearest month.

2. In a certain section of the Hartebeespoort dam it was found that the area covered by water hyacinth (a type of water weed) increased exponentially according the formula

$$n(t) = n_0 e^{rt}$$

   where $n(t)$ was the area covered by water hyacinth in square metres after t days. Initially the area covered by water hyacinth was 100 m^2. After 10 days the area was 150 m^2.

   (a) Determine the value of r in the above equation. Leave your answer in terms of ln.

   (b) Use the formula to determine the area covered by water hyacinth after 30 days, without using a calculator.

3. A radioactive substance decays according to the formula

$$m(t) = m_0 e^{kt}$$

   where $m(t)$ is the amount of substance in grams after t years. The initial amount of substance is 100 grams and after 5 years an amount of 25 grams is left.

   (a) Determine the value of k (leave your answer in terms of ln).

   (b) Determine the amount of substance remaining after 10 years, without using a calculator.

4. The population of a certain country grows according to the formula

$$n(t) = n_0 e^{rt}$$

where $n(t)$ is the number of people in millions at time t in years.

The population in 1980 was 40 million and in 1990 was 50 million.

   (a) Determine the value of k (leave your answer in terms of ln).
   (b) What was the population in the year 2000? Find the answer without using a calculator.

5. If the intensity of an earthquake is 10 times larger than one measuring 5.4 on the Richter scale, what will it measure on the Richter scale?

6. Find the loudness in decibels of a lawnmower if it operates at an intensity of 10^{-4} watts per square metre.

7. If one sound is 100 times as intense as another, what is the difference in the loudness of the two sounds? Express your answer in decibels.

8. (a) The hydrogen ion concentration of a sample of orange juice is $4.1 \times 10^{-4} M$. Calculate the pH of this orange juice.
   (b) Write down the exponential form of the formula $pH = -\log[H^+]$.
   (c) Suppose a certain acid with a pH of 4.0 was diluted until the pH was 5.0. Calculate the hydrogen ion concentration $[H^+]$ of the original acid and of the diluted acid. By what factor did the hydrogen ion concentration decrease? You do not need to use a calculator to find the answer.

4 REVIEW AND TEST

Now that you have reached the end of Chapter 4, work through the Chapter Review: Concept Check and Exercises, and the Chapter Test. You may do as many questions as you have time for.

In the Concept Check, at question 8 you could include the following question.

(c) Suppose an asset is valued at P, and that the rate of appreciation or depreciation is r. If A is the appreciated/depreciated value of the asset after t years, write down the formulas used to calculate appreciation and depreciation.

5 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

Trigonometric functions can be defined in terms of real numbers or in terms of angles. In this chapter we study the real number approach and in the next chapter we consider the angle approach. As the two approaches are independent of each other you may study either chapter first. If you are familiar with the angle approach from work done at school you may find it easier to study Chapter 6 before this chapter. Now read the introductory remarks on the first page of Chapter 5 in SRW.

5.1 THE UNIT CIRCLE

To study SRW, Section 5.1, and additional notes.

Outcomes When you have studied this section you should be able to

- ▶ determine whether or not a point lies on the unit–circle
- ▶ determine the point which lies on the unit circle given one of its coordinates and some other information
- ▶ find on the unit circle the point $P(t)$, the terminal point determined by the real number t
- ▶ determine the coordinates of the terminal point $P(t)$ for some special values of t such as 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ and multiples of these numbers
- ▶ find the reference number $\bar{t}$ associated with the real number t
- ▶ use the reference number to find the terminal point.

Exercises Exercises 5.1: questions 3, 5, 7, 9, 13–53 (odd numbers), 55, 56

The four quadrants

Before studying Section 5.1 make sure that you know the signs of the x and y–coordinates of a point depending on the quadrant in which the point lies. We summarise this information in Figure 5.1.1 below. Note that the x–axis and y–axis divide the Cartesian plane into four quadrants which are called quadrants I, II, III and IV (or first, second, third and fourth quadrants). The points on the coordinate axes are not considered to belong to any specific quadrant.

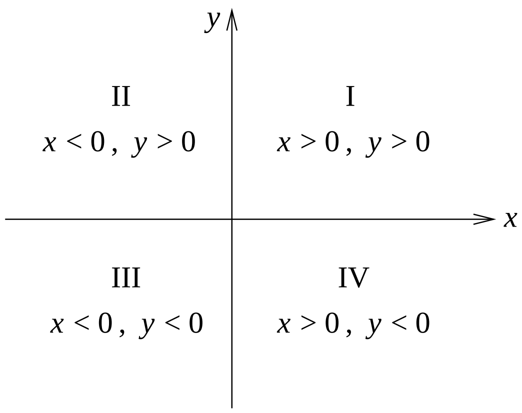


Figure 5.1.1

■ The Unit Circle

Read the subsection in SRW. Note that

- ▶ if the coordinates of a point do **not** satisfy the equation of the unit circle then the point does not lie on the circle
- ▶ we use the information contained in Figure 5.1.1 to solve the problem in Example 2.

■ Terminal Points on the Unit Circle

Notation

Now carefully study this subsection in SRW, and the additional notes below. We use $P(t)$ to denote the terminal point determined by the real number t. Thus, for example, we write $P\left(\frac{\pi}{2}\right) = (0,1)$ to denote the terminal point determined by $t = \frac{\pi}{2}$.

In Example 3(a) we see that $P(3\pi) = P(\pi)$. Since the circumference of the unit circle is 2π we see, for example, that

$$P(\pi) = P(3\pi) = P(5\pi) = P(7\pi) = P(9\pi)\ldots$$

i.e. we will reach $P(\pi)$ if we travel along an arc, in a **counterclockwise** direction, a distance of π units, or (π plus any multiple of 2π) units. This reasoning is true for any point $P(t_0)$ where $0 \leq t_0 \leq 2\pi$, i.e.

$$P(t_0 + 2n\pi) = P(t_0) \quad \text{where } n \in \mathbb{Z} \text{ and } n > 0.$$

Now suppose $-2\pi \leq t_0 \leq 0$. For example, choose $t_0 = -\frac{\pi}{2}$. Again we see that if we travel along an arc, in a **clockwise** direction, for a distance of $\left|-\frac{\pi}{2}\right|$ units, or a distance of $\left(\left|-\frac{\pi}{2}\right| \text{plus any multiple of } 2\pi\right)$ units, we will arrive at the same point, i.e.

$$P\left(-\frac{\pi}{2}\right) = P\left(-\frac{\pi}{2} - 2\pi\right) = P\left(-\frac{\pi}{2} - 4\pi\right) = P\left(-\frac{\pi}{2} - 6\pi\right).$$

Thus for $P(t_0)$ where $-2\pi \leq t_0 \leq 0$ we have

$$P(t_0 + 2n\pi) = P(t_0) \text{ where } n \in \mathbb{Z} \text{ and } n < 0.$$

Srategy to locate $P(t)$ where $t > 2\pi$

If we want to locate the terminal point $P(t)$ where $t > 2\pi$ we first subtract a suitable multiple of 2π from t to obtain t_0, such that $0 \leq t_0 < 2\pi$. Then $P(t) = P(t_0)$ and we can then locate $P(t_0)$ on the unit circle. For example, suppose $t = 7\frac{1}{3}\pi$. We have

$$\begin{aligned} t_0 &= 7\tfrac{1}{3}\pi - 3 \times 2\pi \\ &= 1\tfrac{1}{3}\pi \\ &= \tfrac{4}{3}\pi. \end{aligned}$$

Thus $P\left(7\frac{1}{3}\pi\right) = P\left(\frac{4}{3}\pi\right)$.

Srategy to locate $P(t)$ where $t < -2\pi$

If $t < -2\pi$ we locate the terminal point $P(t)$ by adding a suitable multiple of 2π to t to obtain t_0, such that $-2\pi < t_0 \leq 0$. Then $P(t) = P(t_0)$ and we can locate $P(t_0)$ on the unit circle. For example, suppose $t = -5\frac{1}{6}\pi$. Then we have

$$\begin{aligned} t_0 &= -5\tfrac{1}{6}\pi + 2 \times 2\pi \\ &= -1\tfrac{1}{6}\pi \\ &= -\tfrac{7}{6}\pi. \end{aligned}$$

Thus $P\left(-5\frac{1}{6}\pi\right) = P\left(-\frac{7}{6}\pi\right)$.

Quadrants in which points lie

We now consider the quadrant in which a point $P(t)$ lies if $0 \leq t \leq 2\pi$ or if $-2\pi \leq t \leq 0$. We also consider approximations of π since t may not be given in terms of π but rather as a number such as 2.3. Since $\pi \approx 3.14$ we have

$$\frac{\pi}{2} \approx 1.57, \quad \frac{3\pi}{2} \approx 4.71 \text{ and } 2\pi \approx 6.28.$$

Figure 5.1.2 gives a useful summary.

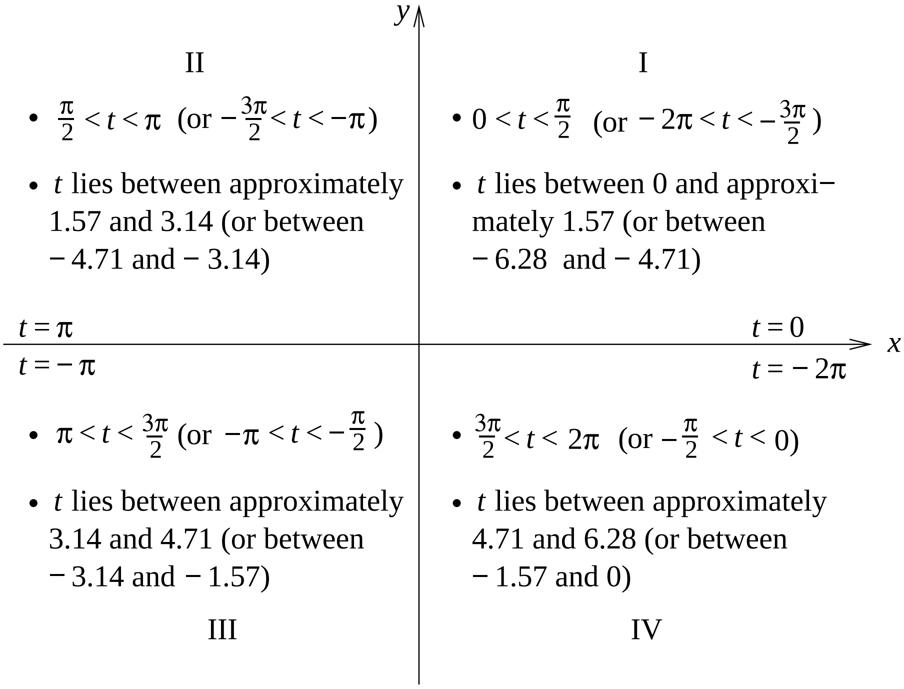


Figure 5.1.2

Thus from Figure 5.1.2 we see that

- ► $P\left(\frac{2\pi}{3}\right)$ lies in quadrant II since $\frac{\pi}{2} < \frac{2\pi}{3} < \pi$
- ► $P\left(-\frac{5\pi}{6}\right)$ lies in quadrant III since $-\pi < -\frac{5\pi}{6} < -\frac{\pi}{2}$
- ► $P(5.23)$ lies in quadrant IV since $4.71 < 5.23 < 6.28$
- ► $P(-4.8)$ lies in quadrant I since $-6.28 < -4.8 < -4.71$.

Note:

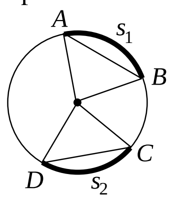
Suppose $t = 10.21$, and we need to locate $P(t)$ on the unit circle. We use the strategy discussed before Figure 5.1.2 and write t in terms of t_0, where $0 \leq t_0 < 6.28$. We then use Figure 5.1.2. Now

$$\begin{aligned} t_0 &= 10.21 - 1 \times 6.28 \\ &= 3.93. \end{aligned}$$

Hence $P(10.21) = P(3.93)$.

Since $P(3.93)$ lies in quadrant III it follows that $P(10.21)$ also lies in this quadrant.

Special Terminal Points



Since arcs s_1 and s_2 have equal lengths, chords AB and CD have equal lengths.

After Example 3 we read how the coordinates of the terminal point $P\left(\frac{\pi}{4}\right)$ are found. It is left as an exercise (questions 55 and 56 of Exercises 5.1) for you to determine the coordinates of the terminal points $P\left(\frac{\pi}{6}\right)$ and $P\left(\frac{\pi}{3}\right)$. We strongly recommend that you first attempt these questions and then compare your solutions with the answers that we provide at the back of this study guide. Hint for question 55: Equal arcs subtend equal chords.

Table 1 in Section 5.1 of SRW gives the terminal points determined by certain special values of t. You should learn this table so that you can easily find the terminal points of related values of t. See, for example, Example 4 and the next subsection.

■ The Reference Number

Study this subsection in SRW. Table 5.1.1 below contains formulas which can be used to find the reference number $\bar{t}$ for a given number t, where $0 \le t \le 2\pi$ or $-2\pi \le t \le 0$. Convince yourself that these formulas are correct by studying the definition of reference number given in SRW, as well as Figure 8 and Example 5 together with Figure 9.

Reference number $\bar{t}$	Value of t	Quadrant in which $P(t)$ lies
$\bar{t} = t$	$0 < t < \frac{\pi}{2}$	I
$\bar{t} = \pi - t$	$\frac{\pi}{2} < t < \pi$	II
$\bar{t} = t - \pi$	$\pi < t < \frac{3\pi}{2}$	III
$\bar{t} = 2\pi - t$	$\frac{3\pi}{2} < t < 2\pi$	IV
$\bar{t} = \lvert t \rvert$	$-\frac{\pi}{2} < t < 0$	IV
$\bar{t} = \pi - \lvert t \rvert$	$-\pi < t < -\frac{\pi}{2}$	III
$\bar{t} = \lvert t \rvert - \pi$	$-\frac{3\pi}{2} < t < -\pi$	II
$\bar{t} = 2\pi - \lvert t \rvert$	$-2\pi < t < -\frac{3\pi}{2}$	I

Table 5.1.1

If we are given a number t, we can use the reference number $\bar{t}$ to determine the terminal point $P(t)$. See the block below Figure 9 and Examples 6 and 7. Figure 5.1.3 on the next page gives the coordinates of $P(t)$ in terms of the coordinates of $P(\bar{t})$ and shows how the coordinates of $P(t)$ depend on the quadrant in which $P(t)$ lies.

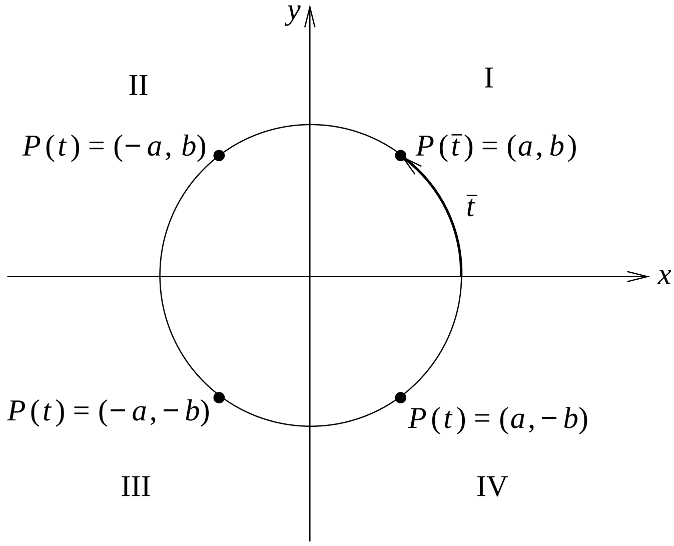


Figure 5.1.3

We expand the block below Figure 9 in SRW in the table below.

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS
To find the terminal point $P(t)$ determined by any value of t, we use the following steps. 1. If $t > 2\pi$ or $t < -2\pi$ then determine t_0 such that $0 \leq t_0 \leq 2\pi$ or $-2\pi \leq t_0 \leq 0$ and $P(t) = P(t_0)$. Use the strategy discussed before Figure 5.1.2 in this study guide. 2. Find the reference number $\bar{t}$. Use Table 5.1.1 given in this study guide. 3. Find the terminal point $P(\bar{t}) = (a,b)$. Use Table 1 in Section 5.1 of SRW. 4. Find the terminal point $P(t)$ according to the quadrant in which $P(t)$ lies: Quadrant I $P(t) = (a,b)$ Quadrant II $P(t) = (-a,b)$ Quadrant III $P(t) = (-a,-b)$ Quadrant IV $P(t) = (a,-b)$

Table 5.1.2

We use these steps in Example 5.1.1.

Example 5.1.1

Find the terminal point $P(t)$ for each of the following values of t.

(a) $t = 3\frac{5}{6}\pi$ (b) $t = -4\frac{2}{3}\pi$

Solution

Step 1 (a) We have $t_0 = 3\frac{5}{6}\pi - 2\pi = 1\frac{5}{6}\pi = \frac{11\pi}{6}$.

Thus $P\left(3\frac{5}{6}\pi\right) = P\left(\frac{11\pi}{6}\right)$.

Step 2 According to Table 5.1.1, since $P\left(\frac{11\pi}{6}\right)$ lies in quadrant IV, we have

$$\bar{t} = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}.$$

Step 3 From Table 1 in Section 5.1 of SRW we have

$$P\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Step 4 Since $P\left(\frac{11\pi}{6}\right)$ lies in quadrant IV we have

$$P\left(\frac{11\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

Hence $P\left(3\frac{5}{6}\pi\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

Step 1 (b) We have $t_0 = -4\frac{2}{3}\pi + 2 \times 2\pi = -\frac{2\pi}{3}$.

Thus $P\left(-4\frac{2}{3}\pi\right) = P\left(-\frac{2\pi}{3}\right)$.

Step 2 According to Table 5.1.1, since $P\left(-\frac{2\pi}{3}\right)$ lies in quadrant III, we have

$$\bar{t} = \pi - \left|-\frac{2\pi}{3}\right| = \pi - \frac{2\pi}{3} = \frac{\pi}{3}.$$

Step 3 From Table 1 in Section 5.1 of SRW we have

$$P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Step 4 Since $P\left(-\frac{2\pi}{3}\right)$ lies in quadrant III we have

$$P\left(-\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Hence $P\left(-4\frac{2}{3}\pi\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

5.2 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

To study SRW, Section 5.2, and additional notes.

Outcomes After studying this section you should be able to

- define the six trigonometric functions
- calculate trigonometric function values (if they exist) for $t = 0$, $t = \frac{\pi}{6}$, $t = \frac{\pi}{4}$, $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$
- determine the domain of each of the trigonometric functions
- calculate trigonometric function values (if they exist) for numbers t,where $t > \frac{\pi}{2}$, that can be related to the special values of t given above
- calculate trigonometric function values of t using a calculator
- apply the even–odd properties of the trigonometric functions to find the trigonometric function values of t,where $t < 0$
- recognise and apply the **reduction formulas** that relate to $f(-t)$, $f(\pi - t)$, $f(\pi + t)$, $f(2\pi - t)$, $f\left(\frac{\pi}{2} - t\right)$ and $f\left(\frac{\pi}{2} + t\right)$, where f is a trigonometric function
- recognise and apply the relationships, which we call the **fundamental identities**, that exist between certain trigonometric functions
- find all the trigonometric function values of t, given the value of one of the trigonometric functions of t and some other information
- write one trigonometric function in terms of another using the fundamental identities.

Exercises Exercises 5.2: questions 1 – 21 (odd numbers), 23 – 26, 27 – 77 (odd numbers)
Additional Exercises 5.2.1: question 1

■ The Trigonometric Functions

Study the subsection in SRW. Note the following.

- ► The abbreviations for the different trigonometric functions are as follows.

Function	**Abbreviation**
sine	sin
cosine	cos
tangent	tan
cosecant	csc or cosec
secant	sec
cotangent	cot

- ► Using set notation we can write the domains of the trigonometric functions as follows.

DOMAINS OF THE TRIGONOMETRIC FUNCTIONS	
Function	**Domain**
sin, cos	$\mathbb{R}$
tan, sec	$\mathbb{R}-\left\{t\in\mathbb{R}\,\middle\|\,t=\frac{\pi}{2}+n\pi,\ n\in\mathbb{Z}\right\}$
cot, csc	$\mathbb{R}-\{t\in\mathbb{R}\,\|\,t=n\pi,\ n\in\mathbb{Z}\}$

■ Values of the Trigonometric Functions

Study the subsection in SRW. We note the following.

- ► The diagram in the margin close to Example 2 helps us to remember which trigonometric functions are positive in each of the quadrants. We can expand this diagram (see Figure 5.2.1) to include all functions that are positive in a particular quadrant. We note, from the definition of the trigonometric functions, that

$$\begin{aligned}\csc t &= \frac{1}{\sin t}\\ \sec t &= \frac{1}{\cos t}\\ \cot t &= \frac{1}{\tan t}\end{aligned}$$

  Hence if $\sin t$ is positive then $\csc t$ will be positive; if $\cos t$ is positive then so will $\sec t$ be positive and if $\tan t$ is positive then $\cot t$ will also be positive. We include this information in Figure 5.2.1.

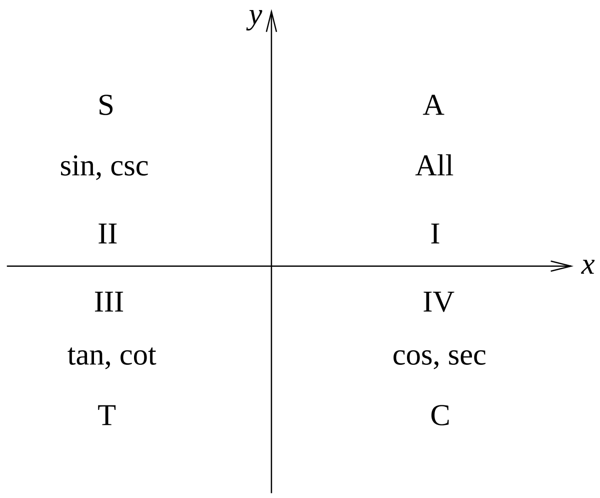


Figure 5.2.1

Reading counterclockwise from quadrant I, SRW says that we can remember this diagram as "**A**ll **S**tudents **T**ake **C**alculus". You could also remember this as "**A**ll **S**tudents **T**ake **C**offee".

► The advantage of the even–odd properties of trigonometric functions given in SRW, below Example 4, is that we can rewrite functions of negative values of t in terms of positive values. We can thus find $\bar{t}$ using only the first four rows of Table 5.1.1 in Section 5.1 of this study guide. We illustrate this by means of an example.

Example 5.2.1

Determine $\sin\left(-5\frac{1}{4}\pi\right)$.

Solution

According to the even–odd properties we have

$$\sin\left(-5\frac{1}{4}\pi\right) = -\sin\left(5\frac{1}{4}\pi\right).$$

Since $5\frac{1}{4}\pi - 4\pi = 1\frac{1}{4}\pi = \frac{5\pi}{4}$ we have

$$P\left(5\frac{1}{4}\pi\right) = P\left(\frac{5\pi}{4}\right).$$

The reference number for $\frac{5\pi}{4}$ is $\bar{t} = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ (see Table 5.1.1). Since $\frac{5\pi}{4}$ is in quadrant III, $\sin\left(\frac{5\pi}{4}\right)$ is negative, i.e. $\sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$. Hence

$$\sin\left(-5\frac{1}{4}\pi\right) = -\sin\left(5\frac{1}{4}\pi\right) = -\sin\left(\frac{5\pi}{4}\right) = -\left(-\sin\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

■ Reduction Formulas

We have just looked at the even–odd properties of the trigonometric functions, in which we expressed $f(-t)$ in terms of $f(t)$. There are several other useful relationships that enable us to express trigonometric functions of $\frac{\pi}{2} - t$, $\frac{\pi}{2} + t$, $\pi - t$, $\pi + t$ and $2\pi - t$ in terms of trigonometric functions of t, for positive values of t. We say that expressions such as $f(\pi + t)$ have been **reduced** to $f(t)$, hence the term **reduction formulas**. For completeness we include this subsection on reduction formulas in the study guide. Although we can **prove the reduction formulas for any positive value of** t, in the discussion that follows we restrict ourselves to $0 < t < \frac{\pi}{2}$.

Relationship between $f(\pi - t)$ and $f(t)$

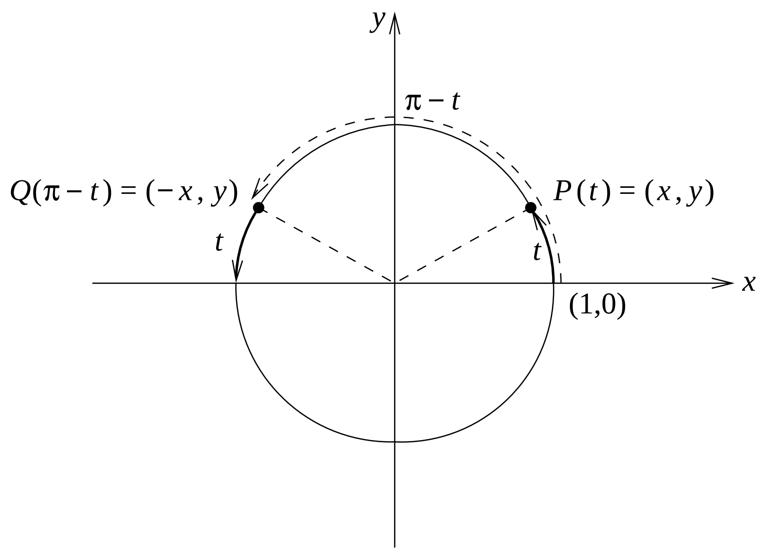


Figure 5.2.2

In Figure 5.2.2 the point $P(t)$ has coordinates (x,y) and the point Q which is determined by $\pi - t$ has coordinates $(-x,y)$.

From the definition of the trigonometric functions it follows that

$$\sin(\pi - t) = y \qquad \cos(\pi - t) = -x \qquad \tan(\pi - t) = \frac{y}{-x}$$

$$\csc(\pi - t) = \frac{1}{y} \qquad \sec(\pi - t) = \frac{1}{-x} \qquad \cot(\pi - t) = \frac{-x}{y}$$

and hence we have the following formulas.

$$\begin{aligned}
\sin(\pi - t) &= \sin t \\
\cos(\pi - t) &= -\cos t \\
\tan(\pi - t) &= -\tan t \\
\csc(\pi - t) &= \csc t \\
\sec(\pi - t) &= -\sec t \\
\cot(\pi - t) &= -\cot t
\end{aligned}$$

Relationship between $f(t+\pi)$ and $f(t)$

See question 83 of Exercises 5.2 in SRW.

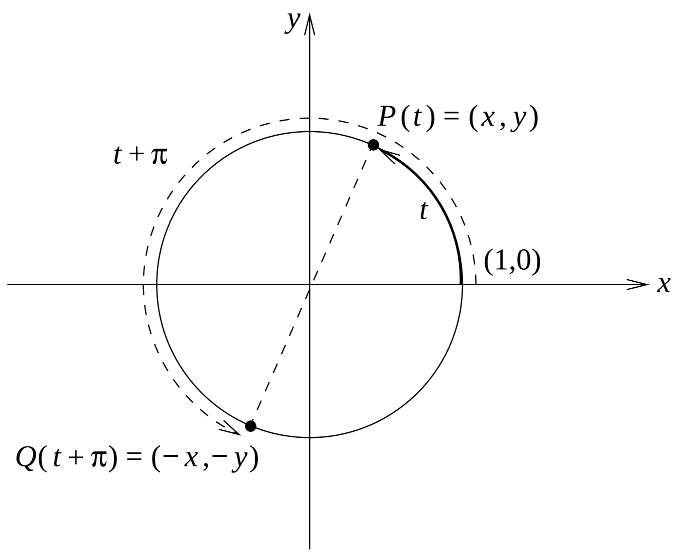


Figure 5.2.3

In Figure 5.2.3 the point $P(t)$ has coordinates (x, y) and the point Q determined by $t+\pi$ has coordinates $(-x, -y)$. If we apply the definition of the trigonometric functions we find the following formulas.

$$\begin{aligned}
\sin(t+\pi) &= -\sin t\\
\cos(t+\pi) &= -\cos t\\
\tan(t+\pi) &= \tan t\\
\csc(t+\pi) &= -\csc t\\
\sec(t+\pi) &= -\sec t\\
\cot(t+\pi) &= \cot t
\end{aligned}$$

Relationship between $f(2\pi - t)$ and $f(t)$

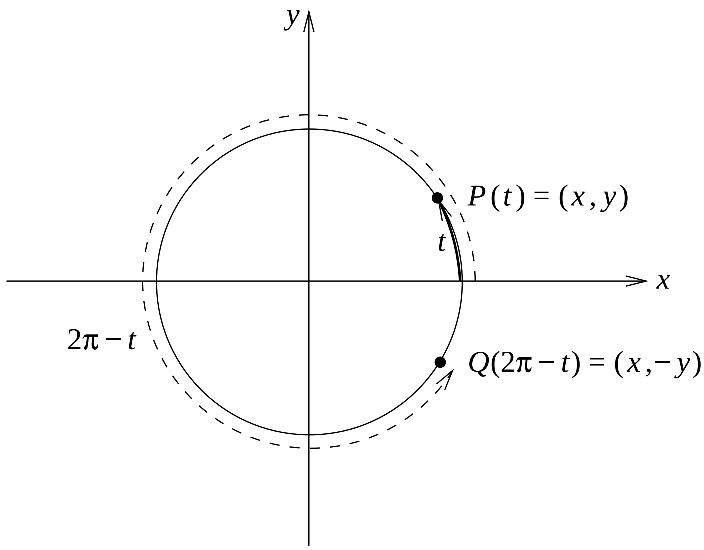


Figure 5.2.4

In Figure 5.2.4 the point $P(t)$ has coordinates (x,y) and the point Q has coordinates $(x,-y)$. After applying the definition of the trigonometric functions we obtain the following formulas.

$$\begin{aligned}
\sin(2\pi-t) &= -\sin t\\
\cos(2\pi-t) &= \cos t\\
\tan(2\pi-t) &= -\tan t\\
\csc(2\pi-t) &= -\csc t\\
\sec(2\pi-t) &= \sec t\\
\cot(2\pi-t) &= -\cot t
\end{aligned}$$

Relationship between $f\left(\frac{\pi}{2}-t\right)$ and $g(t)$

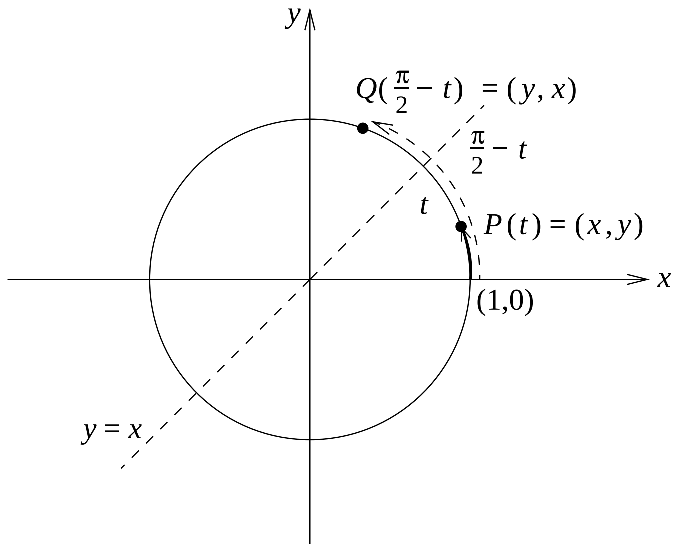


Figure 5.2.5

In Figure 5.2.5 the point $P(t)$ has coordinates (x,y). The arc length from $(1,0)$ to $P(t)$ is t units and the arc length from the point Q to $(0,1)$ is also t units. Thus $Q\left(\frac{\pi}{2}-t\right)$ is the reflection of the point $P(t)=(x,y)$ in the line $y=x$. Thus $Q\left(\frac{\pi}{2}-t\right)=(y,x)$.

From the definition of the trigonometric functions we have

$$\sin\left(\frac{\pi}{2}-t\right)=x \qquad \cos\left(\frac{\pi}{2}-t\right)=y \qquad \tan\left(\frac{\pi}{2}-t\right)=\frac{x}{y}$$

$$\csc\left(\frac{\pi}{2}-t\right)=\frac{1}{x} \qquad \sec\left(\frac{\pi}{2}-t\right)=\frac{1}{y} \qquad \cot\left(\frac{\pi}{2}-t\right)=\frac{y}{x}$$

and hence the following formulas hold.

$$\begin{aligned}
\sin\left(\frac{\pi}{2}-t\right) &= \cos t\\
\cos\left(\frac{\pi}{2}-t\right) &= \sin t\\
\tan\left(\frac{\pi}{2}-t\right) &= \cot t\\
\csc\left(\frac{\pi}{2}-t\right) &= \sec t\\
\sec\left(\frac{\pi}{2}-t\right) &= \csc t\\
\cot\left(\frac{\pi}{2}-t\right) &= \tan t
\end{aligned}$$

Relationship between $f\left(t+\frac{\pi}{2}\right)$ and $g\left(t\right)$

See question 84 of Exercises 5.2 in SRW.

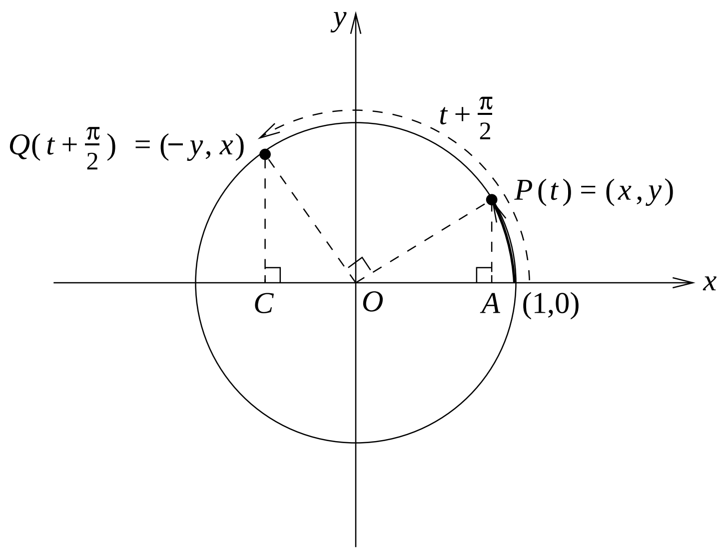


Figure 5.2.6

In Figure 5.2.6 the coordinates of $P\left(t\right)$ are $\left(x,y\right)$. In order to find the coordinates of the point Q determined by $t+\frac{\pi}{2}$ we must show that $\triangle PAO\equiv\triangle OCQ$. We have

$$OQ=OP=1$$

and

$$\angle QCO=\angle OAP=\frac{\pi}{2}.$$

Also

$$\angle QOC+\angle POA+\frac{\pi}{2}=\pi.$$

Hence

$$\begin{aligned}\angle QOC &= \frac{\pi}{2}-\angle POA\\ &= \angle OPA.\end{aligned}$$

Hence

$$\triangle PAO\equiv\triangle OCQ\quad(\text{SAA}).$$

Thus

$$PA = OC$$

and

$$OA = CQ.$$

Thus

$$Q\left(t+\frac{\pi}{2}\right) = (-y, x).$$

From the definition of the trigonometric functions we obtain these formulas.

$$\begin{aligned}
\sin\left(t+\frac{\pi}{2}\right) &= \cos t \\
\cos\left(t+\frac{\pi}{2}\right) &= -\sin t \\
\tan\left(t+\frac{\pi}{2}\right) &= -\cot t \\
\csc\left(t+\frac{\pi}{2}\right) &= \sec t \\
\sec\left(t+\frac{\pi}{2}\right) &= -\csc t \\
\cot\left(t+\frac{\pi}{2}\right) &= -\tan t
\end{aligned}$$

An Easy Way to Remember the Reduction Formulas involving $f(-t)$, $f(\pi - t)$, $f(t+\pi)$ and $f(2\pi - t)$

We have proved all the reduction formulas for values of t such that $0 < t < \frac{\pi}{2}$. Although we have not done so, it is possible to prove that all the reduction formulas given here hold for all $t \in D_f$.

However, if we **assume** $0 < t < \frac{\pi}{2}$ in all the reduction formulas, then

points associated with $\pi - t$ lie in quadrant II
points associated with $\pi + t$ lie in quadrant III
points associated with $2\pi - t$ lie in quadrant IV
points associated with $-t$ lie in quadrant IV

As a way of remembering the formulas we can pretend that we have $0 < t < \frac{\pi}{2}$, and use Figure 5.2.1 (All Students Take Calculus) to find out which expressions are positive and which are negative. Figure 5.2.7 provides a useful "memory hook" for the reduction formulas.

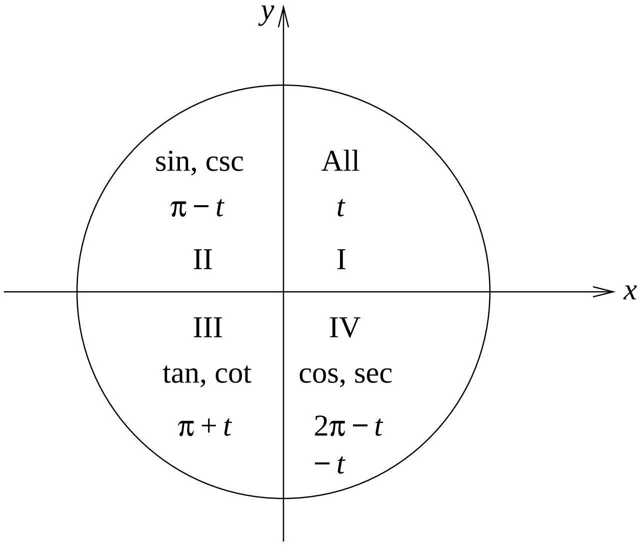


Figure 5.2.7

We interpret Figure 5.2.7 in the following way:
Suppose we want to find $f(2\pi - t)$.
We reason as though $0 < t < \frac{\pi}{2}$.
Then $2\pi - t$ lies in quadrant IV.
Hence $f(2\pi - t) = f(t)$ if f represents cos or sec and $f(2\pi - t) = -f(t)$ if f represents sin, tan, cot or csc.

For example, suppose we want to know the value of $\csc\left(\frac{7\pi}{4}\right)$. We note that $\frac{7\pi}{4}$ lies in quadrant IV since $\frac{3\pi}{2} < \frac{7\pi}{4} < 2\pi$. Now

$$\begin{aligned} 2\pi - t = \frac{7\pi}{4} &\Leftrightarrow t = 2\pi - \frac{7\pi}{4} \\ &\Leftrightarrow t = \frac{\pi}{4}. \end{aligned}$$

Hence

$$\begin{aligned} \csc\left(\frac{7\pi}{4}\right) &= \csc(2\pi - t) \\ &= -\csc t \\ &= -\csc\frac{\pi}{4} \\ &= -\sqrt{2}. \end{aligned}$$

This figure however does not provide us with an easy way of remembering the relationships involving $\frac{\pi}{2} \pm t$.

■ Fundamental Identities

Study the subsection in SRW. We can solve Example 6 in a slightly different way, by using a sketch of the unit circle. See Figure 5.2.8 below.

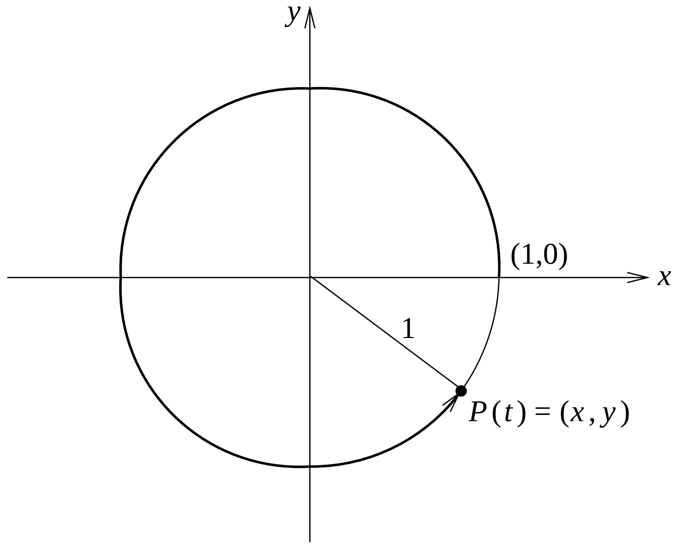


Figure 5.2.8

Since $\cos t = \frac{3}{5}$ we have $x = \frac{3}{5}$.

Since (x,y) lies on the unit circle we have

$$x^2 + y^2 = 1.$$

Thus

$$\left(\frac{3}{5}\right)^2 + y^2 = 1$$

$$\Rightarrow \qquad y^2 = 1 - \frac{9}{25}$$

$$\Rightarrow \qquad y^2 = \frac{16}{25}$$

$$\Rightarrow \qquad y = -\frac{\sqrt{16}}{25} \qquad \text{Since } P(t) \text{ lies in quadrant IV.}$$

$$\Rightarrow \qquad y = -\frac{4}{5}.$$

Thus $\sin t = -\frac{4}{5}$.

We can now use the fundamental identities or the definition of the trigonometric functions to determine the other trigonometric function values of t.

In the next example we use some of the reduction formulas and the fundamental identities to complete a table of different trigonometric values. We do **not** restrict ourselves to t such that $0 < t < \frac{\pi}{2}$.

Example 5.2.2

Complete the following table without using a calculator.

$\tan t$	$\sin\left(\frac{\pi}{2} - t\right)$	$\cos(t - 2\pi)$	$\sin(\pi - t)$	smallest positive value of t
$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$			
1			$-\frac{\sqrt{2}}{2}$	
		$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	

Solution

According to the reduction formulas we have

$$
\begin{aligned}
\sin\left(\frac{\pi}{2} - t\right) &= \cos t && (1)\\
\cos(t - 2\pi) &= \cos(-(2\pi - t))\\
&= \cos(2\pi - t)\\
&= \cos t && (2)
\end{aligned}
$$

and

$$\sin(\pi - t) = \sin t. \qquad (3)$$

According to the fundamental identities we have

$$\tan t = \frac{\sin t}{\cos t}, \qquad (4)$$

hence

$$\sin t = \tan t . \cos t \qquad (5)$$

and

$$\cos t = \frac{\sin t}{\tan t}. \qquad (6)$$

We begin with the first row. We have

$$\tan t = -\frac{\sqrt{3}}{3}$$

and from (1) we have

$$\cos t = \frac{\sqrt{3}}{2}.$$

Thus, from (2), (3) and (5) we have

$$\cos(t - 2\pi) = \frac{\sqrt{3}}{2}$$

and

$$\sin(\pi - t) = \sin t = -\frac{\sqrt{3}}{3} \times \frac{\sqrt{3}}{2} = -\frac{3}{6} = -\frac{1}{2}.$$

We must now find the smallest value of t that satisfies the conditions in row 1. We have that $\tan t < 0$ and $\cos t > 0$. Now tan is negative in quadrants II and IV and cos is positive in quadrants I and IV. Thus the point determined by t lies in quadrant IV. Thus we have

$$t = 2\pi - \bar{t}$$

where $\bar{t}$ is the reference number. Since the reference number lies in quadrant I all the trigonometric functions of $\bar{t}$ are positive. Thus we have

$$\tan \bar{t} = \frac{\sqrt{3}}{3} \quad \text{and} \quad \cos \bar{t} = \frac{\sqrt{3}}{2}.$$

From Table 1 in Section 5.2 of SRW we have

$$\bar{t} = \frac{\pi}{6}.$$

Hence

$$t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

We have now completed the first row.

We now consider the second row. We have $\tan t = 1$ and $\sin(\pi - t) = -\frac{\sqrt{2}}{2}$. Thus from (3) we have

$$\sin t = -\frac{\sqrt{2}}{2}$$

and hence from (6) we obtain

$$\cos t = \frac{-\frac{\sqrt{2}}{2}}{1} = -\frac{\sqrt{2}}{2}.$$

Thus from (1) and (2) we have

$$\sin\left(\frac{\pi}{2}-t\right)=-\frac{\sqrt{2}}{2}$$

and

$$\cos(t-2\pi)=-\frac{\sqrt{2}}{2}.$$

To find the smallest positive value of t we note that $\tan t>0$ and $\cos t<0$. Now $\tan t$ is positive in quadrants I and III and $\cos t$ is negative in quadrants II and III and thus the point determined by t lies in quadrant III. Hence

$$t=\pi+\bar{t}$$

where $\bar{t}$ is the reference number.

Now $\tan\bar{t}=1$ and $\cos\bar{t}=\frac{\sqrt{2}}{2}$ and from Table 1 in Section 5.2 of SRW it follows that

$$\bar{t}=\frac{\pi}{4}.$$

Consequently

$$t=\pi+\frac{\pi}{4}=\frac{5\pi}{4}.$$

In the third row we have $\cos(t-2\pi)=-\frac{1}{2}$ and $\sin(\pi-t)=\frac{\sqrt{3}}{2}$. From (2) and (1) we have

$$\sin\left(\frac{\pi}{2}-t\right)=-\frac{1}{2}$$

and from (3), (2) and (4) we have

$$\tan t=\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}=-\sqrt{3}.$$

Thus it follows that $\tan t<0$ and $\cos t<0$. Now $\tan t$ is negative in quadrants II and IV and $\cos t$ is negative in quadrants II and III and thus the point determined by t is in quadrant II. Hence

$$t=\pi-\bar{t}$$

where $\bar{t}$ is the reference number.

Now $\tan\bar{t}=\sqrt{3}$ and $\cos\bar{t}=\frac{1}{2}$ and thus from Table 1 in Section 5.2 of SRW it follows that $\bar{t}=\frac{\pi}{3}$. Hence

$$t=\pi-\frac{\pi}{3}=\frac{2\pi}{3}.$$

We now complete the table.

$\tan t$	$\sin\left(\frac{\pi}{2}-t\right)$	$\cos(t-2\pi)$	$\sin(\pi-t)$	smallest positive value of t
$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{11\pi}{6}$
1	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$
$-\sqrt{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{2\pi}{3}$

Additional Exercises 5.2.1

1. Complete the following table without using a calculator.

$\sec t$	$\sin t$	$\cos(t-2\pi)$	$\sin(\pi+t)$	smallest positive value of t
		-1		
	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$		
$\frac{2\sqrt{3}}{3}$	$-\frac{1}{2}$			

5.3 TRIGONOMETRIC GRAPHS

To study SRW, Section 5.3, (excluding the subsection "Using Graphing Devices to Graph Trigonometric Functions") and additional notes below.

Outcomes When you have studied this section you should be able to

- draw the graphs of the sine and cosine functions
- apply the principles of transformation of functions (dealt with in Chapter 2) to the graph of the sine or cosine function to draw the graph of $g(x) = a\sin k(x-b)+c$ or $g(x) = a\cos k(x-b)+c$ where a, b, c, k are real constants and $k > 0$
- determine the equation $y = g(x) = a\sin k(x-b)+c$ or $y = g(x) = a\cos k(x-b)+c$ of some given sine or cosine curve.

Exercises Exercises 5.3: questions 1–39 (odd numbers), 41–48
Additional Exercises 5.3.1: questions 1, 2

■ Graphs of the Sine and Cosine Functions

Study the subsection in SRW. Note that the graphs of the sine and cosine functions are similar and that the graph of $y = \sin t$ can be obtained by shifting the graph of $y = \cos t$, $\frac{\pi}{2}$ units to the right. Can you explain why this is so? Consider the reduction formulas and apply one of the transformations dealt with in Section 2.4. We leave this as an exercise for you to do. See Additional Exercises 5.3.1.

Also take note of the symmetry properties of the two graphs.

■ Graphs of Transformations of Sine and Cosine

Make sure that you are able to apply the principles of transformation of functions which are discussed in Section 2.4 before you study this subsection in SRW. In SRW examples of graphs of the following functions have been sketched. Here f denotes either the sine or cosine function.

$y = f(x) + c$	Example 1(a)
$y = af(x)$	Example 1(b), Figure 5 and Example 2
$y = f(kx)$	Figure 8
$y = af(kx)$	Figure 9 and Example 3
$y = f(x-b)$	Figure 12
$y = af[k(x-b)]$	Examples 4 and 5

First study this subsection in SRW and then work through the following example in which we sketch the graph of a function of the form $g(x) = af[k(x-b)] + c$.

Example 5.3.1

Sketch the graph of the function

$$y = g(x) = 3\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 1$$

over the interval $\left[-\frac{\pi}{2}, 3\frac{1}{2}\pi\right]$.

Solution

We first rewrite $g(x) = 3\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 1$ in the form $g(x) = a\cos k(x-b) + c$.

We have

$$\begin{aligned} g(x) = 3\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right) - 1 &= 3\cos\frac{1}{2}\left(x + \frac{\pi}{2}\right) - 1 \\ &= 3\cos\frac{1}{2}\left[x - \left(-\frac{\pi}{2}\right)\right] - 1. \end{aligned}$$

Thus we have

amplitude $= \lvert a\rvert = 3$	
period $= \dfrac{2\pi}{k} = \dfrac{2\pi}{\frac{1}{2}} = 4\pi$	
phase shift $= b = -\dfrac{\pi}{2}$	Shift $\frac{\pi}{2}$ units to the left.
vertical shift $= c = -1$	Shift 1 unit down.

The graph is sketched in Figure 5.3.1.

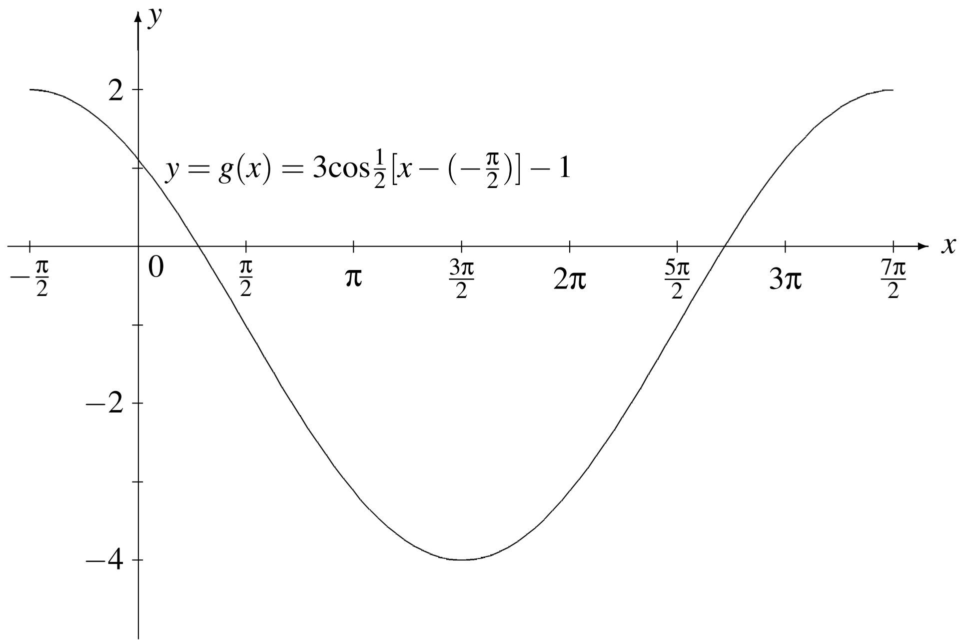


Figure 5.3.1

At this stage we have a strategy for determining the graph of a trigonometric function g of the form $y = g(x) = af[k(x-b)] + c$, where a, b, c, k are real constants and $k > 0$, and f is either the sine or cosine function. It should now be fairly straightforward to apply this strategy in reverse and determine the equation of the function from its graph.

Do you think that the equation that we obtain will always be unique? Think back to the reduction formulas. Notice, for example, that

$$\sin(\pi - x) = \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

and hence $y = \sin(\pi - x)$, $y = \sin x$ and $y = \cos\left(\frac{\pi}{2} - x\right)$ all have the same graph. Thus the equation will not necessarily be unique. Consider the following example.

Example 5.3.2

Determine an equation that defines the function whose graph is sketched in Figure 5.3.2.

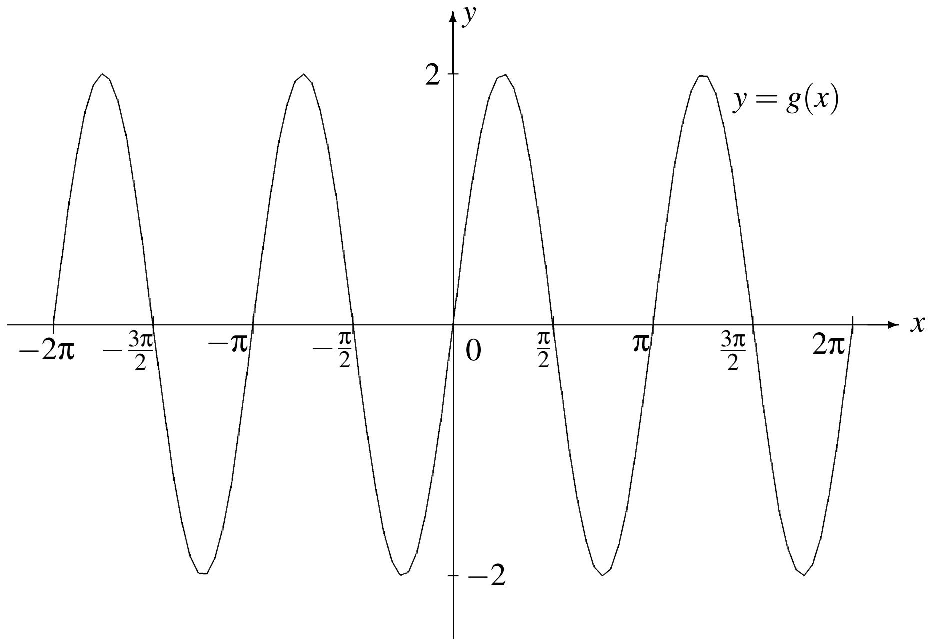


Figure 5.3.2

Solution

We assume that the function g has the equation

$$y = g(x) = af[k(x-b)] + c,$$

where a, b, c and k represent real constants with $k > 0$, and f is either the sine or cosine function. First decide what function this graph most closely resembles. You should have no difficulty in identifying this as a sine curve, derived from the function $y = f(x) = \sin x$.

Since the curve represents the positive sine curve we have $a > 0$. The amplitude is 2, i.e. $|a| = 2$, and hence $a = 2$.

Thus the equation has the form

$$g(x) = 2\sin[k(x-b)] + c.$$

The period of g is π and hence $\frac{2\pi}{k} = \pi$, i.e. $k = 2$. No horizontal or vertical shifting has taken place, and thus $b = 0$ and $c = 0$. The equation of g is thus

$$y = g(x) = 2\sin 2x. \qquad (1)$$

If we want to consider the graph in Figure 5.3.2 as a cosine curve then we will have $a = 2$ and $k = 2$, and thus we have $y = g(x) = 2\cos[2(x-b)] + c$. In this case we see that the graph of $y = 2\cos 2x$ has been shifted $\frac{\pi}{4}$ units to the right to obtain the curve in Figure 5.3.2. Thus $b = \frac{\pi}{4}$. No vertical shifting has taken place and hence $c = 0$. Thus

$$y = g(x) = 2\cos 2\left(x - \frac{\pi}{4}\right) \tag{2}$$

We have thus described the equation that defines the graph in Figure 5.3.2 in two different ways. By using the reduction formulas we can show that (1) and (2) are the same. We have

$$\begin{aligned} 2\cos 2\left(x - \frac{\pi}{4}\right) &= 2\cos\left(2x - \frac{\pi}{2}\right) \\ &= 2\cos\left(-\left(\frac{\pi}{2} - 2x\right)\right) \\ &= 2\cos\left(\frac{\pi}{2} - 2x\right) && \cos(-x) = \cos x. \\ &= 2\sin 2x. && \cos\left(\frac{\pi}{2} - x\right) = \sin x. \end{aligned}$$

We now consider a more complex example.

Example 5.3.3

The graph of

$$y = g(x) = a\cos k(x - b) + c$$

is sketched in Figure 5.3.3.

Find the values of a, b, c and k and write down the equation for g.

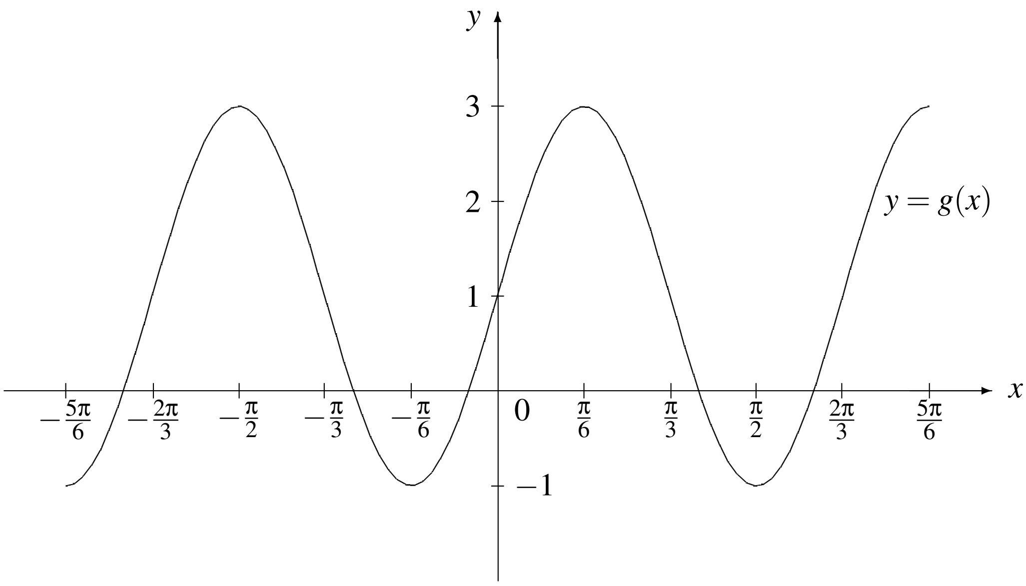


Figure 5.3.3

Solution

We consider the graph in Figure 5.3.3 as a positive cosine curve. Thus $a > 0$. Also $|a| = 2$ and hence $a = 2$. The period of g is $\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right)$, i.e. $\frac{2}{3}\pi$. Hence

$$\frac{2\pi}{k} = \frac{2\pi}{3}, \text{ i.e. } k = 3.$$

The equation for g is

$$y = g(x) = 2\cos 3(x - b) + c.$$

The graph of $y = 2\cos 3x$ is shifted $\frac{\pi}{6}$ units to the right to obtain the graph of $y = 2\cos 3(x - b)$. Hence $b = \frac{\pi}{6}$. Also the graph of $y = 2\cos 3\left(x - \frac{\pi}{6}\right)$ has been shifted 1 unit upwards to obtain the graph of $y = g(x)$ and hence $c = 1$. Thus

$$y = g(x) = 2\cos 3\left(x - \frac{\pi}{6}\right) + 1.$$

Additional Exercises 5.3.1

1. Explain why the graph of $y = \sin x$ is the same as the graph of $y = \cos x$ shifted to the right by $\frac{\pi}{2}$ units.
2. Sketch the graph of $y = 2\sin\left(x - \frac{\pi}{3}\right) + 1$ on the interval $\left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$.

5.4 MORE TRIGONOMETRIC GRAPHS

To study SRW, Section 5.4, and additional notes.

Outcomes When you have studied this section you should be able to

- ► sketch the graphs of the tangent, cotangent, secant and cosecant functions
- ► apply the principles of transformation of functions dealt with in Chapter 2 to draw the graphs of $g(x) = af[k(x-b)] + c$, where $a,\ b$, c and k are real constants with $k > 0$ and f is either the tangent, cotangent, secant or cosecant function
- ► determine the equation $y = g(x) = af[k(x-b)] + c$ of some given tangent, cotangent, secant or cosecant curve.

Exercises Exercises 5.4: questions 1–51 (odd numbers)
Additional Exercises 5.4.1: questions 1, 2

■ Graphs of the Tangent, Cotangent, Secant and Cosecant Function

Please read the first two pages of Section 3.6 which deals with arrow notation and asymptotes before you study this subsection. Note that the period of the secant and cosecant functions is the same as the period of the sine and cosine functions, namely 2π, whereas the period of the tangent and cotangent functions is π. Also pay attention to the type of symmetry that the graphs of the tangent, cotangent, cosecant and secant functions display.

■ Graphs involving Tangent and Cotangent Functions

Study the relevant pages. In this subsection examples of graphs of the following functions are sketched. Here f denotes either the tangent or cotangent function.

$y = af(x)$	Example 1
$y = af(kx)$	Example 2(a)
$y = af[k(x-b)]$	Examples 2(b) and 3

In Example 5.4.1 we sketch the graphs of functions of the form $y = f(x) + c$ and $y = af[k(x-b)] + c$.

Example 5.4.1

Sketch the graphs of the following functions.

(a) $y = \tan x + 2$ (b) $y = -\cot 2\left(x - \frac{\pi}{4}\right) + 1$

Solution

(a) The graph of $y = \tan x + 2$ is obtained by shifting the graph of $y = \tan x$ two units upwards. The graph is sketched in Figure 5.4.1.

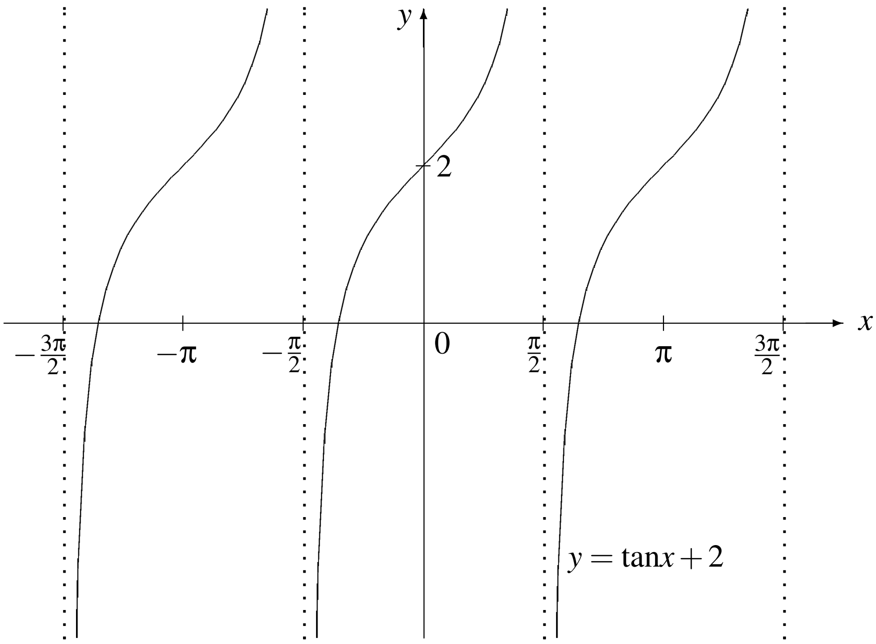


Figure 5.4.1

(b) From the equation $y = -\cot 2\left(x - \frac{\pi}{4}\right) + 1$ we see that the period of the function is $\frac{\pi}{2}$. We obtain the graph of $y = -\cot 2\left(x - \frac{\pi}{4}\right) + 1$ by

- compressing the graph of $y = \cot x$ horizontally by a factor of two to obtain the graph of $y = \cot 2x$
- reflecting the graph of $y = \cot 2x$ in the x–axis to obtain the graph of $y = -\cot 2x$
- shifting the graph of $y = -\cot 2x$, $\frac{\pi}{4}$ units to the right to obtain the graph of $y = -\cot 2\left(x - \frac{\pi}{4}\right)$ and
- shifting the graph of $y = -\cot 2\left(x - \frac{\pi}{4}\right)$ one unit upwards to obtain the graph of $y = -\cot 2\left(x - \frac{\pi}{4}\right) + 1$.

The graph is sketched in Figure 5.4.2.

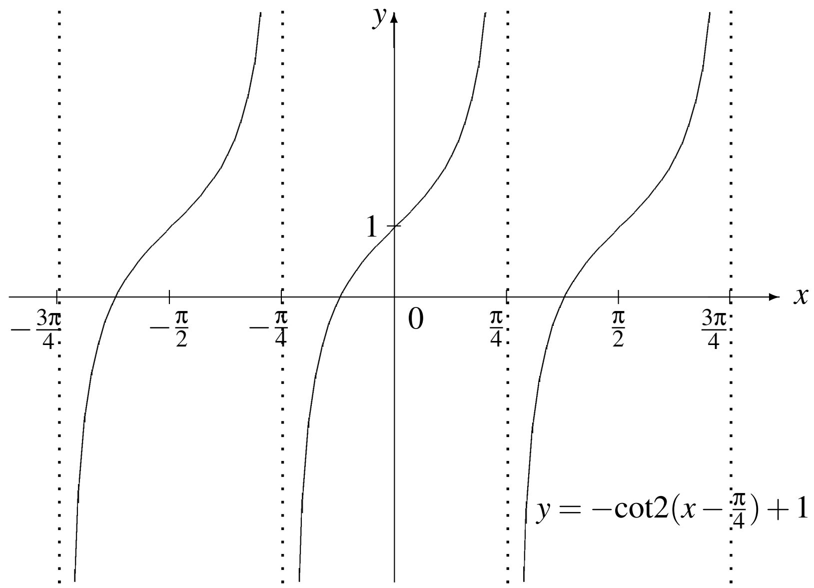


Figure 5.4.2

■ Graphs involving the Cosecant and Secant Functions

Study the relevant pages. In this subsection examples of the graphs of the following functions are sketched. Here f denotes the secant or cosecant function.

$y = af(kx)$	Examples 4(a) and 5
$y = af[k(x-b)]$	Example 4(b)

In Example 5.4.2 we sketch the graphs of functions of the form $y = f(x) + c$ and $y = af[k(x-b)] + c$.

Example 5.4.2

Sketch the graphs of the following functions.

(a) $y = \csc x - 1$ (b) $y = 2\sec\frac{1}{2}\left(x + \frac{\pi}{4}\right) - 1$

Solution

(a) The graph of $y = \csc x - 1$ is obtained by shifting the graph of $y = \csc x$ one unit downwards. The graph is sketched in Figure 5.4.3.

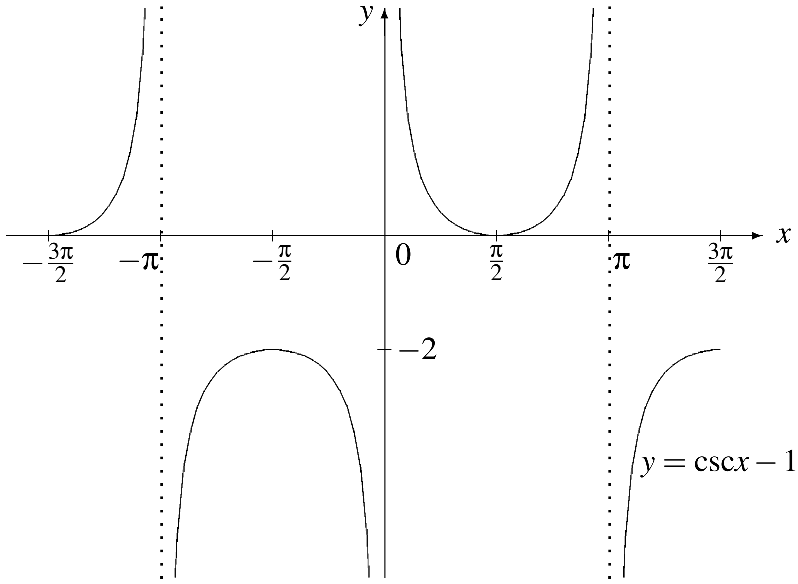


Figure 5.4.3

(b) The period of the function is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. We can obtain the graph of $y = 2\sec\frac{1}{2}\left(x + \frac{\pi}{4}\right) - 1$ by

- stretching the graph of $y = \sec x$ horizontally by a factor of 2 to obtain the graph of $y = \sec\frac{1}{2}x$
- stretching the graph of $y = \sec\frac{1}{2}x$ vertically by a factor of 2 to obtain the graph of $y = 2\sec\frac{1}{2}x$
- shifting the graph of $y = 2\sec\frac{1}{2}x$, $\frac{\pi}{4}$ units to the left to obtain the graph of $y = 2\sec\frac{1}{2}\left(x + \frac{\pi}{4}\right)$ and
- shifting the graph of $y = 2\sec\frac{1}{2}\left(x + \frac{\pi}{4}\right)$ one unit downwards to obtain the graph of $y = 2\sec\frac{1}{2}\left(x + \frac{\pi}{4}\right) - 1$.

The graph is sketched in Figure 5.4.4.

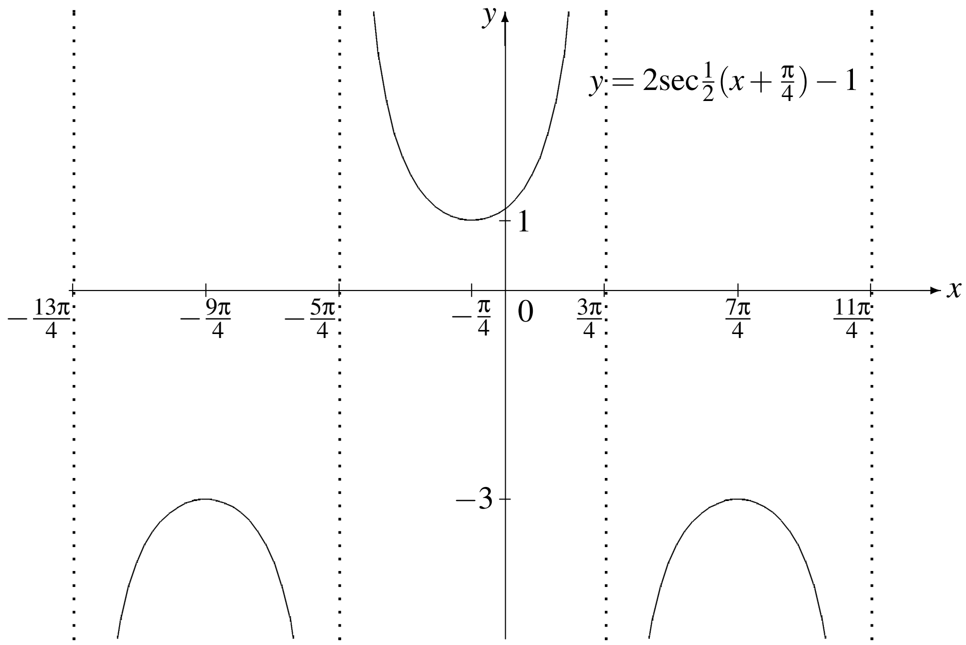


Figure 5.4.4

■ Finding Equations of given Tangent, Cotangent, Secant or Cosecant Curves

Now that we have sketched graphs of the form $y = af\,[k\,(x-b)] + c$, where f is either the tangent, cotangent, secant or cosecant function we consider the reverse procedure. Given a tangent, cotangent, secant or cosecant curve can we find the equation that describes it? In general, it is more difficult to find equations of tangent, cotangent, secant and cosecant curves than it is to find equations of sine and cosine curves, and we often need additional information in order to find the equation.

Example 5.4.3

Figure 5.4.5 is the graph of the function f defined by $y = a\tan kx$ or $y = a\cot kx$, where $k > 0$ and $\left(\frac{\pi}{8}, -1\right)$ is a point on the graph of f. Find a and k and write down the equation for f.

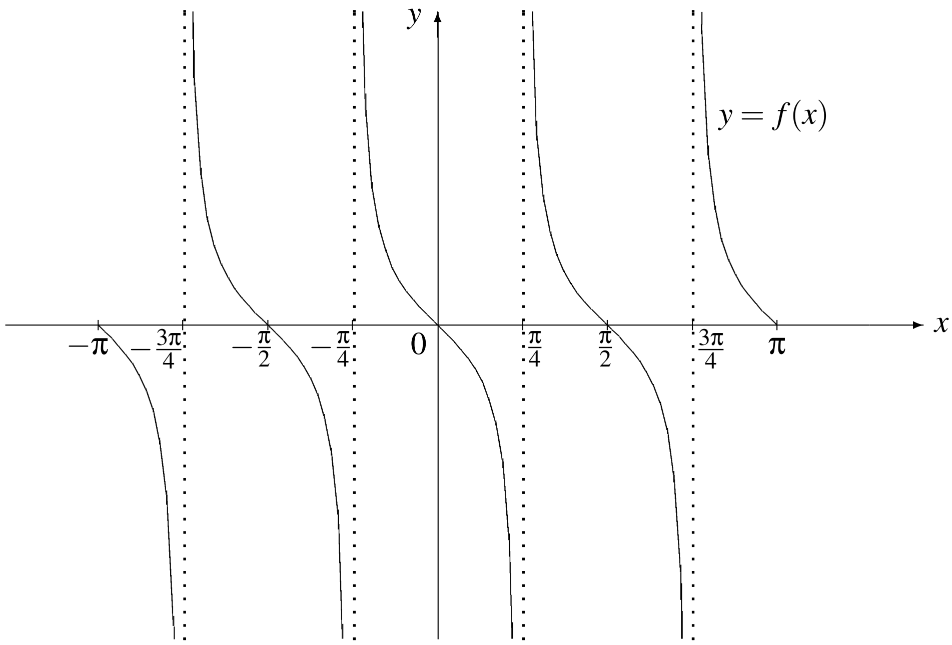


Figure 5.4.5

Solution

Since the graph passes through the origin we have a tangent curve. Thus

$$y = a \tan kx.$$

From the graph we see that the period of the function f is $\frac{\pi}{2}$. Therefore

$$\frac{\pi}{k} = \frac{\pi}{2}$$

i.e. $$k = 2.$$

Thus $f(x) = a \tan 2x$. We are also given that $f\left(\frac{\pi}{8}\right) = -1$. We substitute into $f(x) = a \tan 2x$ and obtain

$$-1 = a \tan 2\left(\frac{\pi}{8}\right)$$

i.e. $$-1 = a \tan \frac{\pi}{4}$$

i.e. $$-1 = a \times 1$$

i.e. $$a = -1.$$

Hence the equation of the function f is

$$y = -\tan 2x.$$

Example 5.4.4

Figure 5.4.6 is the graph of the function h defined by $y = a\sec kx$ or $y = a\csc kx$, where $k > 0$. Find a and k and write down the equation for h.

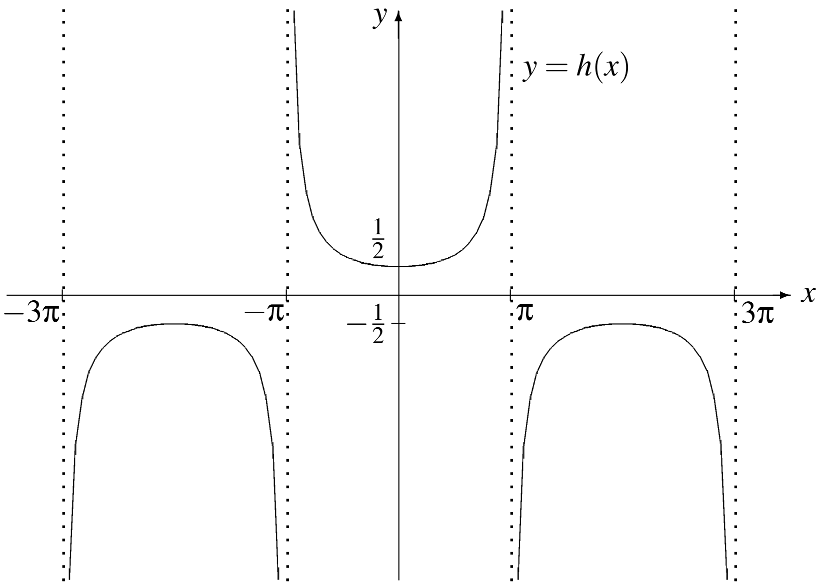


Figure 5.4.6

Solution

Since the graph is symmetric about the y–axis we have a secant curve. Since the graph cuts the y–axis at $\frac{1}{2}$ and not at 1 we have $a = \frac{1}{2}$. The period of the function k is 4π. Hence

$$\frac{2\pi}{k} = 4\pi$$

i.e. $$k = \frac{2\pi}{4\pi}$$

i.e. $$k = \frac{1}{2}.$$

Thus the equation for h is

$$y = \frac{1}{2}\sec\frac{x}{2}.$$

Additional Exercises 5.4.1

1. The graph of $y = h(x)$ is sketched in Figure 5.4.7. Find the equation that defines the function h.

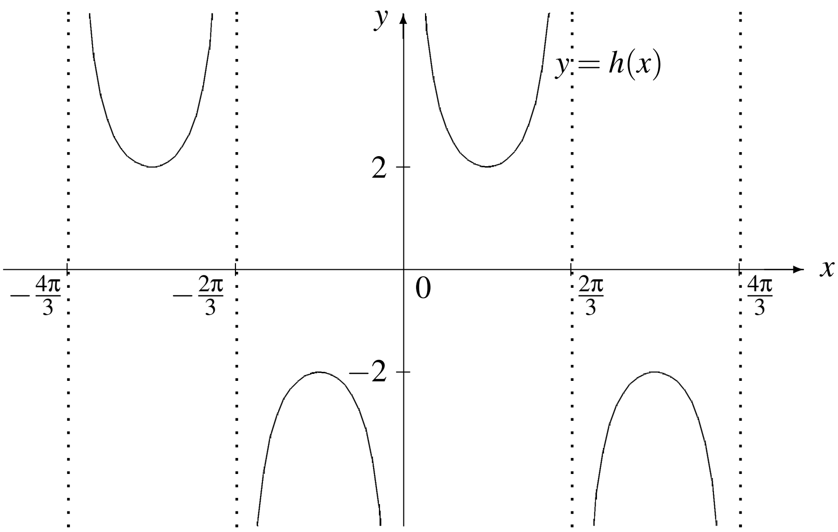


Figure 5.4.7

2. Figure 5.4.8 is a sketch of the graphs of the functions f and g on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The function f is defined by either $y = a\sin kx$ or $y = a\cos kx$, and the function g is defined by either $y = m\tan px$ or $y = m\cot px$, where a, k, m and p are constants and $k > 0$ and $p > 0$. The point $P\left(\frac{\pi}{4}, -2\right)$ lies on the graph of g.

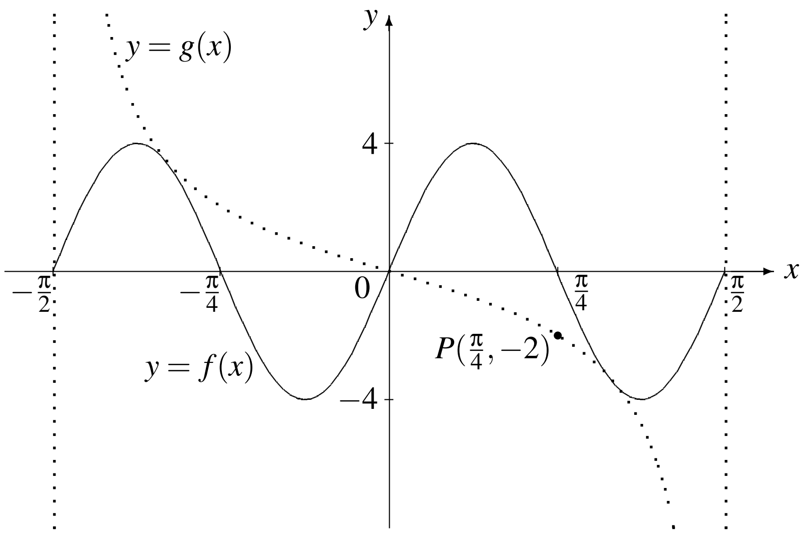


Figure 5.4.8

Find the values of a, k, m and p and write down the equations that define f and g.

5.5 MODELING HARMONIC MOTION

You will deal with these concepts in later mathematics, and you do not need to study this section now.

5 REVIEW AND TEST

Now assess yourself by attempting the questions in the Concept Check, trying a selection of questions from the Exercises and doing the Chapter 5 Test.

6 TRIGONOMETRIC FUNCTIONS OF ANGLES

We have previously noted that you could study Chapter 6 before Chapter 5, or vice versa. The introduction in the Chapter Overview of Chapter 6 explains why.

6.1 ANGLE MEASURE

To study SRW, Section 6.1.

Outcomes After studying this section you should be able to

- define angle measure in terms of degrees and radians
- use degree or radian measure when working with angles; convert from one measure to the other
- find an angle in standard position and angles coterminal with it
- calculate the length of a circular arc
- calculate the area of a sector of a circle.

Exercises Exercises 6.1: questions 1 – 75 (odd numbers)

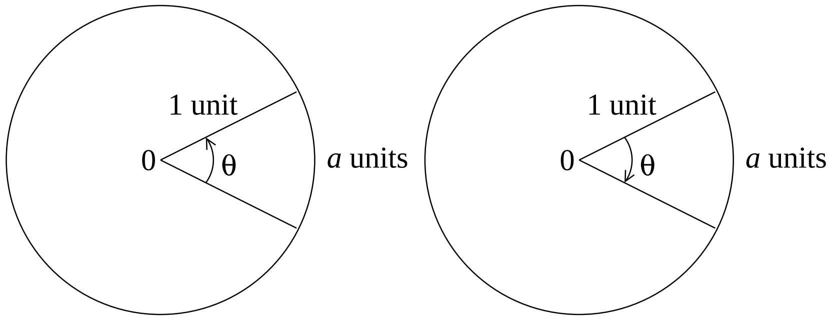
■ Angle Measure

Notation

SRW begins by discussing an angle AOB. This angle can be denoted in several ways, e.g. $A\hat{O}B$, $\angle AOB$, $\angle O$, or $\hat{O}$. When no ambiguity is possible we often use just the one letter; when two angles share the same vertex we need more than just the letter assigned to the vertex to describe the angle. The angle can also be denoted by a letter such as θ, or x. These letters then denote both the **angle** and the **measure of the angle**.

The measure of an angle is discussed on the first two pages of the section, to the end of Example 1. Take note of the convention that **positive angles** are obtained when a ray is rotated in a **counterclockwise** (sometimes referred to as anticlockwise) direction; **negative angles** are obtained when the rotation is **clockwise**.

This convention applies to degree and to radian measure. The definition of radian measure assumes that the angle is measured in a counterclockwise direction. If the angle is measured in a clockwise direction the radian measure is negative. See Figure 6.1.1 below.



Measure of $\theta = a$ radians $(a > 0)$ Measure of $\theta = -a$ radians $(a > 0)$

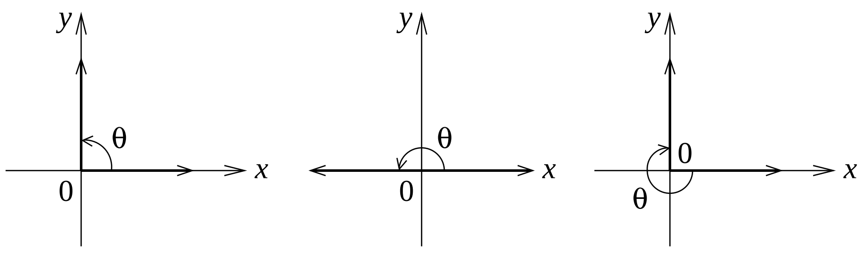
Figure 6.1.1

Make sure you understand the meaning of rotation and revolution.

Note: One complete revolution is 360^o or 2π radians. We use this to find the relationship between degrees and radians, given in the block above Example 1.

■ Angles in Standard Position

Figure 5 shows four different angles, all in **standard position**. The convention of standard position ensures that when we consider an angle we all agree that the angle's initial side is in the same position. Later (in Section 6.3) we shall see that it is also important to know in which quadrant the terminal side of an angle lies. We say that an angle in standard position lies in a given quadrant if its terminal side lies in that quadrant. If the terminal side coincides with either the x– or the y–axis then the angle is called a **quadrantal angle**. See Figure 6.1.2.



$\theta = 90^o$ $\theta = 180^o$ $\theta = -270^o$

Examples of quadrantal angles
Figure 6.1.2

Study Examples 2 and 3, and note that there are positive and negative angles that are coterminal with any given angle. Figure 6 shows one positive and one negative angle coterminal with 30^o.

Example 6.1.1

(a) Find the smallest positive angle that is coterminal with $\theta = 270^o$.

(b) Find the biggest negative angle that is coterminal with $\theta = \pi$ radians.

Solution

(a) To find a positive angle that is coterminal with 270^o we must **add** any multiple of 360^o. The smallest multiple of 360^o will thus give the smallest positive angle coterminal with 270^o, i.e.

$$270^o + 360^o = 630^o$$

hence 630^o is the smallest positive angle coterminal with 270^o.

Note: We need not write the unit radians each time, since we follow the convention that when no unit is given, the angle is assumed to be measured in radians.

(b) To find a negative angle that is coterminal with $\theta = \pi$, we **subtract** any multiple of 2π. The biggest negative angle coterminal with θ will thus be found by subtracting the smallest multiple of 2π. If we subtract bigger multiples of 2π the angle becomes smaller, since

$$\ldots < \pi - 6\pi < \pi - 4\pi < \pi - 2\pi.$$

Now

$$\pi - 2\pi = -\pi$$

and hence $-\pi$ is the biggest negative angle that is coterminal with π.

Example 6.1.2

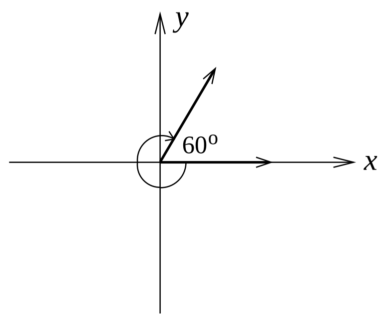
In each of the following, draw the angle in standard position that corresponds to the given rotation, and give the measure of the angle in (i) degrees and (ii) radians.

(a) $\frac{5}{6}$ of a revolution in a clockwise direction

(b) $1\frac{1}{3}$ revolutions in a counterclockwise direction

Solution

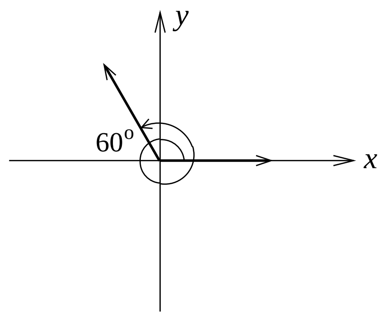
(a)



(i) $\frac{5}{6} \times 360^o = 300^o$
The direction is clockwise, hence the angle is negative; i.e. -300^o.

(ii) $\frac{5}{6} \times 2\pi = \frac{5\pi}{3}$
The direction is clockwise, hence the angle is negative; i.e. $-\frac{5\pi}{3}$.

(b)



(i) $1\frac{1}{3} \times 360^o = 480^o$
The direction is counterclockwise, hence the angle is positive.

(ii) $1\frac{1}{3} \times 2\pi = \frac{8\pi}{3}$
The direction is counterclockwise, hence the angle is positive.

■ Length of a Circular Arc

Study this subsection, from below Figure 8. In Figure 9 we have

- ► a circle with radius r units
- ► a central angle whose radian measure is θ
- ► an arc whose length is s units.

We say that the arc **subtends** the central angle, and we note that the length of the arc is some fraction of the circle's circumference. Clearly, the bigger the angle, the longer the arc length, i.e. the arc length s is dependent on the size of θ, and is the fraction $\frac{\theta}{2\pi}$ of the circle's circumference. This leads to the formula for arc length, namely

$$s = r\theta.$$

Note that this formula relates arc length to the **radian** measure of the central angle, and we cannot use the degree measure of θ in the formula.

The definition of radian measure is based on the **unit circle** (i.e. a circle with radius 1 unit). From the formula

$$s = r\theta$$

we now deduce that

$$\theta = \frac{s}{r}$$

and this enables us to define radian measure in terms of a circle with **any** radius r units. Figure 10 shows how this leads to the term "radian". Study Example 4.

■ Area of a Circular Sector

Study this short subsection, to the end of Example 5.

■ Circular Motion

You need not study this subsection, but you may find it very interesting just to read through.

To check whether you can satisfy the outcomes for this section, try the questions suggested in Exercises 6.1. Question 69 refers to cities that lie on the same **meridian**. This is a word used to describe a **line of longitude**. You may remember that the Greenwich meridian is the zero line of longitude, which divides the earth into a Western and an Eastern hemisphere in the same way that the Equator (the zero line of the latitude) divides the earth into a Northern and Southern hemisphere. Lines of longitude and latitude are imaginary lines on the earth's circumference, which enable us to describe the locations of places. Thus the distance from Pittsburgh to Miami is the length of a circular arc.

In Question 71 we are again dealing with the length of a circular arc, as the sketch shows, even though the answer will be inaccurate since the path of the earth around the sun is not a circle, but an ellipse. The question notes that the ellipse is very nearly circular because it has small **eccentricity**: this means that the earth's "circular" path is only pulled outwards to a small extent.

Question 73 refers to a central angle whose measure is 1 **minute**. The Babylonians introduced degrees as a measure of angles in about 100 BC. As we know, one revolution is defined to be 360°, and an angle whose measure is 1° is obtained by a rotation of $\frac{1}{360}$ of a complete revolution. For occasions when even smaller measurements were required the Babylonians also subdivided degrees into **minutes** and **seconds**, where

- ► 1 minute (denoted by $1'$) $= \dfrac{1}{60}$ of a degree i.e. $60' = 1^{\circ}$
- ► 1 second (denoted by $1''$) $= \dfrac{1}{60}$ of a minute i.e. $60'' = 1'$.

For example, we may (with very accurate measuring instruments) find an angle of 20 degrees 10 minutes and 43 seconds, which we would then write as $20^{\circ}\ 10'\ 43''$. As is the case with time (also a system where 60 is the base unit) we need to be careful that we convert correctly from the decimal expression of a measurement. For example, if we use a calculator and find that an angle measures 30.45° then the angle does not measure 30° and $45'$. We have

$$0.45^{\circ} = (0.45 \times 60)' = 27'$$

i.e.
$$30.45^{\circ} = 30^{\circ}\ 27'.$$

6.2 TRIGONOMETRY OF RIGHT TRIANGLES

To study	SRW, Section 6.2.
Outcomes	After studying this section you should be able to ► define the trigonometric ratios in terms of right triangles ► relate these trigonometric ratios to other trigonometric ratios in terms of reciprocal relations ► calculate trigonometric ratios of given angles, using a calculator if necessary ► use the Pythagorean theorem to determine trigonometric ratios of special angles (called special in the sense that we can easily construct right triangles which contain these angles, and hence find their trigonometric ratios without the use of a calculator) ► solve right triangles (i.e. determine the lengths of all sides and the measures of all angles, from given information) ► apply the techniques of solving right triangles to problems where angles of elevation and angles of depression are involved.
Exercises	Exercises 6.2: questions 1–35 (odd numbers), 39–55 (odd numbers), 59, 61

■ Trigonometric Ratios

The full name of the ratio found by calculating $\frac{\text{hypotenuse}}{\text{opposite}}$ is cosecant. In SRW and hence in this guide we use the abbreviation csc; you may find other books in which the abbreviation cosec is used.

Figure 1 explains the terminology used in the definition in the block next to it. Note that in a right triangle we have one right angle and two other angles, which must both be acute (remember that an acute angle measures less than 90^o). Note also that the ratios depend only on the measure of θ and not on the size of the triangles.

Suppose we call the other acute angle in the right triangle α. Then we see that

- $\theta + \alpha = 90^o$ (θ and α are thus **complementary** angles.)
- the adjacent side for θ is the opposite side for α
- the opposite side for θ is the adjacent side for α
- the hypotenuse for both angles is the same.

From this it follows that

Complementary angles and co–ratios

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \cos\alpha$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \sin\alpha$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \cot\alpha$
- $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \tan\alpha$
- $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \csc\alpha$
- $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \sec\alpha.$

We use the terms opposite and adjacent as they relate to the angle θ.

From the definition of the trigonometric ratios given in the block next to Figure 1, we also have three **reciprocal relationships**, namely

- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$

which can as easily be expressed as

- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}.$

From these reciprocal relationships it is clear that

- $\sin\theta\csc\theta = 1$
- $\cos\theta\sec\theta = 1$
- $\tan\theta\cot\theta = 1.$

Now read carefully the rest of this subsection. In Example 1 all the information required is supplied in Figure 3; in Example 2, the information that $\cos\alpha = \frac{3}{4}$ provides some of the required information; the third side of the triangle must first be calculated using the theorem of Pythagoras, and then it is possible to find all the trigonometric ratios. See Figure 4.

■ Special Triangles

Study the subsection to just after Table 1. Special angles are convenient in that we do not need to use a calculator to find the values of the trigonometric ratios, as we see in Table 1 (although we will need to use a calculator if we need to find an approximate answer for a value such as $\frac{2\sqrt{3}}{3}$).

There are obviously many more angles than the special angles, and for these we need to use a calculator. Since a calculator only has keys for the first three trigonometric ratios (i.e. for sin, cos and tan) we need to convert the last three trigonometric ratios (i.e. cot, sec, csc) to their respective reciprocal ratios.

In SRW, note the distinction between

$\sin 1$ (implying an angle of 1 **radian**)

and

$\sin 1^\circ$ (where an angle of 1 **degree** is involved).

Caution: Remember to check that your calculator is set in the correct mode.

Below Table 1 the use of a calculator to find the value of a trigonometric ratio for any angle is discussed, and this is illustrated in Example 3.

■ Applications of Trigonometry of Right Triangles

Study Example 4. "Solving" this triangle simply means finding the values of a, b and the angle $\angle ABC$. There are times when we may know the lengths of the sides of a triangle. In that case "solving the triangle" implies calculating the measures of the two acute angles. We will consider this later (see Example 6.3.1). Take note of Figure 8. It is useful to remember that, for any angle θ such as the angle shown in the diagram,

$$\begin{aligned}\text{adjacent side} &= r\cos\theta\\ \text{opposite side} &= r\sin\theta.\end{aligned}$$

$\frac{\text{adjacent}}{\text{hypotenuse}} = \cos\theta$
$\Rightarrow$ adjacent
$= (\text{hypotenuse})(\cos\theta)$
$= r\cos\theta$

$\frac{\text{opposite}}{\text{hypotenuse}} = \sin\theta$
$\Rightarrow$ opposite
$= (\text{hypotenuse})(\sin\theta)$
$= r\sin\theta$

In order to consider certain applications of the trigonometry of right triangles we first need to understand the concepts of

- line of sight
- angle of elevation
- angle of depression

which are all explained after Example 4.

Note that an angle of elevation (or depression) yields additional angles when we apply the fact that if parallel lines are cut by a tranversal line, the alternate angles are equal.

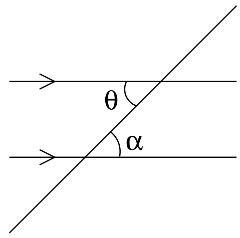


Figure 6.2.1

θ and α are alternate angles. Since the lines are parallel, $\theta = \alpha$.

The importance of these concepts is illustrated in Examples 5, 6 and 7.

Caution: Angles of elevation and depression must always be measured **from the horizontal**. Consider the figure below.

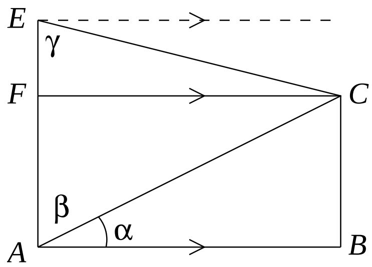


Figure 6.2.2

The angle of elevation of C from A is α, and not β. The angle of depression of A from C is $\angle FCA$. Since $\angle FCA$ and $\angle CAB$ are alternate angles we have $\angle FCA = \alpha$. We are not given the angle of depression of C from E, but we can calculate it by constructing a line through E, parallel to FC. The angle of depression of C from E is then $90^o - \gamma$.

6.3 TRIGONOMETRIC FUNCTIONS OF ANGLES

To study	SRW, Section 6.3.
Outcomes	After studying this section you should be able to ▶ define the six trigonometric functions of any angle ▶ find the reference angle associated with any angle θ ▶ use the reduction formulas, and trigonometric identi– ties such as the reciprocal identities and the Pythagorean identities, to express one trigonometric function in terms of another ▶ calculate the values of the trigonometric functions for any angle (measured in degrees or radians) ▶ find the area of any triangle.
Exercises	Exercises 6.3: questions 1–55, odd numbers only Additional Exercises 6.3.1

■ Trigonometric Functions of Angles

If you have already studied Chapter 5, read the notes in the page size block on the second page of the subsection to see how the two perspectives (trigonometric functions of real numbers and trigonometric functions of angles) are related. Study the subsection to the end of Figure 3.

In the definition of trigonometric **ratios** (in Section 6.2) we used

- ▶ a right triangle
- ▶ the relative position (i.e. adjacent or opposite) of two of the sides of the triangle to the angle, and the lengths of these sides
- ▶ the length of the hypotenuse.

In this subsection we define trigonometric **functions** and we use

- ▶ an angle in standard position, with a point P on the terminal side of the angle
- ▶ the x– and y–coordinates of P
- ▶ the distance r of the point P to the origin.

Why do we call them functions? Recall the definition of a function given in Chapter 2, Section 2.1. In this case we have

- ► a rule that assigns to each element θ in a set A exactly one element $f(\theta)$ in a set B.

The set A consists of **angles** and the set B consists of the sin, cos, tan, etc. of these angles, i.e. B is a set of real numbers. For example,

$$f(\theta) = \sin\theta = \frac{y}{r}; \quad g(\theta) = \cos\theta = \frac{x}{r}; \text{ etc.}$$

From this it follows that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Compare this with the comment in the second paragraph of the subsection "Applications of Trigonometry of Right Triangles" in Section 6.2 of this guide.

In general, in this guide we use θ to represent an angle, and do not imply that the angle is necessarily measured in degrees.

If f represents one of the six trigonometric functions then

$$f(\theta) = f(t)$$

where θ is the degree measure of the angle and t is the radian measure of the angle (although θ could just as easily represent radians, and t, degrees).

Using the definition of trigonometric functions we can now extend Table 1, given in Section 6.2, to include the "special" angles 0^o, 90^o, 180^o, 270^o and 360^o (i.e. the quadrantal angles).

The same reasoning applies if we use a circle with radius $r \neq 1$.

Consider a unit circle , i.e. a circle with radius one unit. For angles of 0^o, 180^o and 360^o (or their radian equivalent), any point on the terminal side of the angle has coordinates $(x,0)$, and the distance r from the origin to the point is $|x|$. Thus, for example,

$$\cos 180^o = \frac{x}{r} = \frac{x}{|x|} = \frac{-1}{1} = -1$$

($x < 0$ since the terminal side of 180^o lies on the left of the y–axis).

Similarly, for angles of 90^o and 270^o (or their radian equivalent), any point on the terminal side of the angle has coordinates $(0,y)$, and the distance r from the origin to the point is $|y|$. Thus, for example ,

$$\csc 270^o = \frac{r}{y} = \frac{|y|}{y} = \frac{1}{-1} = -1$$

($y < 0$ since the terminal side of 270^o lies below the x–axis).

We give a table similar to Table 1 in Section 6.2 for these special angles.

Angle θ		$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
degrees	radians						
0	0	0	1	0	undefined	1	undefined
90	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180	π	0	-1	0	undefined	-1	undefined
270	$\frac{3\pi}{2}$	-1	0	undefined	0	undefined	-1
360	2π	0	1	0	undefined	1	undefined

Table 6.3.1

If you find it difficult to understand the table look at Figure 6.3.1.

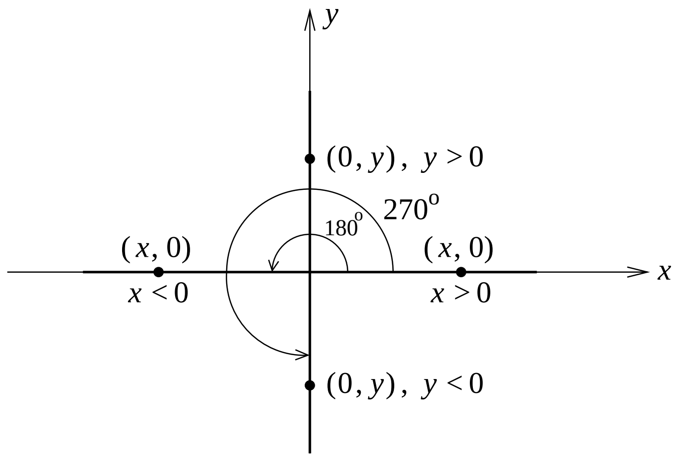
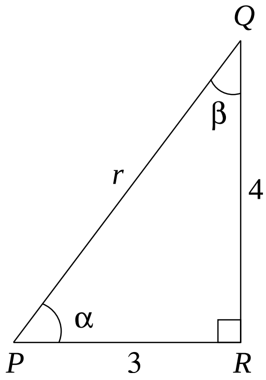


Figure 6.3.1

Our knowledge of trigonometric functions will also be applied when we solve triangles, or equations (see Chapter 7, Section 7.5). Consider the following example.

Example 6.3.1

Solve $\triangle PQR$, sketched in the figure below.



Solution

By Pythagoras

$$r = \sqrt{3^2 + 4^2} = 5.$$

Now $\tan\alpha = \frac{4}{3}$. Hence α is the angle whose tangent value is $\frac{4}{3}$. Since $\frac{4}{3}$ is not associated with one of the special angles contained in Table 1 we need to use a calculator to find α. In order to do this we need to use (although we will not study this now) the inverse tangent function. Recall what you learned about the inverse of a function in Section 2.8. We have

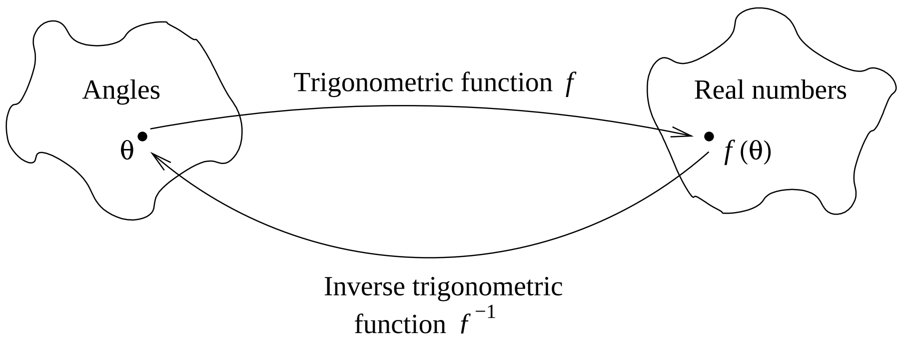


Figure 6.3.2

We know that a function is a **process**. When we "process" θ to obtain $f(\theta)$, we **find the value of the trigonometric function** f **of** θ. When we reverse the process, we go from $f(\theta)$ back to θ, i.e. we start from a trigonometric function value, and find the angle associated with that function value.

We have seen in Section 6.1 that many angles are coterminal with any given angle θ, which means, for example, that

$$\sin 70^o = \sin 430^o = \sin 790^o, \text{ etc.},$$

i.e. sine is not a one–to–one function, which is a fundamental requirement if we want to find an inverse **function**.

However, we already know that in the case of functions that are not one–to–one, we can restrict the domain so that we obtain subsets of the original domain over which the function is one–to–one. This is also the case for trigonometric functions. Although we will not study the inverse trigonometric functions in this module, we note that they exist, and that calculators apply this in order to obtain the value of θ from the trigonometric function value $f(\theta)$.

We have

$$\tan\theta = \tfrac{4}{3}.$$

We now use the key for the inverse tan function, marked $\tan^{-1}$, or arctan, or inv tan, depending on your calculator. If you want to find θ in degrees, then

- make sure your calculator is in degree mode
- enter $\frac{4}{3}$
- press the key for the inverse tan function.

You will then find on the calculator display

53.13

i.e.

$$\theta = \tan^{-1}\tfrac{4}{3} \approx 53.13^{\circ}.$$

Since $\triangle PQR$ is a right triangle we have

$$\alpha + \beta = 90^{\circ}$$

and hence

$$\begin{aligned} \beta &\approx 90^{\circ} - 53.13^{\circ} \\ &= 36.87^{\circ}. \end{aligned}$$

■ Evaluating Trigonometric Functions at any Angle

Since the definition of the trigonometric functions depends on the x– and y–coordinates of the point P on the terminal side of the angle, it is clear that as the signs of x and y change (depending on the quadrants in which P lies) the signs of the trigonometric functions will also change. The "Signs of the Trigonometric Functions" given in the block above Example 1 is important, but can easily be remembered, as we see from the diagram in the margin next to the table.

Study the subsection to the end of Example 1.

Reference angle

Now read the definition of **reference angle**. This is a particularly important concept, and we use it when we solve trigonometric equations (Section 7.5). Study Figure 6, Example 2 and the guidelines for evaluating trigonometric functions given below Example 2. In Examples 3 and 4 we see how the reference angle is used when we calculate the value of a trigonometric function.

The reference angle $\overline{\theta}$ is always acute. We can thus represent it in standard position as an angle in the first quadrant. Thus any point P on the terminal side of $\overline{\theta}$ has coordinates x and y such that $x > 0$ and $y > 0$.

Consider the four figures in Figure 6. For convenience let us consider them, from left to right, as Figures 6(a), 6(b), 6(c) and 6(d). Look at the angles in each of the figures. Take a point on the terminal side of each of these angles and compare its coordinates to the coordinates of P. We summarise the information in the following table.

Figure	Quadrant in which the angle θ lies	Coordinates of a point on the terminal side of θ in terms of the coordinates of $P(x,y)$, where $x > 0$, $y > 0$	$\overline{\theta}$ in terms of θ	
			in degrees	in radians
6(a)	I	(x,y)	$\overline{\theta} = \theta$	$\overline{\theta} = \theta$
6(b)	II	$(-x,y)$	$\overline{\theta} = 180^o - \theta$	$\overline{\theta} = \pi - \theta$
6(c)	III	$(-x,-y)$	$\overline{\theta} = \theta - 180^o$	$\overline{\theta} = \theta - \pi$
6(d)	IV	$(x,-y)$	$\overline{\theta} = 360^o - \theta$	$\overline{\theta} = 2\pi - \theta$

Table 6.3.2

We use this table to

- determine the signs of x and y in order to find the signs of the trigonometric functions
- find the reference angle associated with any angle, according to the quadrant in which its terminal side lies.

Note that for a positive angle θ

when θ lies in quadrant II, $\overline{\theta} = 180^o - \theta$ i.e. $\theta = 180^o - \overline{\theta}$
when θ lies in quadrant III, $\overline{\theta} = \theta - 180^o$ i.e. $\theta = 180^o + \overline{\theta}$
when θ lies in quadrant IV, $\overline{\theta} = 360^o - \theta$ i.e. $\theta = 360^o - \overline{\theta}$.

In Figures 6(a) to (d) the angles are all positive. However, if θ is negative, then

- for $-90^o < \theta < 0^o$, $\overline{\theta}$ is an acute angle in quadrant IV
- for $-180^o < \theta < -90^o$, $\overline{\theta}$ is an acute angle in quadrant III
- for $-270^o < \theta < -180^o$, $\overline{\theta}$ is an acute angle in quadrant II
- for $-360^o < \theta < -270^o$, $\overline{\theta}$ is an acute angle in quadrant I

Note also that this applies equally to angles measured in radians.

and we can apply the same reasoning as above.

Thus, for a negative angle θ

when θ lies in quadrant IV, $\overline{\theta} = -\theta$ i.e. $\theta = -\overline{\theta}$
when θ lies in quadrant III, $\overline{\theta} = \theta + 180^o$ i.e. $\theta = \overline{\theta} - 180^o$
when θ lies in quadrant II, $\overline{\theta} = -\theta - 180^o$ i.e. $\theta = -\overline{\theta} - 180^o$
when θ lies in quadrant I, $\overline{\theta} = 360^o + \theta$ i.e. $\theta = \overline{\theta} - 360^o$.

Make sure you understand how to interpret Table 6.3.2. Consider for example Figure 7.

- θ is a positive angle, and θ lies in quadrant IV.
  In quadrant IV, x values are positive and y values are negative.
- To calculate the reference angle $\overline{\theta}$, we determine $2\pi - \theta$, i.e. $\overline{\theta} = 2\pi - \frac{5\pi}{3}$.
- $\overline{\theta}$ is acute and can be related to quadrant I.
  $P(x, y)$ lies on the terminal side of $\overline{\theta}$, hence $x > 0$, $y > 0$.
- Thus a point on the terminal side of θ has coordinates $(x, -y)$.

The same reasoning applies to any angle coterminal with θ.

From this we conclude that, in order to calculate $f(\theta)$ for any trigonometric function f and any angle θ that is not quadrantal (measured in degrees or radians), we reason in the following way. It is convenient to work with an angle whose measure is between 0^o and 360^o. If two angles, θ and θ_t, are coterminal (i.e. a point on the terminal side of θ_t is also on the terminal side of θ), we know that $f(\theta) = f(\theta_t)$. We thus first convert an angle that measures more than 360^o to an angle θ that is coterminal with it, such that $0^o < \theta < 360^o$. Then we reason as follows.

When we say "θ lies in quadrant IV" we mean that the terminal side of θ lies in quadrant IV.

- Calculate the reference angle $\bar{\theta}$ as follows :

  if θ lies in quadrant II we calculate $\bar{\theta} = 180^{o} - \theta$ (or $\pi - \theta$)

  if θ lies in quadrant III we calculate $\bar{\theta} = \theta - 180^{o}$ (or $\theta - \pi$)

  if θ lies in quadrant IV we calculate $\bar{\theta} = 360^{o} - \theta$ (or $2\pi - \theta$).

- Note the quadrant in which the terminal side of θ lies.
- Find the sign of $f(\theta)$ (using the diagram in the margin on p. 490).
- Calculate $f\left(\bar{\theta}\right)$.
- Calculate $f(\theta)$: $f(\theta)$ and $f\left(\bar{\theta}\right)$ will have the same numerical value but may have different signs.

If we need to calculate $f(\theta)$ where $\theta < 0^{o}$, we adapt the reasoning above so that we have the following. We again first reduce the angle to an angle θ such that $-360^{o} < \theta < 0^{o}$.

This also applies to cases where radian measure is involved.

- Calculate the reference angle $\bar{\theta}$: we apply the same reasoning as before by considering the quadrant in which $|\theta|$ lies, and then calculating $180^{o} - |\theta|$, $|\theta| - 180^{o}$ or $360^{o} - |\theta|$.
- Note the quadrant in which the terminal side of θ lies.
- Find the sign of $f(\theta)$ using the figure in the margin on p. 490.
- Calculate $f\left(\bar{\theta}\right)$.
- Calculate $f(\theta)$: $f(\theta)$ and $f\left(\bar{\theta}\right)$ will have the same numerical value, but possibly different signs.

■ Reduction Formulas

The reduction formulas enable us to express $f(-\theta)$, $f(\pi - \theta)$, $f(\pi + \theta)$ and $f(2\pi - \theta)$ in terms of $f(\theta)$, where f is one of the trigonometric functions. We say that expressions such as $f(\pi + \theta)$ have been **reduced** to $f(\theta)$ and this gives rise to the term **reduction formulas**. We also have relationships between function values of θ and $\frac{\pi}{2} - \theta$, and between function values of θ and $\frac{\pi}{2} + \theta$.

The reduction formulas are given in Tables 6.3.3. to 6.3.8. The relationships follow from the definition of the trigonometric functions.

To give you an idea of how the formulas are derived, we prove the formulas given in the first table, Table 6.3.3. The proofs of all the formulas will hold for all angles, but in this case we prove the relationship between $f(-\theta)$ and $f(\theta)$ for an angle θ that lies in quadrant II. If θ lies in quadrant II, then $-\theta$ will lie in quadrant III. See the figure below.

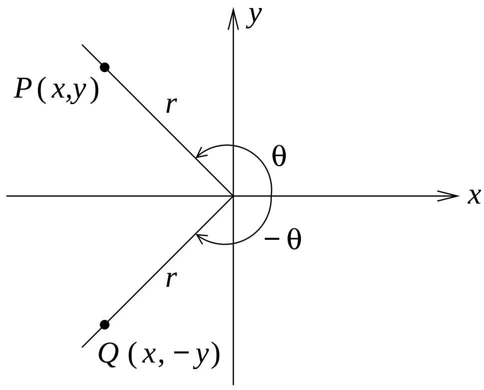


Figure 6.3.3

Now $\sin(-\theta) = \dfrac{-y}{r}$ and $\sin\theta = \dfrac{y}{r}$.

Thus $\sin(-\theta) = -\sin(\theta)$ and, by the reciprocal relationships,

$\csc(-\theta) = -\csc(\theta))$.

Similarly we find that

$\cos(-\theta) = \frac{x}{r} = \cos\theta$ (and thus also $\sec(-\theta) = \sec(\theta)$

and

$\tan(-\theta) = \frac{-y}{x} = -\tan(\theta)$ (and hence $\cot(-\theta) = -\cot(\theta))$.

We summarise these relationships in Table 6.3.3.

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\cot(-\theta) = -\cot\theta$
$\sec(-\theta) = \sec\theta$
$\csc(-\theta) = -\csc\theta$

Table 6.3.3

Relationship between $f(-\theta)$ and $f(\theta)$

From these results we note that

- sin and csc, tan and cot are **odd** functions
- cos and sec are **even** functions.

We also have the relationships in Tables 6.3.4, 6.3.5 and 6.3.6.

$\sin(\pi-\theta)=\sin\theta$
$\cos(\pi-\theta)=-\cos\theta$
$\tan(\pi-\theta)=-\tan\theta$
$\cot(\pi-\theta)=-\cot\theta$
$\sec(\pi-\theta)=-\sec\theta$
$\csc(\pi-\theta)=\csc\theta$

Table 6.3.4

Relationship between $f(\pi-\theta)$ **and** $f(\theta)$

Note that the formulas in Tables 6.3.4 to 6.3.6 can also be expressed in terms of degrees.

$\sin(\pi+\theta)=-\sin\theta$
$\cos(\pi+\theta)=-\cos\theta$
$\tan(\pi+\theta)=\tan\theta$
$\cot(\pi+\theta)=\cot\theta$
$\sec(\pi+\theta)=-\sec\theta$
$\csc(\pi+\theta)=-\csc\theta$

Table 6.3.5

Relationship between $f(\pi+\theta)$ **and** $f(\theta)$

$\sin(2\pi-\theta)=-\sin\theta$
$\cos(2\pi-\theta)=\cos\theta$
$\tan(2\pi-\theta)=-\tan\theta$
$\cot(2\pi-\theta)=-\cot\theta$
$\sec(2\pi-\theta)=\sec\theta$
$\csc(2\pi-\theta)=-\csc\theta$

Table 6.3.6

Relationship between $f(2\pi-\theta)$ **and** $f(\theta)$

Comment

Why do we not consider reduction formulas of the type where $f(\theta \pm 2\pi)$ is reduced to $f(\theta)$? We know that adding (or subtracting) complete revolutions does not change the function value, thus for all trigonometric functions f,

$$f(\theta \pm 2\pi) = f(\theta).$$

Complementary angles and cofunctions

Look again at the relationship between the angles θ and α in a right triangle (see p. 155 of this guide). Since θ and α are **complementary** angles, i.e. $\theta + \alpha = 90^o$, we have $\alpha = 90^o - \theta$. We thus have another set of reduction formulas, given in Table 6.3.7.

Note that these formulas can also be expressed in terms of degrees.

$$\sin\alpha = \sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\alpha = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\alpha = \tan\left(\tfrac{\pi}{2} - \theta\right) = \cot\theta$$

$$\cot\alpha = \cot\left(\tfrac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\alpha = \sec\left(\tfrac{\pi}{2} - \theta\right) = \csc\theta$$

$$\csc\alpha = \csc\left(\tfrac{\pi}{2} - \theta\right) = \sec\theta$$

Table 6.3.7

Relationship between $f(\theta)$ **and** $f(\frac{\pi}{2} - \theta)$

Note that the formulas in Table 6.3.7 link **complementary angles** to **cofunctions**, e.g. the **sine** of $(\frac{\pi}{2} - \theta)$ is linked to the **cosine** of θ.

If you have already studied Chapter 5 (or when you study Chapter 5) you will also be able to explain the relationship between $\sin\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta$ in terms of horizontal shifting of the graph of the sine function.

The last set of reduction formulas relate $f\left(\frac{\pi}{2} + \theta\right)$ to $f(\theta)$. We occasionally require these formulas, and they can be deduced in several ways: from the congruency of right triangles (see Exercises 5.2, question 84), from the graphs of the functions, or by using a reduction formula (as we do in the solutions to question 2 of Additional Exercises 6.3.1). We do not expect you to prove any of these formulas, although you should be able to apply them. The formulas are stated in the next table, Table 6.3.8, and you are asked to deduce two of them in Additional Exercises 6.3.1. Understanding how a formula is deduced means that you do not have to spend too much time memorising it: you know you will be able to work it out if you need to use it.

$$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$

$$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$$

$$\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$$

$$\sec\left(\frac{\pi}{2}+\theta\right) = -\csc\theta$$

$$\csc\left(\frac{\pi}{2}+\theta\right) = \sec\theta$$

Table 6.3.8

Relationship between $f(\theta)$ and $f(\frac{\pi}{2}+\theta)$

The next example shows how the reduction formulas may be applied.

Example 6.3.2

Show that

$$\frac{\tan 45^{\circ}\csc(180^{\circ}+\theta)\cot(-\theta)+\cos(90^{\circ}+\theta)\sin(180^{\circ}+\theta)}{\sec 30^{\circ}\tan(\theta-180^{\circ})\csc^{2}(180^{\circ}-\theta)}$$

is equal to

$$\frac{\sqrt{3}\cos\theta\left(\cos\theta+\sin^{4}\theta\right)}{2\sin\theta}.$$

Solution

Note that one of the terms in this example, $\tan(\theta-180^{\circ})$, does not appear in the form in which the reduction formulas are usually stated. We rewrite it, and have

$$\begin{aligned} \tan(\theta-180^{\circ}) &= \tan[-(180^{\circ}-\theta)] \\ &= -\tan(180^{\circ}-\theta) && \text{By Table 6.3.3.} \\ &= -(-\tan\theta). && \text{By Table 6.3.4.} \end{aligned}$$

Remember that $csc^2\theta = (csc\theta)^2$

Now

$$\frac{\tan 45^{\circ}\csc(180^{\circ}+\theta)\cot(-\theta)+\cos(90^{\circ}+\theta)\sin(180^{\circ}+\theta)}{\sec 30^{\circ}\tan(\theta-180^{\circ})\csc^2(180^{\circ}-\theta)}$$

$$= \frac{1(-\csc\theta)(-\cot\theta)+(-\sin\theta)(-\sin\theta)}{\frac{2}{\sqrt{3}}(-(-\tan\theta))\csc^2\theta} \qquad \text{Tables 6.3.3 to 6.3.8}$$

$$= \frac{\sqrt{3}\left(\csc\theta\cot\theta+\sin^2\theta\right)}{2\tan\theta\csc^2\theta}$$

$$= \frac{\sqrt{3}\left(\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}+\sin^2\theta\right)}{2\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin^2\theta}}$$

$$= \frac{\sqrt{3}\left(\frac{\cos\theta+\sin^4\theta}{\sin^2\theta}\right)}{2\left(\frac{1}{\cos\theta\cdot\sin\theta}\right)}$$

$$= \frac{\sqrt{3}\left(\cos\theta+\sin^4\theta\right)}{\sin^2\theta}\left(\frac{\cos\theta\sin\theta}{2}\right)$$

$$= \frac{\sqrt{3}\cos\theta\left(\cos\theta+\sin^4\theta\right)}{2\sin\theta}.$$

■ Trigonometric identities

Study this subsection in SRW, to the end of Example 7.

Note the difference between equations and identities. An **equation** is a statement that puts two mathematical expressions equal to each other, for example

$$2x-3=x^2+4$$

or

$$\sin^2\theta+2\sin\theta\cos\theta=-\cos^2\theta.$$

An equation may be **solved**; it may have no solution or more than one solution. It may also be undefined for certain values of the variable. For example

$$\frac{1}{x}+3=-2$$

is undefined for $x=0$, and its solution is $x=-\frac{1}{5}$.

Identities are true **for all values** of the variables for which both sides exist. For example, the equation

$$\sin^2\theta+\cos^2\theta=1$$

is true regardless of the value of θ. It is thus called an identity. The equation

$$\tan^2\theta+1=\sec^2\theta$$

is also true for all values of θ, **excluding** the values of θ for which $\tan\theta$ and $\sec\theta$ are undefined. It is thus also an identity.

The Pythagorean identities are also referred to as the squares identities.

Identities can be **proved**, whereas equations must be solved. Many of the proofs of identities you may have encountered before follow from an application of the fundamental identities given in the block above Example 5. You are not expected to be able to prove the Pythagorean identities, but understanding how they are derived will help you remember them.

We deal with proving identities in Chapter 7, Section 7.1.

■ Areas of triangles

Read the whole of this subsection up to the exercises. The "area rule" for triangles given in the block is a simple application of the trigonometry you have studied so far.

Additional Exercises 6.3.1

1. Use $\theta = 60^o$ to **illustrate** that the formulas in Table 6.3.8 are valid.
2. **Prove** that

$$\begin{aligned} \sec\left(\frac{\pi}{2}+\theta\right) &= -\csc\theta \\ \csc\left(\frac{\pi}{2}+\theta\right) &= \sec\theta. \end{aligned}$$

3. Simplify the following expressions.

(a) $\dfrac{\sin^2(180^o+\theta)+\cos(180^o-\theta)\csc(90^o-\theta)}{\tan 60^o\tan(90^o-\theta)\cot(180^o-\theta)}$

(b) $\dfrac{\cos\left(\frac{\pi}{2}-\theta\right)\csc\left(\frac{\pi}{2}+\theta\right)}{\sin(\pi+\theta)\sec(2\pi-\theta)\tan(-\theta)}$

6.4 THE LAW OF SINES

To study	SRW, Section 6.4.
Outcomes	After studying this section you should be able to ► state and prove the Law of Sines ► apply the Law of Sines to solve triangles that are not necessarily right triangles ► apply the Law of Sines to solve real–life problems where triangles are involved.
Exercises	Exercises 6.4: questions 1–39 (odd numbers)

In this section and the next we study two useful laws, the Law of Sines and the Law of Cosines. The Law of Sines is discussed, stated and proved at the beginning of Section 6.4.

In Section 6.5 the proof of the Law of Cosines depends on drawing a system of axes and placing one of the vertices of the triangle at the origin. We can also prove the Law of Sines in the same way.

Proof

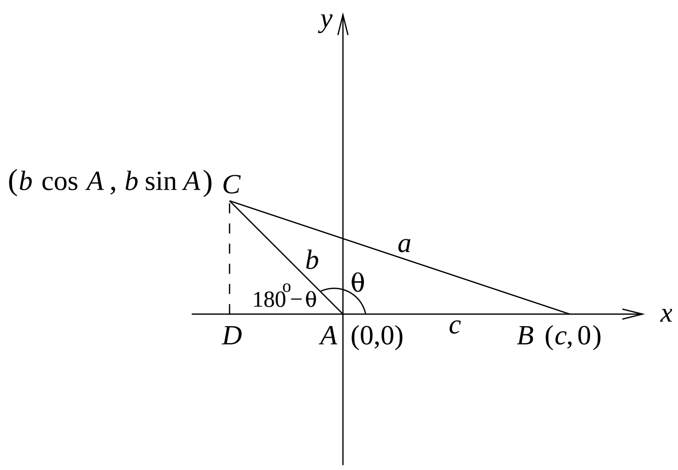


Figure 6.4.1

The vertex C has coordinates (x,y). We first show that $(x,y) = (b\cos A, b\sin A)$.

If we denote $\angle CAB$ by θ, then $\angle CAD = 180^{\circ} - \theta$.

Now

$$\begin{aligned} x &= -\text{ length of } AD \\ y &= \text{ length of } CD. \end{aligned}$$

We have

$$\frac{AD}{b} = \cos(180^{\circ} - \theta)$$

i.e.

$$AD = b\cos(180^{\circ} - \theta) = -b\cos\theta. \qquad \text{By the formulas in Table 6.3.4.}$$

Hence

$$x = b\cos\theta.$$

Also,

$$\frac{CD}{b} = \sin(180^{\circ} - \theta)$$

i.e.

$$CD = b\sin(180^{\circ} - \theta) = b\sin\theta. \qquad \text{By the formulas in Table 6.3.4.}$$

Hence

$$y = b\sin\theta.$$

If we denote $\angle CAB$ by A instead of θ we have $(x,y) = (b\cos A,\ b\sin A)$.

$$\begin{aligned} \text{Area of } \triangle ABC &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times c \times CD \\ &= \tfrac{1}{2} \times c \times b\sin A \end{aligned}$$

i.e.

$$\text{Area of} \triangle ABC = \tfrac{1}{2} bc\sin A.$$

Similarly it can be shown that

$$\begin{aligned} \text{Area of } \triangle ABC &= \tfrac{1}{2} ab\sin C \\ &= \tfrac{1}{2} ac\sin B \end{aligned}$$

and hence

$$\tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ab\sin C \qquad (1)$$

i.e.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

if we multiply each term in equation (1) by $\dfrac{2}{abc}$.

Examples 1 and 2 show how the Law of Sines can be applied. Note the abbreviation "mi" for miles, which is one of the units of length used in the United States of America.

■ The Ambiguous Case

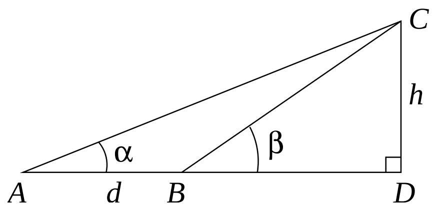
Supplementary angles

Study this subsection carefully. The "caution" note after Example 3 points out the necessity of checking both the angle and its supplement as possibilities. Remember that two angles are **supplementary** if their sum is 180^o (or π radians).

Take note of the answer to question 29, Exercises 6.4, given at the end of this Study Guide.

Additional Exercises 6.4.1

1. Consider the following diagram.



   (a) Use the Law of Sines to write an expression for BC in terms of α, β and d.

   (b) Show that
$$h = d\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

6.5 THE LAW OF COSINES

To study	SRW, Section 6.5.
Outcomes	After studying this section you should be able to ► state and prove the Law of Cosines ► apply the Law of Cosines to solve triangles and real–life problems in which triangles are involved ► apply the Law of Cosines to obtain Heron's formula for the area of a triangle ► use Heron's formula to calculate area.
Exercises	Exercises 6.5: questions 1–49 (odd numbers)

We see that the Law of Sines is insufficient, and we thus need the Law of Cosines as well in certain cases. The Law and its proof are given at the beginning of the section.

After the proof, the Law of Cosines is stated in words. It is always a useful technique to try to express a mathematical statement as a language statement. This process helps to make the mathematics clearer and easier to remember.

Before Example 1, note the comment regarding the Pythagorean Theorem being a special case of the Law of Cosines.

Study Examples 1–4. In Example 1, note that after

$$c^2 \approx 173730.2367$$

we have

$$c \approx \sqrt{173730.2367}.$$

We know that

$$a^2 = b \Leftrightarrow a = \pm\sqrt{b}$$

but in this case, since c represents a length, and length is always positive, we may ignore the negative possibility. Note the caution stated after Example 3. Sketching the triangle and reviewing your answer in terms of the sketch are two important activities.

■ **Area of Triangles**

Heron's formula is an interesting application of the area rule of triangles and the Law of Cosines. Study the derivation of the formula. It is a useful formula to apply when only the three sides of a triangle are known. Example 5 illustrates how Heron's formula is used.

Now do the questions set for Exercises 6.5. In question 39 **divergent** roads are mentioned. Two roads diverge if they move further and further apart from each other.

6 REVIEW AND TEST

The Chapter Review again provides a useful summary (Concept Check) of important concepts. You may try the Exercises and Test if you have time now, or try selected odd–numbered questions when you prepare for the exams.

7 ANALYTIC TRIGONOMETRY

You may have studied Chapter 5 first, or Chapter 6 first, but, regardless of the order in which you studied these chapters, you should now have a good understanding of the graphical and geometric properties of trigonometric functions. In this chapter we study the algebraic properties of trigonometric functions.

7.1 TRIGONOMETRIC IDENTITIES

To study SRW, Section 7.1.

Outcomes After studying this section you should be able to

- ► simplify trigonometric expressions
- ► prove trigonometric identities.

Exercises Exercises 7.1: questions 1–93 (odd numbers), 99(c)

Apart from the fundamental trigonometric identities given in this section you may also need the Reduction Formulas in Tables 6.3.3 to 6.3.8.

■ Simplifying Trigonometric Expressions

Work through Examples 1 and 2. Later, in Section 7.5, we see how the identities are also used to solve equations.

■ Proving Trigonometric Identities

Examples 3 – 6 illustrate the different methods we use to prove identities. We have three options.

- ► Transform the more complicated side into the other side (Examples 3 and 4).
- ► Introduce something extra and then transform (Example 5).
- ► Transform both sides to the same expression (Example 6).

Take note of the Guidelines for Proving Trigonometric Identities and the "caution" note given before Example 3. Also take note of the **incorrect** way of "proving" an identity. Suppose we want to prove that

$$\frac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2 .$$

Caution: The **incorrect** way of "proving" an identity

We **may not** reason as follows.

$$\frac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \sec^2 x - 2\sec x \tan x + \tan^2 x$$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \tan^2 x + 1 - 2\sec x \tan x + \tan^2 x$$

Using the squares identity $\tan^2 x + 1 = \sec^2 x$.

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = 2\tan^2 x - 2\sec x \tan x + 1$$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{2\sin^2 x}{\cos^2 x} - \frac{2\sin x}{\cos^2 x} + 1$$

Writing all expressions in terms of $\sin$ and $\cos$.

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{2\sin^2 x - 2\sin x + \cos^2 x}{\cos^2 x}$$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{2\sin^2 x - 2\sin x + 1 - \sin^2 x}{\cos^2 x}$$

Using the squares identity $\sin^2 x + \cos^2 x = 1$ in two different steps.

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{\sin^2 x - 2\sin x + 1}{1 - \sin^2 x}$$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{(\sin x - 1)^2}{(1-\sin x)(1+\sin x)}$$

$a^2 - b^2 = (a-b)(a+b)$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{(1-\sin x)^2}{(1-\sin x)(1+\sin x)}$$

$(a-b)^2 = (b-a)^2$

$$\Rightarrow \quad \frac{1-\sin x}{1+\sin x} = \frac{1-\sin x}{1+\sin x}$$

$$\Rightarrow \quad \text{LHS} = \text{RHS}$$

Can you understand why this is wrong? In order to **prove** that LHS = RHS we may not begin by **assuming** (as we do above) that LHS = RHS, and then simplify until we find that the two expressions are in fact equal. We need to work separately with the two sides, and show in the end that they are equal.

Now try a selection of the questions in Exercises 7.1. Note that the proof of 99(c) and alternative proofs to some of the other questions are given at the end of this Study Guide. Example 7 concerns trigonometric substitution which is useful in calculus.

7.2 ADDITION AND SUBTRACTION FORMULAS

To study	SRW, Section 7.2 (excluding Expressions of the Form $A\sin x + B\cos x$).
Outcomes	After studying this section you should be able to ► use addition and subtraction formulas to calculate the value of a trigonometric expression, to prove identities and to simplify trigonometric expressions.
Exercises	Exercises 7.2: questions 1–39 (odd numbers), 49

The formulas are stated in the block at the beginning of the section. You may read through the proof of the addition formula for cosine given below the block, but you are not expected to prove the formula. Examples 1 and 2 show how the formulas can be used to find the exact value of an expression (assuming we can relate the angles involved to special angles, and assuming we do not have access to a calculator). Examples 3, 4 and 5 show how the formulas are used to prove an identity. Later, in Section 7.5, we see how the formulas are also used to solve equations.

Caution: It is **not true** that $f(x+y) = f(x) + f(y)$, where f is any one of the trigonometric functions.

■ Expressions of the Form $A\sin x + B\cos x$

You may omit this subsection.

Now do the questions from Exercises 7.2. Note that proofs for some of the questions, and the solution for question 49, are also given at the end of this Study Guide.

7.3 DOUBLE–ANGLE, HALF–ANGLE AND PRODUCT–SUM FORMULAS

To study SRW, Section 7.3.

Outcomes After studying this section you should be able to

- use double–angle and half–angle formulas and the formulas for lowering powers to
  - calculate values of expressions
  - simplify expressions
  - prove identities
- use product–to–sum formulas to
  - calculate values of expressions
  - simplify expressions
  - prove identities.

Exercises Exercises 7.3: questions 1 – 73 (odd numbers, every third question, i.e. 1, 7, 13, etc.)

The double– and half–angle formulas, and the product–to–sum formulas, are useful when we solve equations.

■ Double–Angle Formulas

The formulas for $\sin 2x$, $\cos 2x$ and $\tan 2x$ are stated in SRW, in the first block of Section 7.3. Clearly we do not need formulas for $\csc 2x$, $\sec 2x$ or $\cot 2x$, since these expressions are related to the above expressions through the reciprocal identities.

Work through the proofs for the double–angle formulas and make sure you understand how the formulas are derived, so that you can work them out yourself if necessary.

Example 1 shows how the double–angle formula is used to find $\sin 2x$ and $\cos 2x$ if we know the value of one of the trigonometric functions of x (in this case $\cos x$). Example 2 shows how we may apply the double–angle formula for cosine to deduce a triple–angle formula; clearly we could continue in this way and find formulas for $\cos 4x$, $\cos 5x$, etc.

In Example 3 the process of transforming one side of an identity into a simpler expression requires the use of a double–angle formula.

■ Half–Angle Formulas

We first derive three formulas for lowering even powers of sin, cos and tan and use these to determine the half–angle formulas (see the relevant blocks in SRW).

To prove $\sin^2 x = \dfrac{1-\cos 2x}{2}$, we reason as follows.

By the double–angle formula for cos we have

$$\cos 2x = 1 - 2\sin^2 x$$

i.e. $$2\sin^2 x = 1 - \cos 2x$$

i.e. $$\sin^2 x = \frac{1-\cos 2x}{2}.$$

Once again, avoid memory overload by making sure you know how the formulas are derived so that you do not necessarily have to rely on remembering them.

Since a detailed proof is not given, we show the derivation of the formula for $\sin\frac{u}{2}$, namely

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}.$$

Proof

Since $$\sin^2 x = \frac{1-\cos 2x}{2}$$

we let $x = \frac{u}{2}$ and we then have

$$\begin{aligned} \sin^2\frac{u}{2} &= \frac{1-\cos 2\left(\frac{u}{2}\right)}{2} \\ &= \frac{1-\cos u}{2} \end{aligned}$$

i.e. $$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}.$$

The formula for $\cos\frac{u}{2}$ is found in a similar way.

At the end of the derivation of the formula for $\tan\frac{u}{2}$, we have the following two statements, just above Example 5 in SRW.

$1-\cos u$ is non–negative for all values of u.

$\sin u$ and $\tan\frac{u}{2}$ always have the same sign.

Can you explain these statements?

In this discussion we exclude values of u for which $\sin u = 0$ and for which $\tan \frac{u}{2}$ is undefined, i.e. all multiples of $n\pi$.

Note that $|\cos\theta| \leq 1$ for all θ, i.e. $-1 \leq \cos\theta \leq 1$. Thus, if $\cos u$ is negative,

$$-1 \leq \cos u < 0$$

and hence

$$1 - \cos u = 1 - \text{ (a negative number) } > 1, \quad \text{i.e. } 1 - \cos u > 0.$$

Also, if $\cos u$ is non–negative, then

$$0 \leq \cos u \leq 1$$

and hence

$$1 - \cos u = 1 - \text{ (a number smaller than or equal to 1)} \geq 0.$$

If we consider the various options for u and $\frac{u}{2}$, then (assuming u is a positive angle, not quadrantal) we have the following.

- $0^\circ < u < 90^\circ \Rightarrow 0^\circ < \frac{u}{2} < 45^\circ$

Thus u and $\frac{u}{2}$ lie in quadrant I; $\sin u > 0$ and $\tan \frac{u}{2} > 0$.

- $90^\circ < u < 180^\circ \Rightarrow 45^\circ < \frac{u}{2} < 90^\circ$

Thus u lies in quadrant II (and $\sin u > 0$); $\frac{u}{2}$ lies in quadrant I (and $\tan \frac{u}{2} > 0$).

- $180^\circ < u < 270^\circ \Rightarrow 90^\circ < \frac{u}{2} < 135^\circ$

Thus u lies in quadrant III (and $\sin u < 0$); $\frac{u}{2}$ lies in quadrant 2 (and $\tan \frac{u}{2} < 0$).

- $270^\circ < u < 360^\circ \Rightarrow 135^\circ < \frac{u}{2} < 180^\circ$

Thus u lies in quadrant IV (and $\sin u < 0$); $\frac{u}{2}$ lies in quadrant 2 (and $\tan \frac{u}{2} < 0$).

If u is a quadrantal angle (and $u \neq n\pi$), then

- $u = 90^\circ \Rightarrow \frac{u}{2} = 45^\circ; \quad \sin 90^\circ = 1 = \tan 45^\circ$
- $u = 270^\circ \Rightarrow \frac{u}{2} = 135^\circ; \quad \sin 270^\circ = -1 = \tan 135^\circ.$

Thus, in both cases we see that $\sin u$ and $\tan \frac{u}{2}$ have the same sign. In the equation

$$\tan \frac{u}{2} = \pm \frac{|1 - \cos u|}{|\sin u|}$$

we thus have

Note that $\pm\frac{1-\cos u}{-\sin u}$ is the same as $\mp\frac{1-\cos u}{\sin u}$.

$$\begin{aligned}\tan\frac{u}{2} &= \pm\frac{1-\cos u}{|\sin u|} \qquad |1-\cos u| = 1-\cos u \text{ since } 1-\cos u \geq 0.\\ &= \begin{cases} \pm\left(\frac{1-\cos u}{\sin u}\right) & \text{if } \sin u > 0\\ \pm\left(\frac{1-\cos u}{-\sin u}\right) & \text{if } \sin u < 0.\end{cases}\end{aligned}$$

Thus, regardless of the sign of $\sin u$, we have $\tan\frac{u}{2} = \pm\frac{1-\cos u}{\sin u}$.

Since $\tan\frac{u}{2}$ and $\sin u$ are simultaneously positive or simultaneously negative, and $1-\cos u$ is always non–negative, we have

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}.$$

Now study Examples 5 and 6 which show how we apply the half–angle formulas.

■ Product–to–Sum Formulas

There are two types of these formulas, namely

- ► product–to–sum formulas
- ► sum–to–product formulas.

Note how the formulas are derived, and the way in which they can be applied to manipulate trigonometric expressions (Examples 7, 8 and 9).

Try as many of the questions set for this section as you can. If you find you are consistently unable to get the correct answers, do more of the problems at the beginning of the exercises, and see whether you can find out where the misconceptions lie. The proofs for questions 61, 63, 67 and 73 are given at the end of this Study Guide as well.

7.4 INVERSE TRIGONOMETRIC FUNCTIONS

You will deal with these concepts in later mathematics, and you do not need to study this section now.

7.5 TRIGONOMETRIC EQUATIONS TRIGONOMETRIC INEQUALITIES

To study	SRW, Section 7.5, and additional notes.
Outcomes	After studying this section you should be able to ► use algebraic methods and apply trigonometric identities to solve trigonometric equations ► apply algebraic and trigonometric principles, and graphs of trigonometric functions, to solve trigonometric inequalities.
Exercises	Exercises 7.5: questions 1 – 47 (odd numbers, every third question); 57–71 (odd numbers, every third question) Additional Exercises 7.5.1

The title of this section includes trigonometric inequalities, which are not covered in SRW, but we will deal with them in this study guide.

■ Trigonometric Equations

Section 7.5 begins by explaining what a trigonometric equation is, and again emphasises the difference between an equation and an identity. In any mathematical problem it is always helpful to be able to visualise the situation, and a graph of the trigonometric functions (and occasionally other functions as well) involved makes it easier to interpret the problem, as we see in Figure 1 (for Example 1). From the sketch we see that the equation has **infinitely many solutions**.

In Example 1, after simplification we have $\sin x = \frac{1}{2}$. It is sometimes convenient to add a step here (not included in SRW). We use the concept of a **reference angle** (measured in degrees or radians).

From Table 1 in Section 6.2 we see that the reference angle is $30°$, or $\frac{\pi}{6}$ radians. Thus, in the interval $[0, 2\pi)$, the values of x for which the equation is true are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Now read the rest of Example 1. (Except in some applied problems, we always use **radian measure** for the variable.)

What would happen if we were asked to solve $2\sin x = 4$? By algebraic simplification this equation becomes $\sin x = 2$; we know that $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$, and thus this equation has **no solution**.

Since trigonometric functions are periodic it is clear that (unless the domain is restricted) trigonometric equations must either have infinitely many solutions, or no solution.

We refer to only one solution as a **unique** solution. A trigonometric equation will have **only one solution** if it is solved on a particular restricted domain. For example, suppose in Example 1 we were asked to solve $2\sin x - 1 = 0$ for $x \in \left[0, \frac{\pi}{2}\right]$ We see from Figure 1 that in the interval $\left[0, \frac{\pi}{2}\right]$, the graph of $y = \sin x$ cuts the line $y = \frac{1}{2}$ in only one point, namely where $x = \frac{\pi}{6}$, and hence $x = \frac{\pi}{6}$ is the only solution of the equation. Of course, if we restrict the domain to a bigger interval we will have more than one solution, but still a finite number of solutions.

Example 3 shows (as we know from previous sections) that the points of intersection of the graphs of two functions are found by equating the two functions. In this example this leads to a trigonometric equation.

Consider Example 4. In this example we must first apply algebraic principles before we can solve the equation. Now consider the following example.

Example 7.5.1

Solve the trigonometric equation $\tan^4 2x - 9 = 0$.

Solution

$$\begin{array}{rrcll} & \tan^4 2x - 9 & = & 0 & \text{Given equation.} \\ \Rightarrow & \tan^4 2x & = & 9 & \text{Add 9.} \\ \Rightarrow & \tan 2x & = & \sqrt{3} \ \text{ or } \ \tan 2x = -\sqrt{3} & \text{Take fourth roots.} \end{array}$$

The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ contains one complete period of the tangent function. In this interval, we have

$$2x = \frac{\pi}{3} \quad \text{or} \quad 2x = -\frac{\pi}{3}.$$

Since the tangent function is periodic with period π, all the solutions are given by

$$\begin{array}{lccccl} & 2x = \frac{\pi}{3} + k\pi & \text{or} & 2x = -\frac{\pi}{3} + k\pi & \\ \text{i.e.} & x = \frac{\pi}{6} + \frac{k\pi}{2} & \text{or} & x = -\frac{\pi}{6} + \frac{k\pi}{2} & \text{Divide by 2.} \end{array}$$

for any integer k.

Some textbooks refer to a "double argument" when they refer to the $2x$ in $\tan 2x$. The word argument refers to the element on which the function (in this case tan) operates. The important differences between equations in which there is a single, double, halved, etc. argument are evident when we move from the reference angle to the general solution. Note that in Example 7.5.1 the reference angle is $\frac{\pi}{3}$. Using the sign of $\tan 2x$ and considering the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we find two possible answers, namely $\frac{\pi}{3}$ and $-\frac{\pi}{3}$. Since the period for tan is π, the general solution for $2x$ is $2x = \frac{\pi}{3} + k\pi$ or $2x = -\frac{\pi}{3} + k\pi$, $k \in \mathbb{Z}$. Solving these equations for x gives us the two cases for the general solution for x.

In some cases it may be possible or necessary to rewrite the equation in which a double (or halved) argument appears as an equation just in x, and solve in the normal way. Consider the example below.

Example 7.5.2

Solve the following equation.

$$\cos 2x + \sin x = 0.$$

Solution

$$\begin{array}{lrcll}
 & \cos 2x + \sin x & = & 0 & \\
\Leftrightarrow & 1 - 2\sin^2 x + \sin x & = & 0 & \text{Double–angle formula.} \\
\Leftrightarrow & -2\sin^2 x + \sin x + 1 & = & 0 & \\
\Leftrightarrow & 2\sin^2 x - \sin x - 1 & = & 0 & \\
\Leftrightarrow & (2\sin x + 1)(\sin x - 1) & = & 0 & \\
\Leftrightarrow & 2\sin x + 1 = 0 & \text{or} & \sin x - 1 = 0 & \\
\Leftrightarrow & \sin x = -\frac{1}{2} & \text{or} & \sin x = 1 &
\end{array}$$

Reference angle for $\sin x = -\frac{1}{2}$: From Table 1 in Section 6.2 we know that the reference angle is $\frac{\pi}{6}$ radians. Thus, in the interval $[0, 2\pi)$, we have $x = \frac{7\pi}{6}$ or $x = \frac{11\pi}{6}$.

For $\sin x = 1$:
From Table 6.3.1 we know that $x = \frac{\pi}{2}$.
The general solution of the equation is thus

$$x = \frac{7\pi}{6} + 2k\pi, \text{ or } x = \frac{11\pi}{6} + 2k\pi, \text{ or } x = \frac{\pi}{2} + 2k\pi, \text{ where } k \in \mathbb{Z}.$$

In Example 7.5.3 we have a halved argument, which we solve directly, without first writing the equation in terms of x.

Example 7.5.3

Solve the equation

$$\tan\frac{x}{2}\tan x + \tan x - \sqrt{3}\tan\frac{x}{2} = \sqrt{3}$$

for $x \in (-\pi, \pi)$.

Solution

We exclude values of x for which $\tan\frac{x}{2}$ and $\tan x$ are undefined. For $x \neq \frac{(n+1)\pi}{4}$, $n \in \mathbb{Z}$, we have

The expression on the left side of this equation is factorised by means of grouping.

$$\begin{array}{lrcl} & \tan\frac{x}{2}\tan x + \tan x - \sqrt{3}\tan\frac{x}{2} & = & \sqrt{3} \\ \Leftrightarrow & \tan\frac{x}{2}\tan x + \tan x - \sqrt{3}\tan\frac{x}{2} - \sqrt{3} & = & 0 \\ \Leftrightarrow & \tan x\left(\tan\frac{x}{2}+1\right) - \sqrt{3}\left(\tan\frac{x}{2}+1\right) & = & 0 \\ \Leftrightarrow & \left(\tan x - \sqrt{3}\right)\left(\tan\frac{x}{2}+1\right) & = & 0 \\ \Leftrightarrow & \tan x - \sqrt{3} = 0 & \text{or} & \tan\frac{x}{2}+1 = 0 \\ \Leftrightarrow & \tan x = \sqrt{3} & \text{or} & \tan\frac{x}{2} = -1. \end{array}$$

We first consider $\tan x = \sqrt{3}$. The reference angle (see Figure 6 on p. 403) is $\frac{\pi}{3}$. On the interval $(-\pi, \pi)$ we have

$$\begin{array}{lrcl} & \tan x & = & \sqrt{3} \\ \Leftrightarrow & x = \frac{\pi}{3} & \text{or} & x = -\frac{2\pi}{3}. \end{array}$$

Now we consider $\tan\frac{x}{2} = -1$. If $x \in (-\pi, \pi)$ it follows that $\frac{x}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We look at the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and we see that

$$\begin{array}{lrcl} & \tan\frac{x}{2} & = & -1 \\ \Rightarrow & \frac{x}{2} & = & -\frac{\pi}{4} \\ \Rightarrow & x & = & -\frac{\pi}{2}. \end{array}$$

The solution of the equation on the interval $(-\pi, \pi)$ is

$$x = \frac{\pi}{3} \quad \text{or} \quad x = -\frac{2\pi}{3} \quad \text{or} \quad x = -\frac{\pi}{2}.$$

Now study Example 5. Notice that we cannot manipulate the equation into a form in which it can be solved until we apply one of the Pythagorean identities.

Study Examples 6, 7 and 8 and take special note of the "caution" comment after Example 7. We prefer that you do not check your answer as SRW does in Example 7. It is mathematically better practice to set out the checking process as follows.

Check your answers:

$$x = \frac{\pi}{2} \Rightarrow \begin{cases} \text{LHS} = \cos x + 1 = \cos\frac{\pi}{2} + 1 = 0 + 1 = 1 \\ \text{RHS} = \sin x = \sin\frac{\pi}{2} = 1 \end{cases}$$

LHS $=$ RHS, and thus $x = \frac{\pi}{2}$ is a solution of the equation.

$$x = \frac{3\pi}{2} \Rightarrow \begin{cases} \text{LHS} = \cos x + 1 = \cos\frac{3\pi}{2} + 1 = 0 + 1 = 1 \\ \text{RHS} = \sin x = \sin\frac{3\pi}{2} = -1 \end{cases}$$

LHS $\neq$ RHS, thus $x = \frac{3\pi}{2}$ is not a solution of the equation.

$$x = \pi \Rightarrow \begin{cases} \text{LHS} = \cos x + 1 = \cos\pi + 1 = -1 + 1 = 0 \\ \text{RHS} = \sin x = \sin\pi = 0 \end{cases}$$

LHS = RHS, and thus $x = \pi$ is a solution of the equation.

Example 6 is similar to our previous Example 7.5.2.

For the time being you may just read Examples 10 and 11. Although we have not studied inverse trigonometric functions in this module, we have in fact been applying an inverse process, as we see in the following sketch.

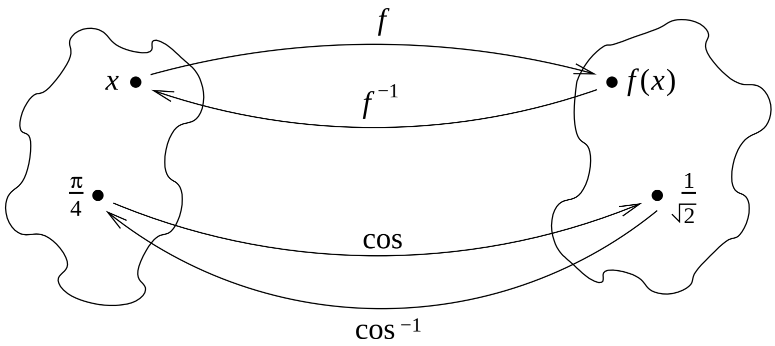


Figure 7.5.1

Suppose f is any one of the trigonometric functions. In this figure we consider the function f defined by $f(x) = \cos x$.

If $\cos x = \frac{1}{\sqrt{2}}$, then we find the reference angle by "working backwards" (i.e. using the inverse function for the function cos) and recognising that the angle associated with a cosine value of $\frac{1}{\sqrt{2}}$ is the angle $\frac{\pi}{4}$.

Using a calculator to solve trigonometric equations is necessary when angles other than the special angles are involved, since we cannot simply recall the angle associated with a specific function value. Some of you will already have solved such equations before, without necessarily having studied inverse trigonometric functions. However, you will return to these later, in other mathematics modules.

Now try the suggested questions in Exercises 7.5.

■ Trigonometric Inequalities

Consider the following figure.

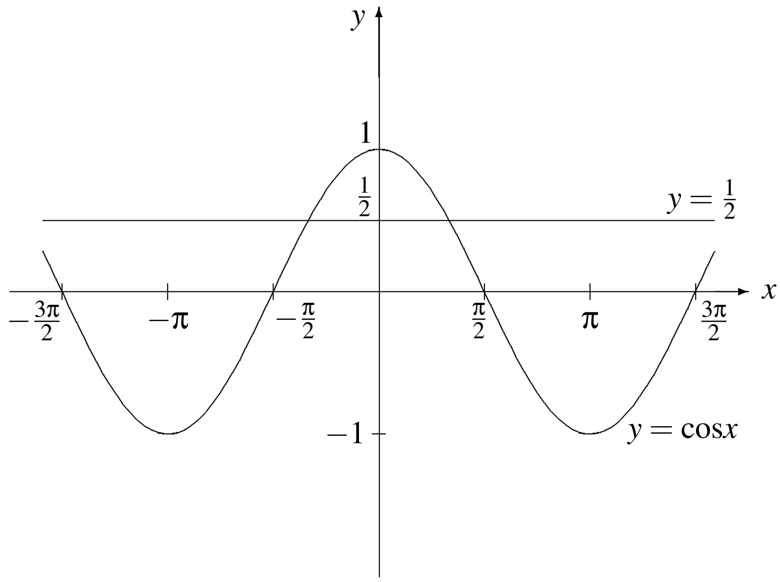


Figure 7.5.2

It is clear that there are infinitely many points of intersection of the graphs of $y = \frac{1}{2}$ and $y = \cos x$, and thus infinitely many solutions of the equation $\cos x = \frac{1}{2}$.

We now consider the **inequality** $\cos x \geq \frac{1}{2}$. We can interpret the statement

$$\text{solve } \cos x \geq \tfrac{1}{2}$$

as

find the values of x for which the graph of $y = \cos x$ lies above the line defined by $y = \frac{1}{2}$.

We see that the graph of $y = \cos x$ lies above the horizontal line defined by $y = \frac{1}{2}$ on an infinite number of intervals, again determined by the periodicity of the cosine function and the fact that the domain of both functions is $\mathbb{R}$.

If we consider only the interval $[-2\pi, 2\pi]$, we see that

$$\left\{x \in \mathbb{R} \mid \cos x \geq \frac{1}{2}\right\} = \left[-2\pi, -\frac{5\pi}{3}\right] \cup \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right].$$

In this example, because we can work with special angles, it is possible to solve the inequality directly from the graphs. However, we do not always want to solve trigonometric inequalities graphically, neither will we always deal with special angles. We will then need to use a calculator, but such inequalities are not dealt with in this module.

Consider Example 7.5.4.

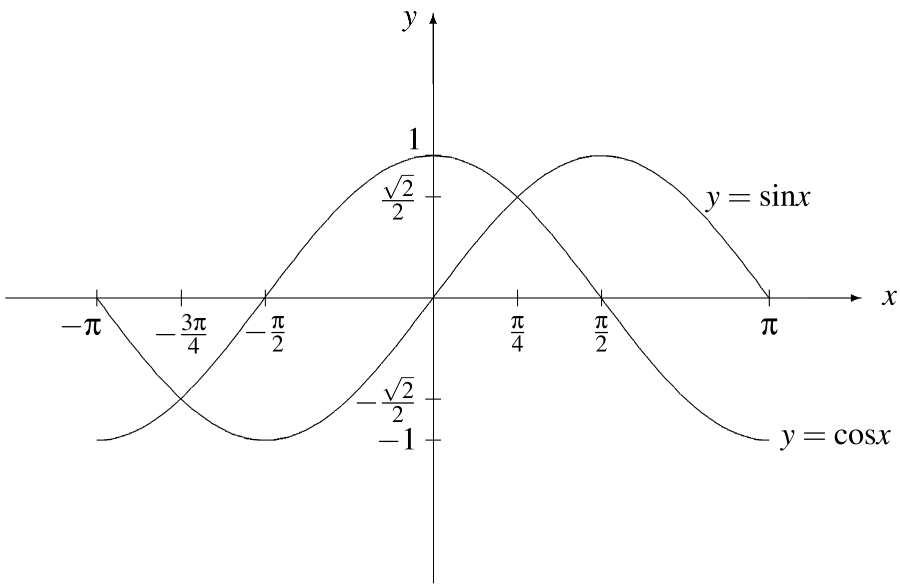
Example 7.5.4

Use appropriate graphs to solve

$$\cos x - \sin x \geq 0$$

for $x \in (-\pi, \pi)$.

Solution



From the figure we see that when $x \in (-\pi, \pi)$

$$\begin{aligned} & \cos x - \sin x \geq 0 \\ \Leftrightarrow \quad & \cos x \geq \sin x \\ \Leftrightarrow \quad & -\tfrac{3\pi}{4} \leq x \leq \tfrac{\pi}{4}. \end{aligned}$$

We must also be able to solve trigonometric inequalities algebraically. In most inequalities we will use a combination of algebraic methods as well as the properties of trigonometric functions.

We begin with an example.

Example 7.5.5

Solve the inequality

$$\sin x \cos 2x > 0$$

for $x \in (-\pi, 0)$.

Solution

Keep in mind what the graphs of $y = \sin x$ and $y = \cos 2x$ look like. The period of the function defined by $y = \sin x$ is 2π. The period of the function defined by $y = \cos 2x$ is half that of the period of the cosine function defined by $y = \cos x$, and hence π.

$$\sin x \cos 2x > 0 \Leftrightarrow (\sin x > 0 \text{ and } \cos 2x > 0) \text{ or } (\sin x < 0 \text{ and } \cos 2x < 0)$$

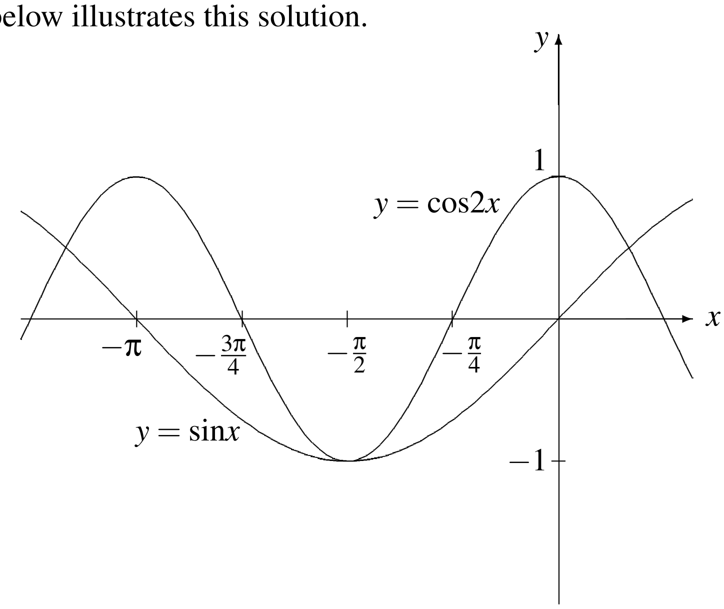
For $x \in (-\pi, 0)$ there are no values of x such that $\sin x > 0$. Hence there are no values of x that satisfy $\sin x > 0$ and $\cos 2x > 0$.

Next we consider $\sin x < 0$ and $\cos 2x < 0$. We see from the graphs that $\sin x < 0$ for all $x \in (-\pi, 0)$, and $\cos 2x < 0$ for all $x \in \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$.

We have

$$\begin{array}{ll} & \sin x < 0 \text{ and } \cos 2x < 0 \\ \Rightarrow & x \in (-\pi, 0) \text{ and } x \in \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right) \\ \Rightarrow & x \in \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right). \end{array}$$

The figure below illustrates this solution.



In the final example you need to solve the inequality for x. However, the argument is $\frac{x}{2}$. This means that the equation must first be solved for $\frac{x}{2}$, and the answers must then be manipulated further.

Example 7.5.6

Solve the inequality

$$\sin\tfrac{x}{2} - 2\cos^2\tfrac{x}{2} + 1 \geq 0$$

for $x \in [0, 3\pi]$.

$$\begin{array}{lrcl}
 & \sin\frac{x}{2} - 2\cos^2\frac{x}{2} + 1 & \geq & 0 \\
\Leftrightarrow & \sin\frac{x}{2} - 2\left(1 - \sin^2\frac{x}{2}\right) + 1 & \geq & 0 \\
\Leftrightarrow & 2\sin^2\frac{x}{2} + \sin\frac{x}{2} - 1 & \geq & 0 \\
\Leftrightarrow & \left(2\sin\frac{x}{2} - 1\right)\left(\sin\frac{x}{2} + 1\right) & \geq & 0 \\
\Leftrightarrow & \sin\frac{x}{2} \geq \frac{1}{2} \quad \text{or} \quad \sin\frac{x}{2} \leq -1 & & \qquad (1)
\end{array}$$

If you are not sure how we obtained (1), make use of the substitution $\sin\frac{x}{2} = k$, and sketch the parabola defined by $y = 2k^2 + k - 1$ to determine the values of k for which the parabola lies above or on the horizontal axis, or use a sign table.

From the $60° - 30°$ (or $\frac{\pi}{3} - \frac{\pi}{6}$) triangle we know that for $x \in \left[0, \frac{3\pi}{2}\right]$,

$$\sin\tfrac{x}{2} = \tfrac{1}{2} \quad \Leftrightarrow \quad \tfrac{x}{2} = \tfrac{\pi}{6} \text{ or } \tfrac{x}{2} = \tfrac{5\pi}{6}, \text{ and hence } x = \tfrac{\pi}{3} \text{ or } x = \tfrac{5\pi}{3};$$

$$\sin\tfrac{x}{2} = -1 \quad \Leftrightarrow \quad \tfrac{x}{2} = \tfrac{3\pi}{2}, \text{ and hence } x = 3\pi.$$

We thus have

$$\sin\tfrac{x}{2} \geq \tfrac{1}{2} \Leftrightarrow \tfrac{\pi}{3} \leq x \leq \tfrac{5\pi}{3}$$

and

$$\sin\tfrac{x}{2} \leq -1 \Leftrightarrow x = 3\pi.$$

(Remember that the range of the sine function is $[-1, 1]$, hence there are no values of $\frac{x}{2}$ such that $\sin\frac{x}{2} < -1$; when $\sin\frac{x}{2} = -1$ we have $\frac{x}{2} = \frac{3\pi}{2}$.)

The solution of (1) is then

$$\tfrac{\pi}{3} \leq x \leq \tfrac{5\pi}{3} \quad \text{or} \quad x = 3\pi.$$

Additional Exercises 7.5.1

In questions 1 – 5 solve the inequalities over the given intervals.

1. $2\cos 2x > 1, \quad 0 \leq x < 2\pi$
2. $\sin 2x > 2\cos x, \quad 0 < x \leq 2\pi$
3. $2\cos x < 1, \quad 0 \leq x < 2\pi$
4. $\tan x < \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$
5. $\sin x < 2\sin^2 x, \quad 0 \leq x < 2\pi$

7 REVIEW AND TEST

Now that you have reached the end of Chapter 7, work through the Chapter Review: Concept Check and Exercises, and the Chapter 7 Test. Since we have not studied all the sections in this chapter, you should omit items 7 to 9 in the Concept Check, questions 25–30, 45–48, 65–78 in the Exercises, and questions 7, 8, 11 in the Test. You may do as many of the odd numbered questions as you have time for.

ANSWERS

Remember that the answers to odd numbered questions are given at the back of SRW.

CHAPTER 1

Additional Exercises 1.1.1

1. (a) $|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1 \end{cases}$

   (b) $|3x-2| = \begin{cases} 3x-2 & \text{if } x \geq \frac{2}{3} \\ -3x+2 & \text{if } x < \frac{2}{3} \end{cases}$

   (c) $|-5x+4| = \begin{cases} -5x+4 & \text{if } x \leq \frac{4}{5} \\ 5x-4 & \text{if } x > \frac{4}{5} \end{cases}$

   (d) $|-4x-3| = \begin{cases} -4x-3 & \text{if } x \leq -\frac{3}{4} \\ 4x+3 & \text{if } x > -\frac{3}{4} \end{cases}$

   Alternatively

   $|-4x-3| = |4x+3| = \begin{cases} 4x+3 & \text{if } x \geq -\frac{3}{4} \\ -4x-3 & \text{if } x < -\frac{3}{4}. \end{cases}$

2. If we substitute a for $3x-2$ then

$$|3x-2| = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

Hence, if we substitute $3x-2$ back for a then

$$|3x-2| = \begin{cases} 3x-2 & \text{if } 3x-2 \geq 0 \\ -3x+2 & \text{if } 3x-2 < 0 \end{cases}$$

which is **NOT** the same as

$$|3x-2| = \begin{cases} 3x-2 & \text{if } x \geq 0 \\ -3x+2 & \text{if } x < 0. \end{cases}$$

Exercises 1.2

10. (a) $\frac{1}{5}$ (b) 1 000 (c) 27

12. (a) $\frac{27}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{100}$

14. (a) 8 (b) -4 (c) -2

16. (a) 14 (b) 4 (c) 6

26. $\sqrt[4]{3}$

34. $8s^9t^3$

40. $\dfrac{d^7}{c^6}$

44. $\dfrac{3c^7}{4a^3}$

48. xy^2

50. $a^2\sqrt[3]{b^2}$

64. $\dfrac{x^3}{y^{\frac{9}{5}}z^6}$

68. $\dfrac{y^{\frac{8}{3}}}{z^2}$

Exercises 1.4

12. $\dfrac{x-4}{x+2}$

22. x

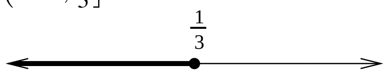
26. 1

36. $\dfrac{x^2+3x+12}{(x-4)(x+6)}$

44. $\dfrac{2(x+1)}{(x-2)(x+2)}$

Exercises 1.7

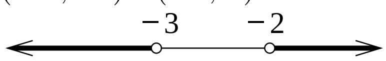
18. $\left(-\infty, \frac{1}{3}\right]$



20. $(-\infty, 11]$

24. $(-1, 4]$

34. $(-\infty, -3) \cup (-2, \infty)$



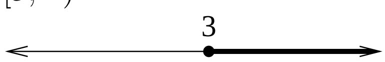
36. $(-1, 2)$

38. $(-\infty, -2] \cup [\frac{1}{2}, \infty)$

Additional Exercises 1.7.1

1. $\left(-\frac{2}{3}, 1\right]$

2. $[3, \infty)$



3. $x \leq -3$ or $x \geq -2$

4. $-1 < x < \frac{3}{2}$

5. $-1 < x < 2$ or $2 < x < 5$

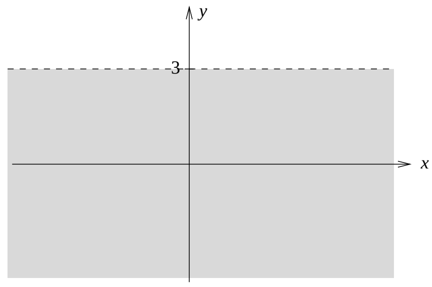
6. $x \geq \frac{5}{2}$ or $x \leq \frac{1}{2}$

7. $\frac{2}{5} < x < \frac{1}{2}$ or $\frac{1}{2} < x < \frac{2}{3}$

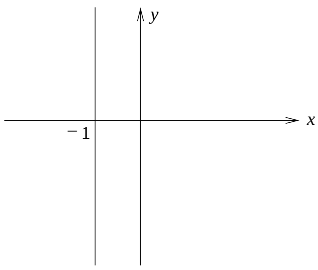
8. $0 < x \leq \frac{10}{3}$ or $x \geq 10$

Exercises 1.8

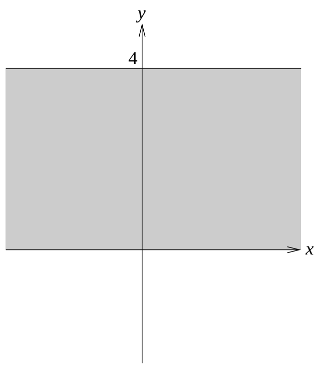
18.



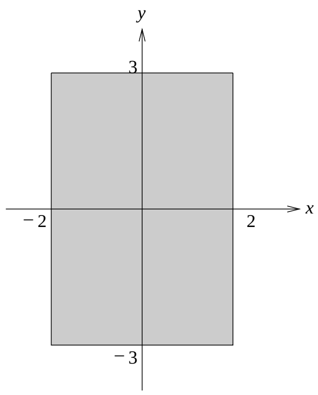
20.



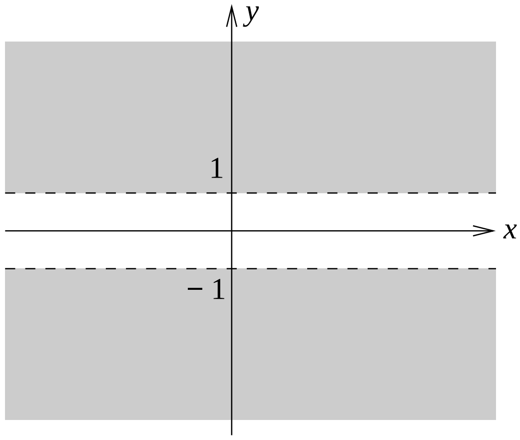
22.



26.



Additional example: Sketch the region given by the set $\{(x,y) \mid |y| > 1\}$.
Now $|y| > 1 \Leftrightarrow y > 1$ or $y < -1$.



82. $(x+1)^2 + (y+4)^2 = 64$

84. $(x-2)^2 + (y-5)^2 = 25$

88. $(x+1)^2 + (y-1)^2 = 10$

90. $x^2 + y^2 + 6y + 2 = 0 \Leftrightarrow x^2 + (y+3)^2 = 7$
Centre $(0,-3)$; radius $= \sqrt{7}$

92. $x^2 + y^2 + \frac{1}{2}x + 2y + \frac{1}{16} = 0 \Leftrightarrow \left(x + \frac{1}{4}\right)^2 + (y+1)^2 = 1$
Centre $\left(-\frac{1}{4}, -1\right)$; radius $= 1$

Exercises 1.10

8. $\frac{4}{7}$

12. $2x - y + 4 = 0$

18. $7x + 2y + 31 = 0$

20. $x - y - 1 = 0$

22. $2x - 5y + 20 = 0$

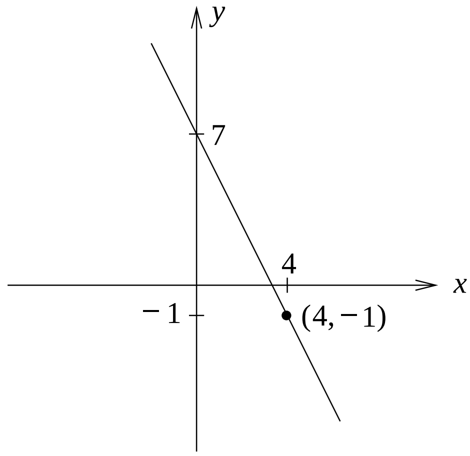
24. $3x - 4y + 24 = 0$

26. $x = 4$

28. $2x + 3y - 18 = 0$

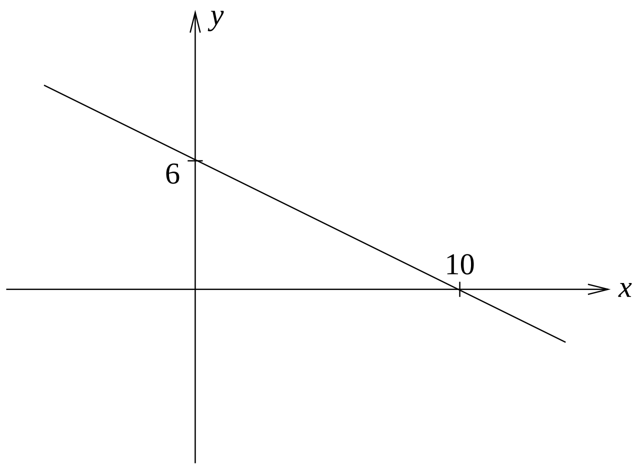
32. $6x + 3y - 1 = 0$

36. (a)

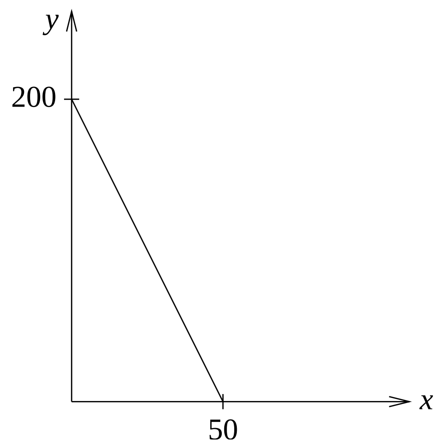


(b) $2x + y - 7 = 0$

46. Slope $= -\frac{3}{5}$; y–intercept $= 6$



64. (a)



(b) The slope of the line is -4 and it represents the decrease in the number of spaces rented for every dollar increase in rental.

The y–intercept is the number of spaces at the flea market, 200.

The x–intercept is the cost per space when the manager rents no spaces.

CHAPTER 2

Exercises 2.1

4. $f(x) = \frac{1}{3}(\sqrt{x}+8)$

6. Divide by 3, then subtract 4.

16. $h(1) = 2, \quad h(-1) = -2, \quad h(2) = 2\frac{1}{2}, \quad h\left(\frac{1}{2}\right) = 2\frac{1}{2},$
$h(x) = x + \frac{1}{x}, \quad h\left(\frac{1}{x}\right) = x + \frac{1}{x}$

30. $f(a) = a^2+1, \quad f(a+h) = a^2+2ah+h^2+1,$
$\frac{f(a+h)-f(a)}{h} = 2a+h$

40. $D_f = [0,5] \quad R_f = [1,26]$

42. $D_f = (-\infty,2)\cup(2,\infty) \quad R_f = (-\infty,0)\cup(0,\infty)$

44. $D_f = \{x \in \mathbb{R} \mid x \neq -3 \text{ and } x \neq 2\} = (-\infty,-3)\cup(-3,2)\cup(2,\infty)$

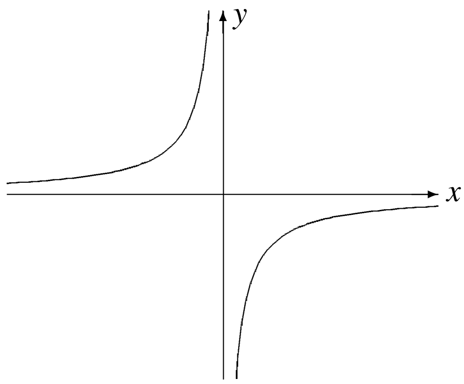
46. $D_f = \{x \in \mathbb{R} \mid x \geq -9\} = [-9,\infty) \quad R_f = [0,\infty)$

48. $D_g = \left\{x \in \mathbb{R} \mid x \leq \frac{7}{3}\right\} = \left(-\infty, \frac{7}{3}\right] \quad R_g = [0,\infty)$

50. $D_G = (-\infty,-3]\cup[3,\infty) \quad R_G = [0,\infty)$

Additional Exercises 2.4.1

1. $f(-x) = -\frac{1}{-x} = \frac{1}{x} = -f(x)$
Thus f is odd.



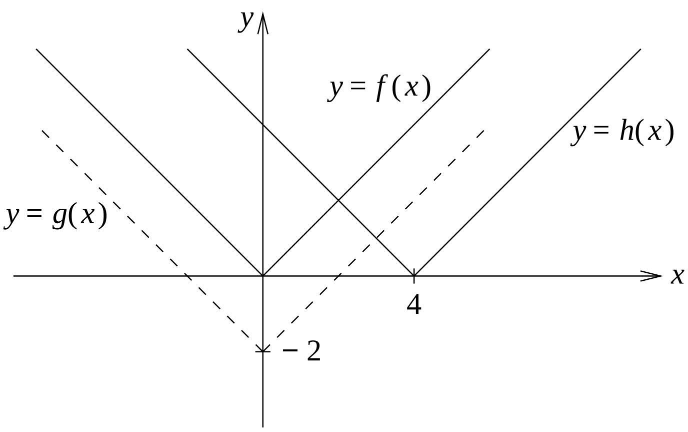
2. (a) The graph of g is obtained by

- stretching the graph of f horizontally by a factor of 4 to obtain the graph of $y = k(x) = f\left(\frac{1}{4}x\right)$
- shifting the resulting graph one unit to the left to obtain the graph of $y = k(x+1) = f\left(\frac{1}{4}(x+1)\right)$.

(b) The graph of h is obtained by

- shifting the graph of f one unit to the left to obtain the graph of $y = j(x) = f(x+1)$
- stretching the resulting graph horizontally by a factor of 4 to obtain the graph of $y = h(x) = j\left(\frac{1}{4}x\right) = f\left(\frac{1}{4}x+1\right)$.

3.



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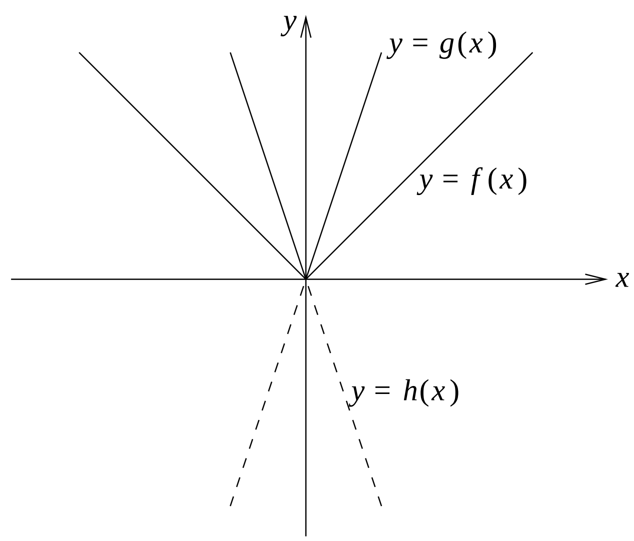

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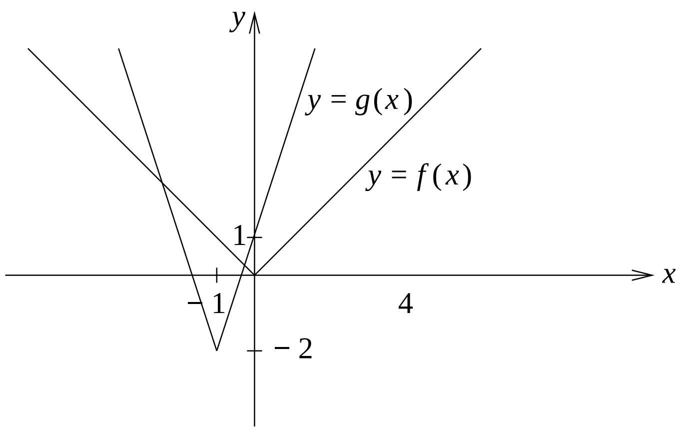

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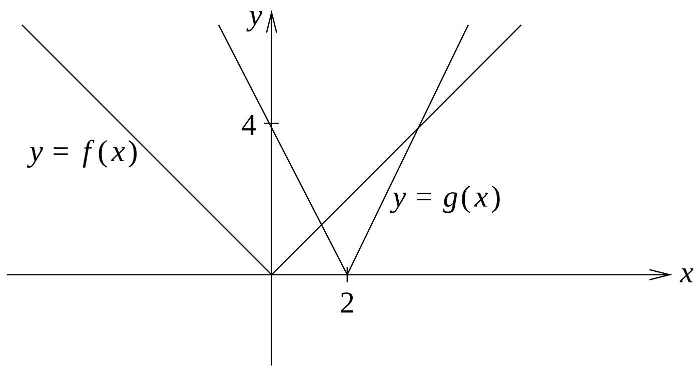

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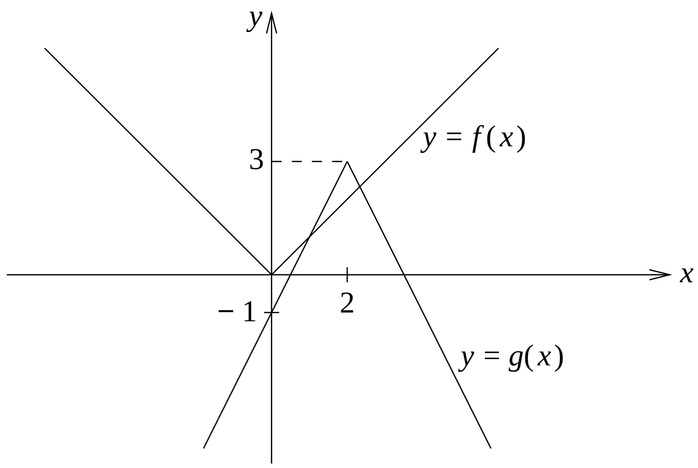


8. $y = g(x) = 3|x+1| - 2$

Shift the graph of $y = f(x)$ one unit the left to obtain the graph of $y = |x+1|$. Stretch this graph vertically by a factor of three to obtain the graph of $y = 3|x+1|$ and then shift the graph two units downwards to obtain the graph of $y = g(x) = 3|x+1| - 2$.

9. $y = g(x) = -2|x-3| + 1$

   Shift the graph of $y = f(x)$ three units to the right to obtain the graph of $y = |x-3|$. Stretch this graph vertically by a factor of two to obtain the graph of $y = 2|x-3|$. Reflect this graph in the x–axis to obtain the graph of $y = -2|x-3|$. Shift this graph one unit upwards to obtain the graph of $y = g(x) = -2|x-3| + 1$.

10. $y = g(x) = |1-x| + 2 = |x-1| + 2$

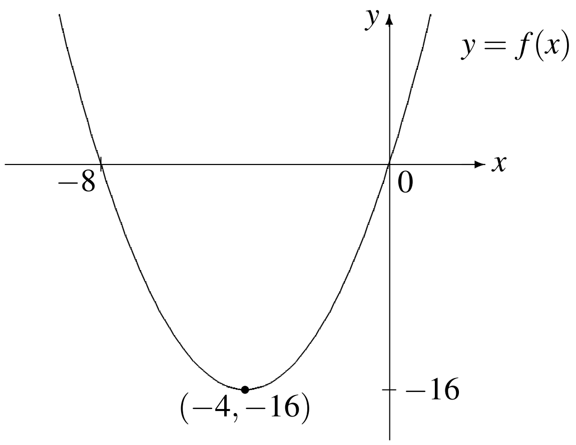
    Shift the graph of $y = f(x)$ one unit to the right to obtain the graph of $y = |x-1|$. Shift this graph two units upwards to obtain the graph of $y = g(x) = |x-1| + 2$.

Exercises 2.5

6. (a) $f(x) = (x+4)^2 - 16$

   (b) Vertex $= (-4,-16)$; x–intercepts $x = -8$ and $x = 0$; y–intercept $y = 0$

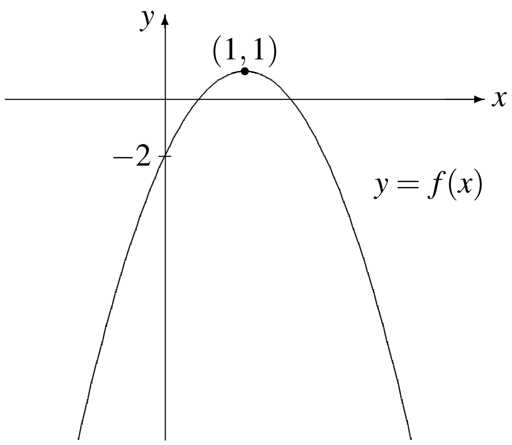
   (c)



14. (a) $f(x) = -3(x-1)^2 + 1$

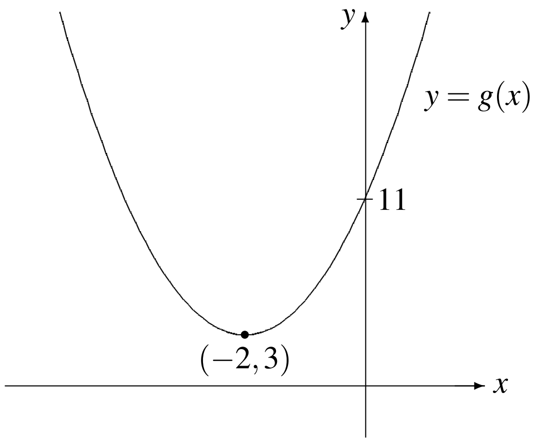
    (b) Vertex $= (1,1)$; x–intercepts $x = 1 - \sqrt{\frac{1}{3}}$ and $x = 1 + \sqrt{\frac{1}{3}}$; y–intercept $y = -2$

(c)



26. (a) $g(x) = 2(x+2)^2 + 3$

(b)



(c) Minimum value is 3.

30. Maximum value is $f\left(\frac{3}{2}\right) = \frac{13}{4}$.

32. Minimum value is $f(-2) = 73$.

40. $y = -3(x-3)^2 + 4 = -3x^2 + 18x - 23$

42. $D_f = \mathbb{R}$; $R_f = [-4, \infty)$

Exercises 2.7

2. $(f+g)(x)=4x^2+2x-1 \quad D_{f+g}=\mathbb{R}$
$(f-g)(x)=-2x^2+2x+1 \quad D_{f-g}=\mathbb{R}$
$(fg)(x)=3x^4+6x^3-x^2-2x \quad D_{fg}=\mathbb{R}$
$\left(\frac{f}{g}\right)(x)=\frac{x^2+2x}{3x^2-1} \quad D_{\frac{f}{g}}=\mathbb{R}-\left\{-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\}$

4. $(f+g)(x)=\sqrt{9-x^2}+\sqrt{x^2-4} \quad D_{f+g}=[-3,-2]\cup[2,3]$
$(f-g)(x)=\sqrt{9-x^2}-\sqrt{x^2-4} \quad D_{f-g}=[-3,-2]\cup[2,3]$
$(fg)(x)=\sqrt{-x^4+13x^2-36} \quad D_{fg}=[-3,-2]\cup[2,3]$
$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-4}} \quad D_{\frac{f}{g}}=[-3,-2)\cup(2,3]$

6. $(f+g)(x)=\frac{x+2}{x+1} \quad D_{f+g}=\mathbb{R}-\{-1\}$

$(f-g)(x)=\frac{2-x}{x+1} \quad D_{f-g}=\mathbb{R}-\{-1\}$

$(fg)(x)=\frac{2x}{(x+1)^2} \quad D_{fg}=\mathbb{R}-\{-1\}$

$\left(\frac{f}{g}\right)(x)=\frac{2}{x} \quad D_{\frac{f}{g}}=\mathbb{R}-\{-1,0\}$

Additional Exercises 2.8.1

1. Suppose f has an inverse function other than f^{-1}, which we call g. Then

$$\begin{aligned} & \left(f^{-1}\circ(f\circ g)\right)(x)=\left(\left(f^{-1}\circ f\right)\circ g\right)(x) \\ \Leftrightarrow\quad & f^{-1}\left((f\circ g)(x)\right)=\left(f^{-1}\circ f\right)(g(x)) \\ \Leftrightarrow\quad & f^{-1}(x)=g(x). \end{aligned}$$

Since f^{-1} and g are both inverses of f.

Hence $f^{-1}=g$.

2. **Option 1:**
$D_{g_r}=\{x\in\mathbb{R}\mid x\geq -3\} \quad R_{g_r}=\{y\in\mathbb{R}\mid y\leq 1\}$
$g_r^{-1}(x)=-3+\sqrt{3(1-x)}$
$D_{g_r^{-1}}=\{x\in\mathbb{R}\mid x\leq 1\}=R_{g_r}$
$R_{g_r^{-1}}=\{y\in\mathbb{R}\mid y\geq -3\}=D_{g_r}$
Check that $\left(g_r^{-1}\circ g_r\right)(x)=x$ for every $x\in D_{g_r}$
and $\left(g_r\circ g_r^{-1}\right)(x)=x$ for every $x\in D_{g_r^{-1}}$.

Option 2:

$D_{g_r} = \{x \in \mathbb{R} \mid x \leq -3\} \quad R_{g_r} = \{y \in \mathbb{R} \mid y \leq 1\}$

$g_r^{-1}(x) = -3 - \sqrt{3(1-x)}$

$D_{g_r^{-1}} = \{x \in \mathbb{R} \mid x \leq 1\} = R_{g_r}$

$R_{g_r^{-1}} = \{y \in \mathbb{R} \mid y \leq -3\} = D_{g_r}$

Check that $\left(g_r^{-1} \circ g_r\right)(x) = x$ for every $x \in D_{g_r}$

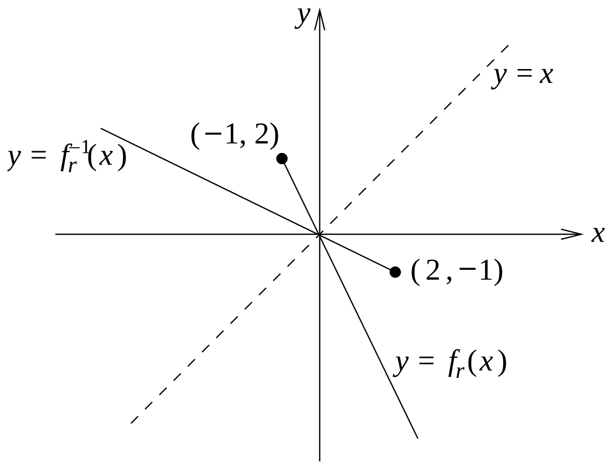
and $\left(g_r \circ g_r^{-1}\right)(x) = x$ for every $x \in D_{g_r^{-1}}$.

3. $D_{f_r} = \{x \in \mathbb{R} \mid x \geq -1\} \quad R_{f_r} = \{y \in \mathbb{R} \mid y \leq 2\}$

   $f_r^{-1}(x) = -\frac{1}{2}x$

   $D_{f_r^{-1}} = \{x \in \mathbb{R} \mid x \leq 2\} = R_{f_r}$

   $R_{f_r^{-1}} = \{y \in \mathbb{R} \mid y \geq -1\} = D_{f_r}$



Additional Exercises 2.9.1

1. (a) $a = -1, \ p = -2, \ q = 3, \ m = 1, \ c = 4$

   $y = f(x) = -|x+2| + 3$

   $y = g(x) = x + 4$

   (b) $A = (-5, 0); \ B = (1, 0); \ P = \left(-\frac{3}{2}, \frac{5}{2}\right)$

   (c) The lines do not intersect since they are parallel and not coincident, i.e. they both have the same slope (namely 1) but different x–intercepts.

   (d) $|x+2| > 3 \Leftrightarrow f(x) < 0 \Leftrightarrow x < -5$ or $x > 1$

2. (a) $A=(-1,-2)$ $B=(-5,0)$ $C=(3,0)$ $D=\left(0,-\frac{3}{2}\right)$
$E=(2,3)$ $F=(-1,0)$ $G=(5,0)$ $H=(0,1)$
$I=\left(-\frac{7}{3},-\frac{4}{3}\right)$ $J=\left(\frac{13}{3},\frac{2}{3}\right)$
(b) (i) $-1<x<5$ (ii) $x<-5$ or $x>3$
(c) $d(x)=-|x-2|-\frac{1}{2}|x+1|+5$ $d(1)=3$

3. (a) $y=f(x)=-\frac{1}{2}x^2+\frac{3}{2}x+2$
$y=g(x)=\frac{5}{2}|x+2|-3$
(b) $\frac{3}{8}$
(c) $x\leq -2$ or $x\geq 0$
(d) $D_{g_r}=\{x\in\mathbb{R}\mid x\leq -2\}$
$g_r^{-1}(x)=-\frac{2}{5}(x+8)$

4. (a) $c=-\frac{2}{3}$; $p=4$; $q=2$
$y=g(x)=-\frac{2}{3}|x-4|+2$
(b) 7
(c) $y=k(x)=\frac{1}{3}x^2-\frac{2}{3}x-\frac{2}{3}$
(d) $\frac{4}{3}$ units
(e) (i) $D_{k_r}=\{x\in\mathbb{R}\mid x\geq 1\}$ or $D_{k_r}=\{x\in\mathbb{R}\mid x\leq 1\}$
(ii) If $D_{k_r}=\{x\in\mathbb{R}\mid x\geq 1\}$ then $k_r^{-1}(x)=1+\sqrt{3x+3}$.
If $D_{k_r}=\{x\in\mathbb{R}\mid x\leq 1\}$ then $k_r^{-1}(x)=1-\sqrt{3x+3}$.

CHAPTER 4

Exercises 4.1

16. $\dfrac{1}{5}=a^{-1}\Rightarrow a=5$
Thus, $f(x)=5^x$.

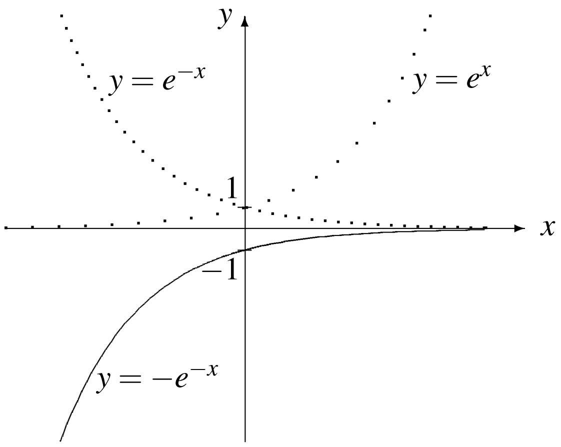
18. $8=a^{-3}\Rightarrow a=\dfrac{1}{2}$
Thus, $f(x)=\left(\frac{1}{2}\right)^x$.

20. $f(x)=-5^x$ matches V

22. $f(x)=5^x+3$ matches VI

24. $f(x)=5^{x+1}-4$ matches IV

36. $y=-e^{-x}$
Consider the graph of $y=e^x$.
Reflect it in the y–axis to give the graph of $y=e^{-x}$. Reflect this graph in the x–axis to give the graph of $y=-e^{-x}$.



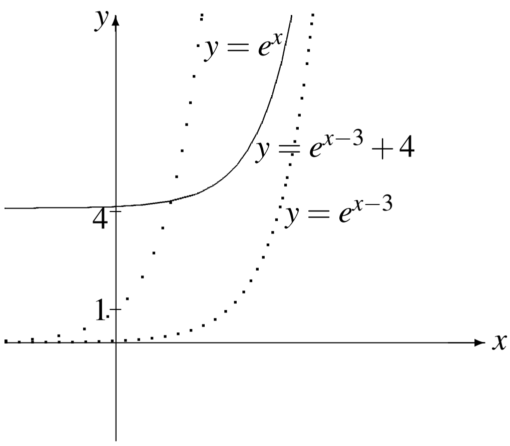
For $y = -e^{-x}$: domain $= \mathbb{R}$; range $= \{y \in \mathbb{R} \mid y < 0\}$; horizontal asymptote is the x–axis.

38. We have $y = e^{x-3} + 4$.

Consider the graph of $y = e^x$.

Shift it horizontally, three units to the right, to give the graph of $y = e^{x-3}$.

Shift this graph vertically upwards, by four units, to give the graph of $y = e^{x-3} + 4$.



For $y = e^{x-3} + 4$: domain $= \mathbb{R}$; range $= \{y \in \mathbb{R} \mid y > 4\}$; horizontal asymptote is the line defined by $y = 4$.

80. Suppose we begin with an amount of money, P.

(i) If interest is calculated semi–annually, it is calculated twice each year. In the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

we thus have

$$\begin{aligned} A(t) &= P\left(1+\frac{9\frac{1}{4}}{100\times 2}\right)^{2t} \\ &= P\left(1+\frac{37}{4\times 2\times 100}\right)^{2t} \\ &= P\left(1+\frac{37}{800}\right)^{2t} \\ &= P(1+0,04625)^{2t} \\ &= P(1,04625)^{2t} \\ &\approx P(1,0946)^{t}. \end{aligned}$$

(ii) If interest is calculated continuously we have

$$\begin{aligned} A(t) &= Pe^{rt} \\ &= Pe^{0.09t} \\ &\approx P(1,0942)^{t}. \end{aligned}$$

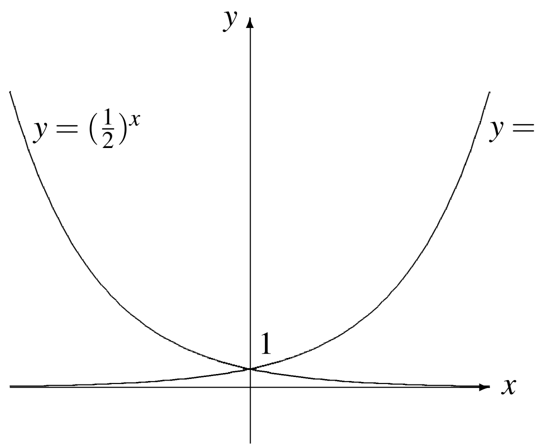
The exponential function is an increasing function, i.e.

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1).$$

Since $1,0946 > 1,0942$ it follows that $(1.0946)^t > (1.0942)^t$ and thus the investment in the first case (interest compounded semi–annually, at $9\frac{1}{4}\%$ per year) is a slightly better investment than the second case (interest compounded continuously, at 9% per year).

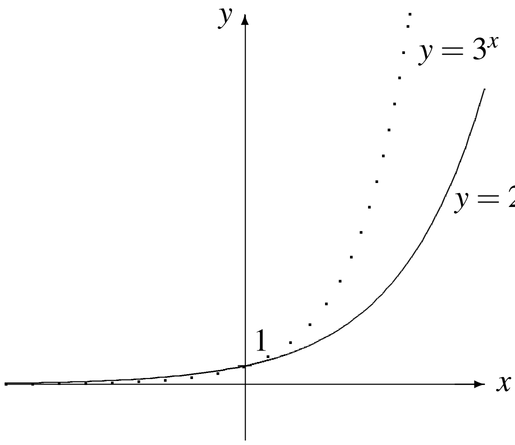
Additional Exercises 4.1.1

1. (a)

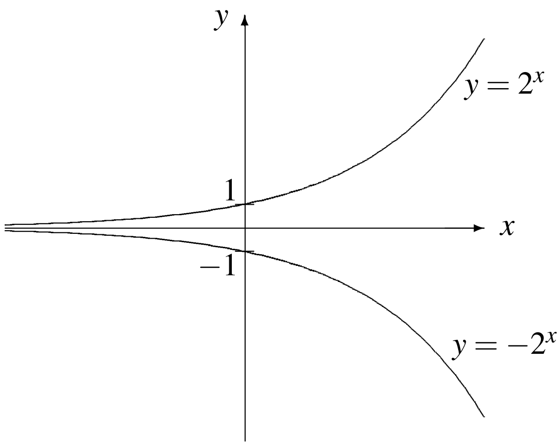


Note that the graph of $\left(\frac{1}{2}\right)^x = 2^{-x}$, i.e. the graph of $y = 2^{-x}$, is a reflection in the y–axis of the graph of $y = 2^x$.

(b)

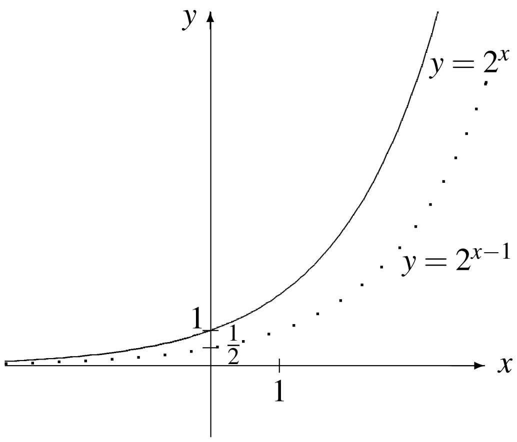


(c)



The graph of $y = -2^x$ is a reflection in the x–axis of the graph of $y = 2^x$.

(d)



The graph of $y = 2^{x-1}$ is obtained by horizontally shifting the graph of $y = 2^x$, one unit to the right.

2. (a) We have $g(x) = 3^{-x}$. Let us define the function f by $f(x) = g(-x)$. Then

$$f(x) = 3^{-(-x)} = 3^x$$

and the graph of f is obtained by reflecting the graph of g in the y–axis. Now let k be the function defined by $k(x) = -f(x)$. Then

$$k(x) = -3^x$$

and the graph of k is obtained by reflecting the graph of f in the x–axis.

(b) We have $g(x) = 3^{-x}$.
Let m be the function defined by $m(x) = g(-x)$. Then

$$m(x) = g(-x) = 3^{-(-x)} = 3^x$$

i.e. we obtain the graph of m by reflecting the graph of g in the y–axis.
Let p be the function defined by $p(x) = m(2x)$. Then

$$p(x) = m(2x) = 3^{2x}$$

i.e. we obtain the graph of p by horizontally compressing the graph of m by a factor of $\frac{1}{2}$.
Let f be the function defined by $f(x) = p\left(x - \frac{1}{2}\right)$. Then

$$f(x) = p\left(x - \tfrac{1}{2}\right) = 3^{2\left(x-\frac{1}{2}\right)} = 3^{2x-1}$$

i.e. we obtain the graph of f by shifting the graph of p horizontally, half a unit to the right.

(c) We have $g(x) = 3^{-x}$.
Let k be the function defined by $k(x) = g(x-2)$. Then

$$k(x) = g(x-2) = 3^{-(x-2)}$$

i.e. we obtain the graph of k by shifting the graph of g horizontally, two units to the right.
Let ℓ be the function defined by $\ell(x) = k(2x)$. Then

$$\ell(x) = k(2x) = 3^{-(2x-2)} = 3^{-2(x-1)}$$

i.e. we obtain the graph of ℓ by horizontally compressing (shrinking) the graph of k by a factor of $\frac{1}{2}$.

3. (a) We begin with $y = e^x$.
Shift two units to the right: $y = e^{x-2}$.
Stretch the graph vertically by a factor of 3: $y = 3e^{x-2}$.
Shift the graph vertically downwards by one unit: $y = 3e^{x-2} - 1$
Thus the equation of the new graph is

$$y = 3e^{x-2} - 1.$$

(b) Its horizontal asymptote is the line defined by $y = -1$, because the horizontal asymptote of the natural exponential graph is the x–axis, defined by $y = 0$.

(c) domain $= \mathbb{R}$
range $= \{y \in \mathbb{R} \mid y > -1\}$

4. $y = f(x) = e^{\frac{x}{3}} - 3$

Shift upwards, 3 units: $y = k(x) = f(x) + 3 = e^{\frac{x}{3}} - 3 + 3 = e^{\frac{x}{3}}$.

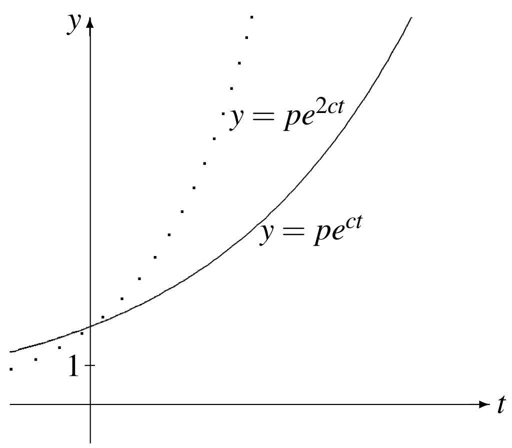
Shrink horizontally, by a factor of 3: $y = k(3x) = e^{\frac{3x}{3}} = e^{x}$.

Reflect in the y–axis: $y = g(x) = k(-x) = e^{-x}$.

Thus the equation of the graph is $y = g(x) = e^{-x}$.

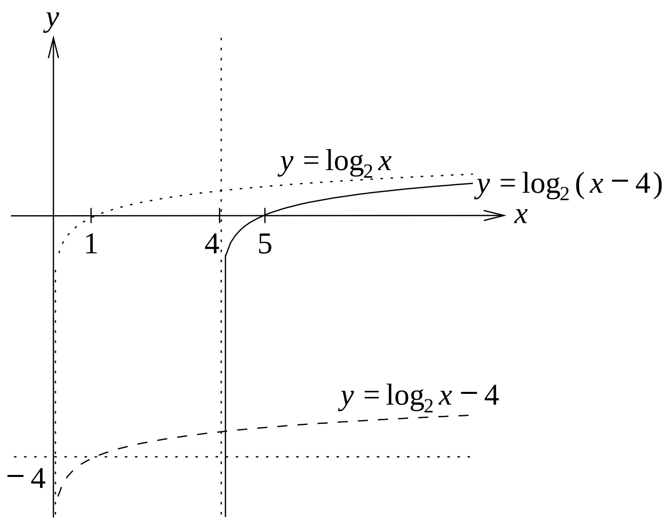
5. (a) Growth rate $= 2c$
Thus $y = \ell(t) = pe^{2ct}$.

(b) $\ell(t) = pe^{2ct} = pe^{c(2t)} = k(2t)$
The graphs of k and ℓ are shown in the following sketch.

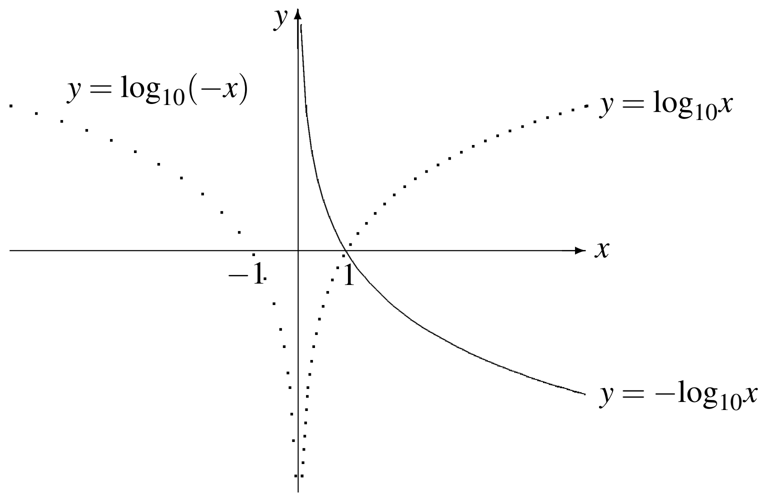


Additional Exercises 4.2.1

1. (a), (b)

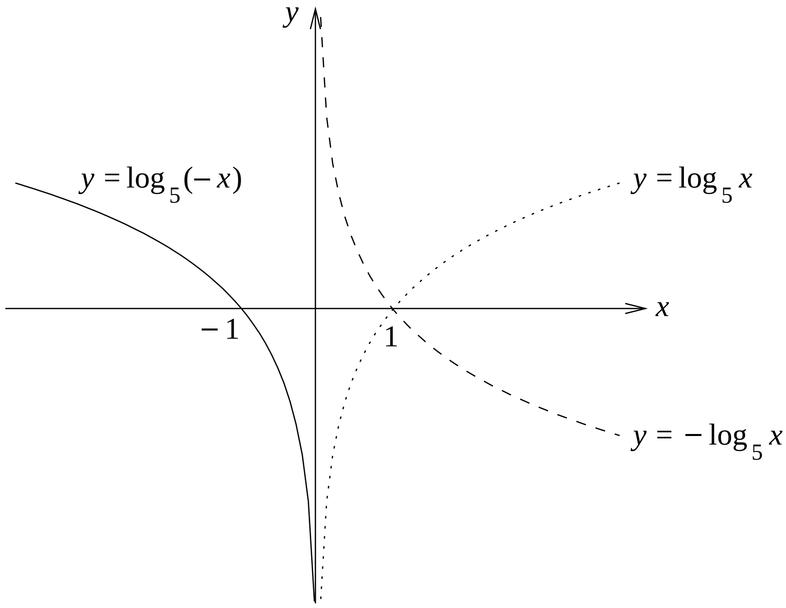


2. (a), (b)

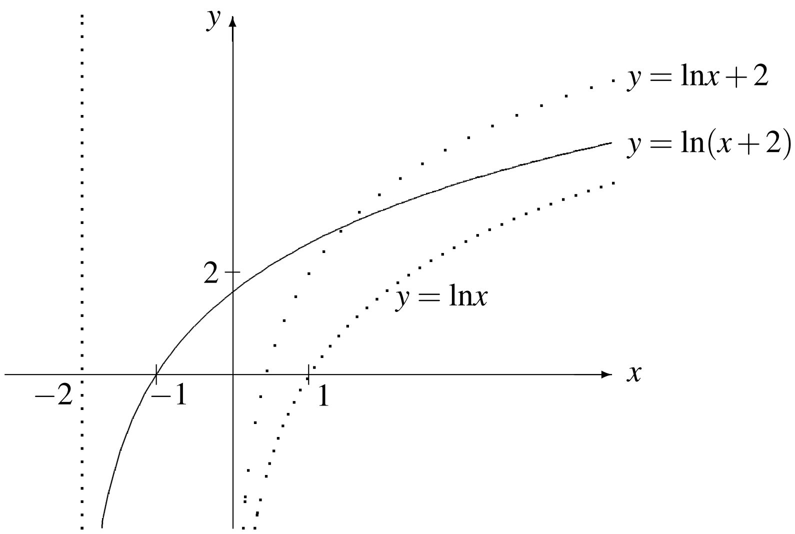


For $g(x) = -\log_{10} x$, $D_f = (0, \infty)$, $R_f = \mathbb{R}$ and the asymptote is the vertical line $x = 0$.

3. (a), (b)

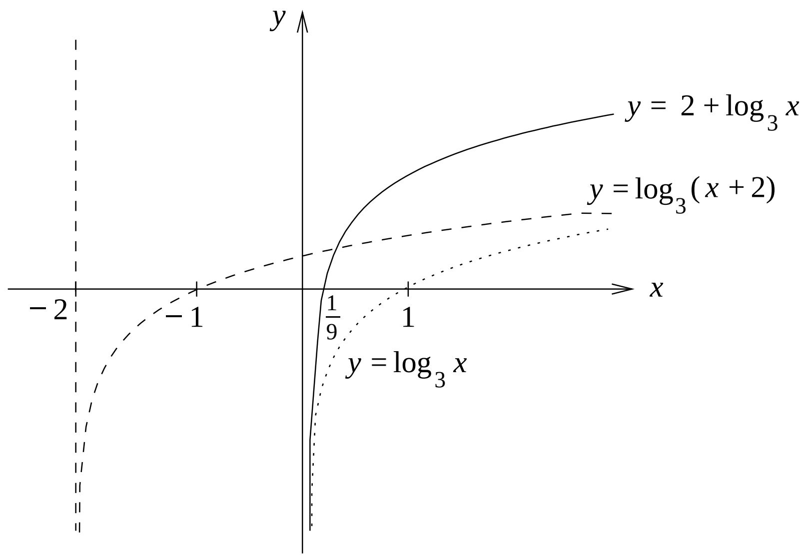


4. (a), (b)

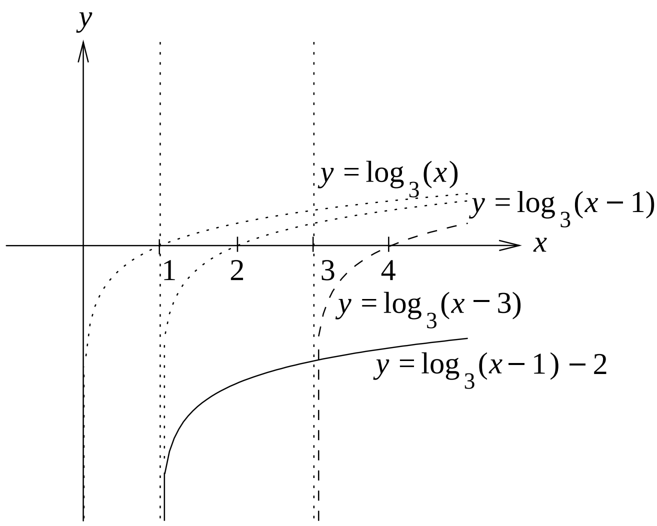


For $f(x) = \ln(x+2)$, $D_f = (-2, \infty)$, $R_f = \mathbb{R}$ and the asymptote is the vertical line $x = -2$.

5. (a), (b)

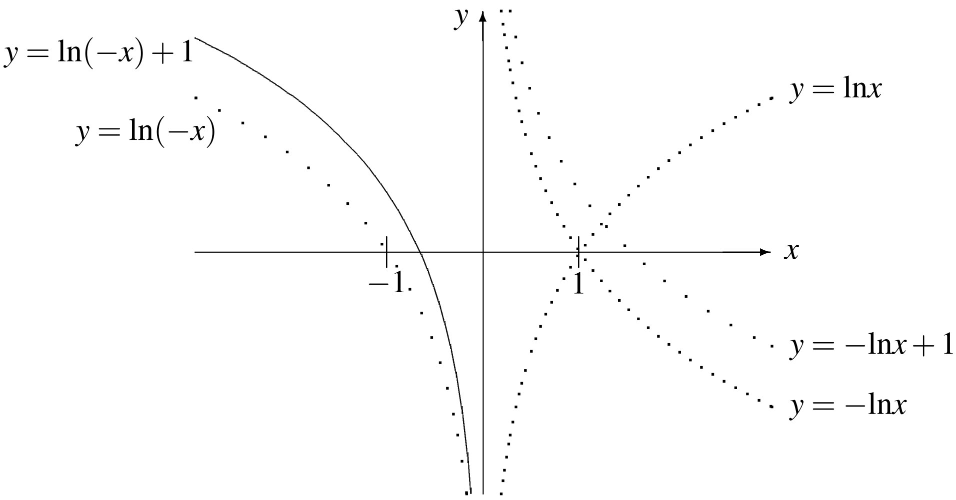


6. (a), (b)



For the function defined by $y = \log_3(x-1) - 2$, the domain is $(1, \infty)$, the range is $\mathbb{R}$, and the asymptote is the vertical line $x = 1$.

7. (a), (b) For (a): Domain $= (-\infty, 0)$; Range $= \mathbb{R}$; Asymptote, $x = 0$



8. Property 2

   To prove: $\ln e = 1$.

   Proof:

   Suppose $\ln e = x$.

   Then

   $e = e^x$ Changing from logarithmic to exponential form.

   and thus

   $x = 1$. One–to–one property.

   Hence $\ln e = 1$.

   Property 3

   To prove: $\ln e^x = x$.

   Proof:

   Suppose $\ln e^x = a$.

   Then

   $e^x = e^a$ Changing from logarithmic to exponential form.

   i.e.

   $x = a$. One–to–one property.

   Thus $\ln e^x = x$.

   Property 4

   To prove: $e^{\ln x} = x$.

   Proof:

   Suppose $e^{\ln x} = b$.

Then

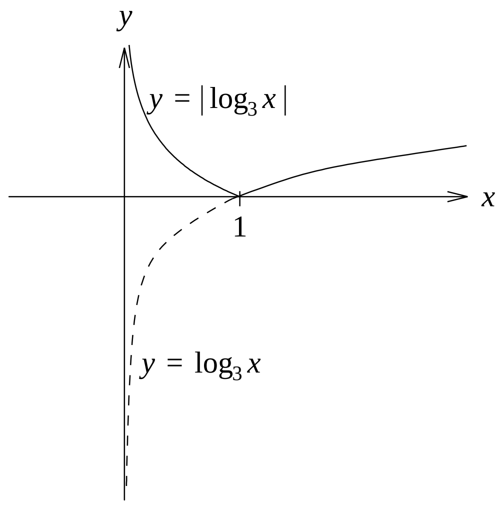
$$\ln b = \ln x \qquad \text{Changing from exponential to logarithmic form.}$$

and thus

$$b = x. \qquad \text{One–to–one property.}$$

Hence $e^{\ln x} = x$.

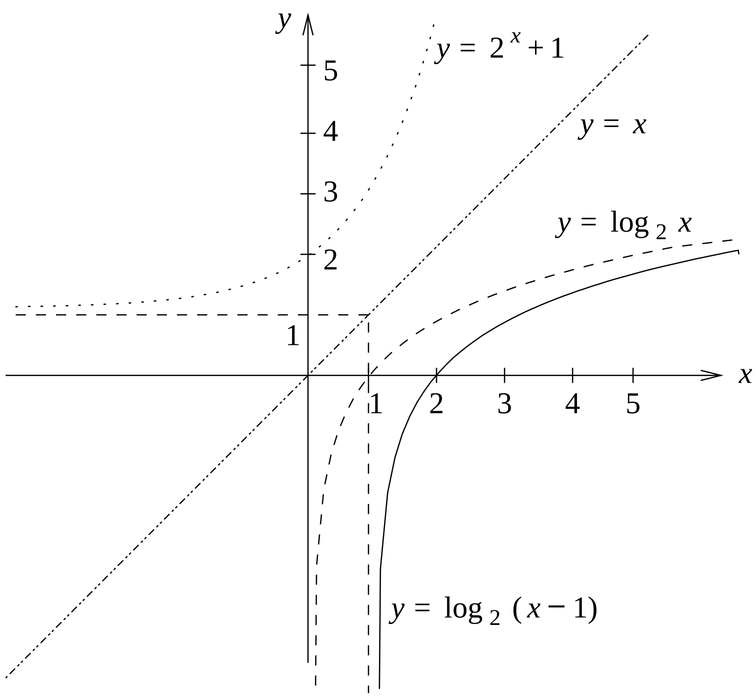
9.



Note that $y = |\log_3 x| \Rightarrow y = \begin{cases} \log_3 x & \text{if} \quad \log_3 x \geq 0 \\ -\log_3 x & \text{if} \quad \log_3 x < 0. \end{cases}$

Thus, when $\log_3 x \geq 0$ we leave the graph of $y = \log_3 x$ as it is; when $\log_3 x < 0$ we reflect that branch of the graph of $y = \log_3 x$ in the x–axis.

10.



We have the function f defined by $y = \log_2 (x-1)$.

Thus

$$x-1 = 2^y$$

i.e.

$$x = 2^y+1$$

i.e.

$$f^{-1}(y) = 2^y+1$$

and hence

$$f^{-1}(x) = 2^x+1.$$

Now $\left(f^{-1}\circ f\right)(x) = f^{-1}(f(x))$, which is defined for all $x \in D_f$ such that $f(x) \in D_{f^{-1}}$.

The function f^{-1} is an exponential function, hence $D_{f^{-1}} = \mathbb{R}$; the function f is a logarithmic function, which operates on $x-1$, hence

$$D_f = \{x \in \mathbb{R} \mid x-1>0\} = \{x \in \mathbb{R} \mid x>1\}.$$

Thus, for $x>1$,

$$\begin{aligned}\left(f^{-1}\circ f\right)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(\log_2(x-1)) \\ &= 2^{\log_2(x-1)}+1 \\ &= x-1+1 \\ &= x.\end{aligned}$$

11. (a) Since the vertical asymptote of the graph of f is the line defined by $x=1$, it follows that the graph of f is obtained from a logarithmic graph which has been shifted horizontally, to the right, by one unit. Thus $a=1$.

(b) Logarithmic graphs of the form $y=\log_a x$ cut the x –axis at $x=1$. The graph of f has the form $y=\log_b(x-1)$; the shift to the right implies that it cuts the x–axis at $x=2$. Hence $A=(2,0)$.

(c) The point $(5,2)$ lies on the graph of f.

Thus

$$2 = \log_b(5-a).$$

Now $a=1$ and thus	$2 = \log_b(5-1)$
i.e.	$2 = \log_b 4$
i.e.	$b^2 = 4 = 2^2$
i.e.	$b = 2.$

The equation of f is thus $y=\log_2(x-1)$.

The point $(5,2)$ also lies on the graph of g, and thus

$$\begin{aligned} 2 &= \frac{1}{5+k} \\ \text{i.e.} \quad 2(5+k) &= 1 \\ \text{i.e.} \quad 10+2k &= 1 \\ \text{i.e.} \quad 2k &= -9 \\ \text{i.e.} \quad k &= -4\frac{1}{2}. \end{aligned}$$

(d) $h(x) = f(-x) = \log_2(-x-1)$.

12. (a) $A = (1,0)$

(b) $q = \frac{1}{3}$; $a = \frac{5}{12}$; $b = -4\frac{5}{12}$; $c = 4$

(c) $y = k(x) = \left(\frac{1}{3}\right)^x$

Exercises 4.3

Using the Laws of Logarithms to Evaluate Expressions

$\log_2 112 - \log_2 7 = \log_2 \frac{112}{7} = \log_2 16 = \log_2 2^4 = 4$

$\log\sqrt{0.1} = \log 0.1^{\frac{1}{2}} = \frac{1}{2}\log 0.1 = \frac{1}{2}\log 10^{-1} = -\frac{1}{2}$

$\log_{12} 9 + \log_{12} 16 = \log_{12}(9\cdot 16) = \log_{12} 144 = \log_{12} 12^2 = 2$

$e^{3\ln 5} = \left(e^{\ln 5}\right)^3 = 5^3 = 125$

$\log_2 8^{33} = \log_2\left(2^3\right)^{33} = \log_2 2^{99} = 99$

16. $\log_5\left(\dfrac{x}{2}\right) = \log_5 x - \log_5 2$

18. $\ln\sqrt{z} = \dfrac{1}{2}\ln z$

20. $\log_6 \sqrt[4]{17} = \frac{1}{4}\log_6 17$

22. $\log_2(xy)^{10} = 10\log_2(xy) = 10(\log_2 x + \log_2 y)$

24. $\log_a\left(\dfrac{x^2}{yz^3}\right) = \log_a x^2 - \log_a\left(yz^3\right) = 2\log_a x - (\log_a y + 3\log_a z)$

26. $\ln\sqrt[3]{3r^2s} = \frac{1}{3}\ln\left(3r^2s\right) = \frac{1}{3}\left[\ln(3s) + 2\ln r\right] = \frac{1}{3}(\ln 3 + \ln s + 2\ln r)$

28. $\log\dfrac{a^2}{b^4\sqrt{c}} = \log a^2 - \log\left(b^4\sqrt{c}\right) = 2\log a - \left(4\log b + \frac{1}{2}\log c\right)$

30. $\log_5 \sqrt{\dfrac{x-1}{x+1}} = \frac{1}{2}\log_5\left(\dfrac{x-1}{x+1}\right) = \frac{1}{2}\left[\log_5(x-1) - \log_5(x+1)\right]$

32. $\ln \dfrac{3x^2}{(x+1)^{10}} = \ln\left(3x^2\right) - \ln(x+1)^{10} = \ln 3 + 2\ln x - 10\ln(x+1)$

34. $\log \dfrac{x}{\sqrt[3]{1-x}} = \log x - \log \sqrt[3]{1-x} = \log x - \frac{1}{3}\log(1-x)$

36.

$$\begin{aligned}\log\sqrt{x\sqrt{y\sqrt{z}}} &= \frac{1}{2}\log\left(x\sqrt{y\sqrt{z}}\right)\\ &= \frac{1}{2}\left(\log x + \log\sqrt{y\sqrt{z}}\right)\\ &= \frac{1}{2}\left[\log x + \frac{1}{2}\log\left(y\sqrt{z}\right)\right]\\ &= \frac{1}{2}\left[\log x + \frac{1}{2}\left(\log y + \frac{1}{2}\log z\right)\right]\\ &= \frac{1}{2}\log x + \frac{1}{4}\log y + \frac{1}{8}\log z\end{aligned}$$

38.

$$\begin{aligned}\log\frac{10^x}{x(x^2+1)(x^4+2)} &= \log 10^x - \log\left[x\left(x^2+1\right)\left(x^4+2\right)\right]\\ &= x - \left[\log x + \log\left(x^2+1\right) + \log\left(x^4+2\right)\right]\end{aligned}$$

40. $\log 12 + \frac{1}{2}\log 7 - \log 2 = \log\left(12\sqrt{7}\right) - \log 2 = \log\frac{12\sqrt{7}}{2} = \log\left(6\sqrt{7}\right)$

42. $\log_5\left(x^2-1\right) - \log_5(x-1) = \log_5\dfrac{x^2-1}{x-1} = \log_5(x+1)$

44. $\ln(a+b) + \ln(a-b) - 2\ln c = \ln\left[(a+b)(a-b)\right] - \ln\left(c^2\right) = \ln\dfrac{a^2-b^2}{c^2}$

46. $2\left[\log_5 x + 2\log_5 y - 3\log_5 z\right] = 2\log_5\dfrac{xy^2}{z^3} = \log_5\left(\dfrac{xy^2}{z^3}\right)^2 = \log_5\dfrac{x^2y^4}{z^6}$

48. $\log_a b + c\log_a d - r\log_a s = \log_a\left(bd^c\right) - \log_a s^r = \log_a\dfrac{bd^c}{s^r}$

50. $\log_5 2 = \dfrac{\log 2}{\log 5} \approx 0.430677$

52. $\log_6 92 = \dfrac{\log 92}{\log 6} \approx 2.523658$

54. $\log_6 532 = \dfrac{\log 532}{\log 6} \approx 3.503061$

56. $\log_{12} 2.5 = \dfrac{\log 2.5}{\log 12} \approx 0.368743$

60. $(\log_2 5)(\log_5 7) = \dfrac{\log 5}{\log 2} \cdot \dfrac{\log 7}{\log 5} = \dfrac{\log 7}{\log 2} = \log_2 7$

Additional Exercises 4.4.1

1. (a) After simplifying you should obtain $4^{4x+1} = 3^{4x+1}$; now take $\log_a$ on both sides (where $a = 3$ or $a = 4$). You will need to apply the change of base formula, and you will then obtain $x = -\frac{1}{4}$.

   (b) $x = \frac{1}{2}$ or $x = 3$

   (c) This reduces to the quadratic equation $2 \cdot 2^{2x} + 7 \cdot 2^x - 4 = 0$; the solution is $x = -1$.

   (d) Note the caution in the comment that follows Example 4.4.1. The solution is $x = 0$.

   (e) Remember that the equation is only defined for $x > 0$.
   This reduces to the quadratic equation $3x^2 - 8x - 3 = 0$; the solution is $x = 3$.

   (f) This equation is only defined for $x > 4$. The equation reduces to the quadratic equation $x^2 - 7x + 10 = 0$; the solution is $x = 5$.

   (g) First apply a change of base formula. Solution: $x = 9$ or $x = \frac{1}{3}$.

   (h) Use the change of base formula. You should obtain the quadratic equation $k^2 + 3k - 10 = 0$ where $k = \ln x$. The solutions are $x = e^{-5}$ or $x = e^2$.

   (i) Use the change of base formula. We obtain $x = 3\sqrt{3}$ (i.e. $x \approx 5.20$) or $x = \frac{1}{27}$ (i.e. $x \approx 0.04$).

2. We begin with the LHS.

$$
\begin{aligned}
\frac{3^{2x+1} + 3^{2x-1}}{4 \cdot 9^{x+1} + 6 \cdot 9^{x-1}} &= \frac{3^{2x} \cdot 3 + 3^{2x} \cdot 3^{-1}}{4 \cdot 3^{2(x+1)} + 6 \cdot 3^{2(x-1)}} \\
&= \frac{3^{2x}(3 + \frac{1}{3})}{4 \cdot 3^{2x} \cdot 3^2 + 6 \cdot 3^{2x} \cdot 3^{-2}} \\
&= \frac{\frac{10}{3}\left(3^{2x}\right)}{3^{2x}(36 + \frac{2}{3})} \\
&= \frac{\frac{10}{3}}{\frac{110}{3}} \\
&= \frac{10}{110} \\
&= \frac{1}{11}
\end{aligned}
$$

Since LHS = RHS, this equation holds for all $x \in \mathbb{R}$.

3. (a) We obtain the quadratic inequality $k^2+2k-8<0$ where $k=2^x$. The solution of the inequality gives us $-4<2^x<2$. Since $2^x>-4$ for all x, we consider only $2^x<2$, and we obtain $x<1$.
   (b) Remember that $\log_{\frac{1}{9}}$ is a decreasing function. The solution is $x>\frac{1}{9}$.
   (c) Note that the inequality is only defined for $x>4$. The solution is $4<x<5$.
   (d) Note that the inequality is only defined for $x>2$.The solution is $x>3$.

Additional Exercises 4.5.1

1. (a) 2 years and 4 months
   (b) 8 years and 9 months
2. (a) $r=\frac{1}{10}\ln\frac{3}{2}$
   (b) 337.5 m^2
3. (a) $k=\frac{1}{5}\ln\frac{1}{4}$
   (b) 6.25 g
4. (a) $r=\frac{1}{10}\ln\frac{5}{4}$
   (b) 62.5 million
5. 6.4
6. 80 decibels
7. 20 decibels
8. (a) 3.4
   (b) $[\text{H}^+]=10^{-\text{pH}}$
   (c) Original acid: $[\text{H}^+]=10^{-4}$ M; Diluted acid: $[\text{H}^+]=10^{-5}$ M
   $[\text{H}^+]$ decreased by a factor of 10 (i.e. the $[\text{H}^+]$ of the diluted acid is $\frac{1}{10}$ that of the original acid).

CHAPTER 5

Exercises 5.1

55. In the figure the point $P(x,y)$ has been reached by moving $\frac{\pi}{6}$ units along the unit circle in a counterclockwise direction. The arc length from $(1,0)$ to $P(x,y)$ is the same as the arc length from $(1,0)$ to the point $Q(x,-y)$, i.e. $\frac{\pi}{6}$ units. Thus the arc length from Q to P is $\frac{\pi}{3}$ units. Now the arc length from P to R is also $\frac{\pi}{3}$ units since the arc length from the initial point $(1,0)$ to the point $R(0,1)$ is $\frac{\pi}{2}$ units and the arc length from $(1,0)$ to P is $\frac{\pi}{6}$ units. Thus the arc lengths from P to Q and from P to R are the same and hence the distances PQ and PR are the same. (Equal arcs subtend equal chords).

Now

$$\begin{aligned} PQ &= \sqrt{(x-x)^2+(y-(-y))^2} \\ &= \sqrt{(2y)^2} \\ &= 2y. && \text{Since } y>0. \end{aligned}$$

Also

$$PR = \sqrt{(x-0)^2+(y-1)^2}.$$

Now

$$\begin{aligned} & PQ = PR \\ \Leftrightarrow\quad & 2y = \sqrt{x^2+(y-1)^2} \\ \Rightarrow\quad & (2y)^2 = \left(\sqrt{x^2+(y-1)^2}\right)^2 \\ \Leftrightarrow\quad & 4y^2 = x^2+y^2-2y+1 \\ \Leftrightarrow\quad & 4y^2 = 1-2y+1 && \text{Since } x^2+y^2=1. \\ \Leftrightarrow\quad & 4y^2+2y-2 = 0 \\ \Leftrightarrow\quad & 2y^2+y-1 = 0 \\ \Leftrightarrow\quad & (2y-1)(y+1) = 0 \\ \Leftrightarrow\quad & y = \tfrac{1}{2} \text{ or } y = -1. \end{aligned}$$

Since $P(x,y)$ is in the first quadrant, y is positive and thus $y = \frac{1}{2}$. We now substitute $y = \frac{1}{2}$ into the equation of the unit circle. We obtain

$$x^2 + \left(\frac{1}{2}\right)^2 = 1$$

i.e. $$x^2 = 1 - \frac{1}{4}$$

i.e. $$x^2 = \frac{3}{4}$$

i.e. $$x = \pm\frac{\sqrt{3}}{2}.$$

Since the x–coordinate of P is positive we have $x = \frac{\sqrt{3}}{2}$ and thus $P\left(\frac{\pi}{6}\right) = P(x,y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

56. The arc length from $(1,0)$ to Q is $\frac{\pi}{6}$ and from $(1,0)$ to P is $\frac{\pi}{3}$. Thus the arc length from P to $(0,1)$ is $\frac{\pi}{6}$. Hence P is the reflection of Q in the line $y = x$. Since $Q = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (found in 55) we have $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
    Thus $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Additional Exercises 5.2.1

1.

$\sec t$	$\sin t$	$\cos(t-2\pi)$	$\sin(\pi+t)$	smallest positive value of t
-1	0	-1	0	π
$-\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$
$\frac{2\sqrt{3}}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{11\pi}{6}$

Exercises 5.2

24. $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, $\csc\frac{\pi}{2} = 1$, $\cot\frac{\pi}{2} = 0$, $\tan\frac{\pi}{2}$ and $\sec\frac{\pi}{2}$ are undefined.

26. $\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, $\csc\frac{3\pi}{2} = -1$, $\cot\frac{3\pi}{2} = 0$, $\tan\frac{3\pi}{2}$ and $\sec\frac{3\pi}{2}$ are undefined.

Exercises 5.3

42. (a) amplitude $= 2$, period $= \dfrac{2\pi}{k} = \pi$, phase shift $= b = 0$, $a = 2$

(b) $y = 2\cos 2x$

44. (a) amplitude $= 3$, period $= \dfrac{2\pi}{k} = 4\pi$, phase shift $= b = 0$, $a = 3$

(b) $y = 3\sin\frac{1}{2}x$; alternatively, $b = \pi$, $y = 3\cos\frac{1}{2}(x-\pi)$

46. (a) amplitude $= \frac{1}{10}$, period $= \dfrac{2\pi}{k} = \pi$, phase shift $= b = 0$, $a = -\frac{1}{10}$

(b) $y = -\frac{1}{10}\sin 2x$; alternatively $b = -\frac{\pi}{4}$, $a = \frac{1}{10}$, $y = \frac{1}{10}\cos 2\left(x+\frac{\pi}{4}\right)$

48. (a) amplitude $= 5$, period $= \dfrac{2\pi}{k} = 1$, phase shift $= b = 0$, $a = 5$

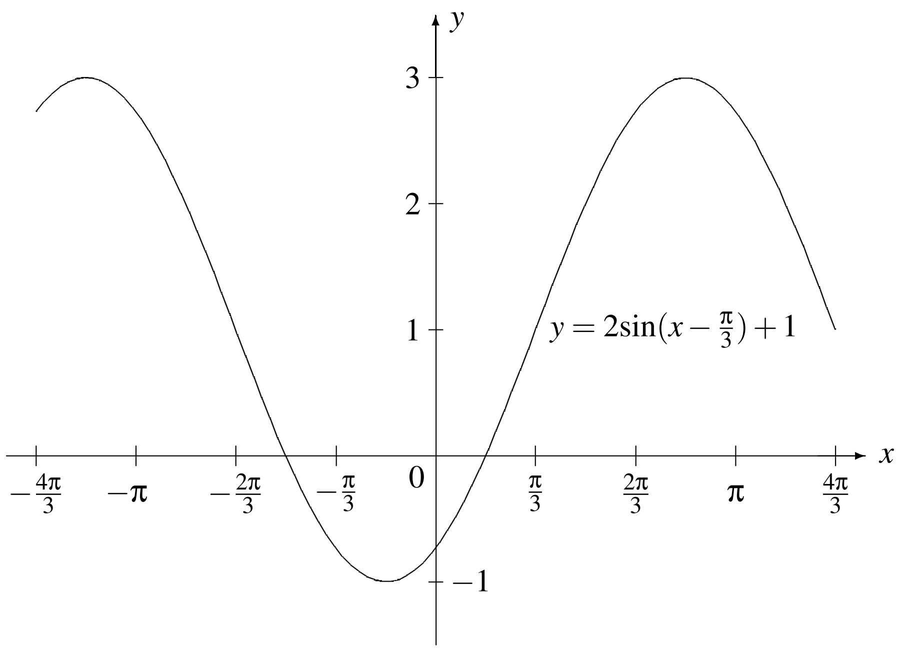
(b) $y = 5\cos 2\pi x$

Additional Exercises 5.3.1

1.
$$\begin{aligned}\sin x &= \cos\left(\frac{\pi}{2} - x\right)\\ &= \cos\left(-\left(x - \frac{\pi}{2}\right)\right)\\ &= \cos\left(x - \frac{\pi}{2}\right)\end{aligned}$$

Thus we can obtain the graph of $y = \sin x$ by shifting the graph of $y = \cos x$ to the right by $\frac{\pi}{2}$ units.

2.



Additional Exercises 5.4.1

1. $y = h(x) = 2\csc\dfrac{3x}{2}$
2. $y = f(x) = 4\sin 4x$
   $y = g(x) = -2\tan x$

CHAPTER 6

Additional Exercises 6.3.1

1. $\sin(90^o + 60^o) = \sin 150^o$

   Using the reduction formula

   $$\sin(180^o - 30^o) = \sin 30^o \qquad \text{(Table 6.3.4)}$$

   we have

   $$\sin 150^o = \sin 30^o \qquad \begin{array}{l}(150^o \text{ lies in quadrant II} \\ \text{sin is positive in quadrant II})\end{array}$$

   and thus

   $$\sin 150^o = \frac{1}{2} \qquad \text{(using special triangles),}$$

   that is

   $$\sin(90^o + 60^o) = \frac{1}{2}.$$

   Now

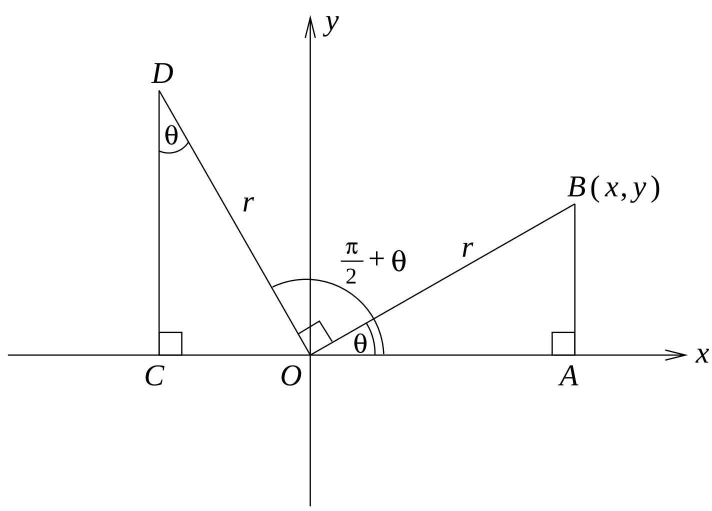
   $$\cos 60^o = \frac{1}{2} \qquad \text{(using special angles).}$$

   Thus, for $\theta = 60^o$,

   $$\sin(90^o + 60^o) = \cos 60^o.$$

   The illustrations for the other cases follow in a similar way.

2.



In the figure $\angle AOD = \theta + \frac{\pi}{2}$.

Since $\theta + \frac{\pi}{2} + \angle COD = \pi$, we have $\angle COD = \frac{\pi}{2} - \theta$, and hence $\angle CDO = \theta$.

The coordinates of B are $(x, y) = (r\cos\theta, r\sin\theta)$.

Now

$$\begin{aligned}\cos\left(\tfrac{\pi}{2}+\theta\right) &= \cos(\pi - \angle COD) = -\cos\angle COD = -\sin\theta\\ \sin\left(\tfrac{\pi}{2}+\theta\right) &= \sin(\pi - \angle COD) = \sin\angle COD = \cos\theta\end{aligned}$$

The coordinates of D are thus

$$\left(r\cos(\tfrac{\pi}{2}+\theta), r\sin(\tfrac{\pi}{2}+\theta)\right) = (-r\sin\theta, r\cos\theta) = (-y, x).$$

We thus have

$$\begin{aligned}\sec(\frac{\pi}{2}+\theta) &= \frac{r}{-y} = -\left(\frac{r}{y}\right) = -\csc\theta\\ \csc(\frac{\pi}{2}+\theta) &= \frac{r}{x} = \sec\theta.\end{aligned}$$

3. (a) $$\frac{\sin^2(180^o+\theta) + \cos(180^o-\theta)\csc(90^o-\theta)}{\tan 60^o \tan(90^o-\theta)\cot(180^o-\theta)}$$

$$= \frac{(-\sin\theta)^2 + (-\cos\theta)\sec\theta}{\sqrt{3}\cot\theta(-\cot\theta)}$$

$$= \frac{\sin^2\theta - 1}{-\sqrt{3}\cot^2\theta}$$

$$= \frac{-\left(1-\sin^2\theta\right)\sin^2\theta}{-\sqrt{3}\cos^2\theta}$$

$$= \frac{-\cos^2\theta . \sin^2\theta}{-\sqrt{3}\cos^2\theta}$$

$$= \frac{1}{\sqrt{3}}\sin^2\theta$$

(b) $$\frac{\cos\left(\frac{\pi}{2}-\theta\right)\csc\left(\frac{\pi}{2}+\theta\right)}{\sin(\pi+\theta)\sec(2\pi-\theta)\tan(-\theta)}$$

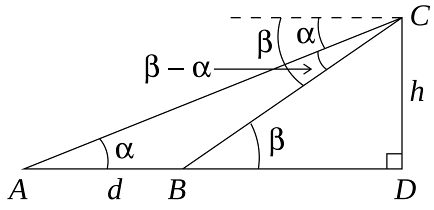
$$= \frac{\sin\theta\sec\theta}{-\sin\theta\sec\theta(-\tan\theta)}$$

$$= \frac{1}{\tan\theta}$$

$$= \cot\theta$$

Additional Exercises 6.4.1

1. (a)



The expression we require for BC must be given in terms of α, β and d.

In $\triangle ABC$ we have

$$\begin{aligned}\angle ABC &= 180^{\circ} - \beta \\ \angle ACB &= \beta - \alpha.\end{aligned}$$

Using the Law of Sines we have

$$\frac{BC}{\sin\alpha} = \frac{d}{\sin(\beta-\alpha)}$$

i.e.

$$BC = \frac{d\sin\alpha}{\sin(\beta-\alpha)}.$$

(b) In $\triangle BCD$

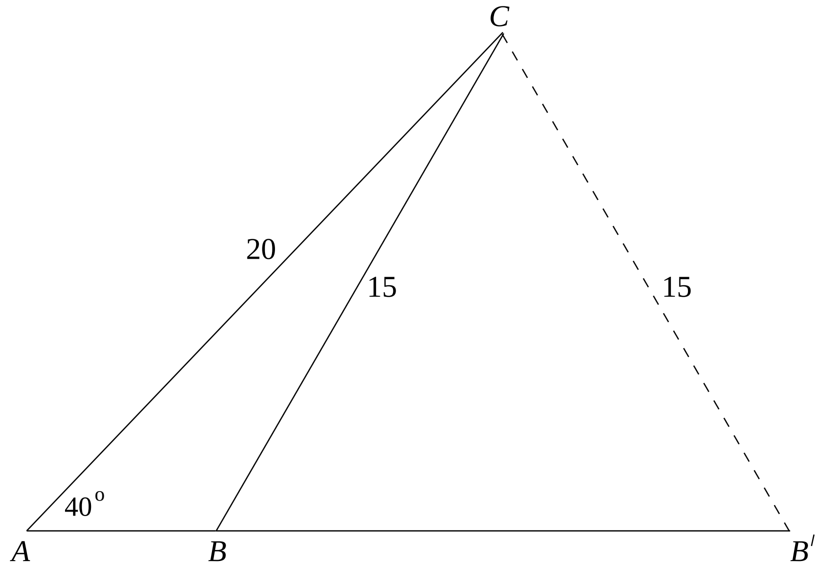
$$\frac{h}{BC} = \sin\beta$$

i.e.

$$\begin{aligned}h &= BC\sin\beta \\ &= \frac{d\sin\alpha}{\sin(\beta-\alpha)}\cdot\sin\beta \\ &= \frac{d\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.\end{aligned}$$

Exercises 6.4

29. (a)



In $\triangle ABC$

$$\frac{\sin B}{20} = \frac{\sin A}{15}$$

i.e.

$$\sin B = \frac{20 \sin 40^o}{15} \approx 0.86$$

i.e.

$$B \approx 58.99^o \text{ or } 121.01^o.$$

Using $B \approx 121.01^o$ we have $\triangle ABC$, in which

$$C \approx 180^o - (40^o + 121. - 01^o)$$

i.e. $C \approx 18.99^o$.

Thus

$$\frac{AB}{\sin C} = \frac{15}{\sin 40^o}$$

$$\Rightarrow \quad AB = \frac{15 \sin C}{\sin 40^o}$$

$$\Rightarrow \quad AB \approx \frac{15 \sin 18.99^o}{\sin 40^o} \approx 7.59.$$

Using $B' = 58.99^o$ we have $\triangle A'B'C'$, in which

$$C' = 180^o - (40^o + 58.99^o)$$

i.e. $C' \approx 81.01^o$.

Thus

$$\begin{aligned} \frac{A'B'}{\sin C'} &= \frac{15}{\sin 40^o} \\ \Rightarrow \quad A'B' &= \frac{15 \sin C'}{\sin 40^o} \\ &\approx \frac{15 \sin 81.01^o}{\sin 40^o} \\ &\approx 23.05. \end{aligned}$$

We thus have the two possible cases. $\triangle ABC$ and $\triangle A'B'C'$ with measurements as above.

(b) By the area rule,

$$\begin{aligned} \text{area } \triangle ABC &= \frac{1}{2} AB.AC \sin 40^o \\ &\approx \frac{1}{2} \times 7.59 \times 20 \times \sin 40^o \\ &\approx 48.79 \end{aligned}$$

$$\begin{aligned} \text{area } \triangle A'B'C' &= \frac{1}{2} A'B'.A'C' \sin 40^o \\ &\approx \frac{1}{2} \times 23.05 \times 20 \times \sin 40^o \\ &\approx 148.16. \end{aligned}$$

Thus

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C} \approx \frac{48.79}{148.16} \approx 0.33.$$

Also

$$\frac{\sin C}{\sin C'} = \frac{\sin 18.90^o}{\sin 81.01^o} \approx \frac{0.3254}{0.9877} \approx 0.33,$$

and we have

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle A'B'C} = \frac{\sin C}{\sin C'}.$$

CHAPTER 7

Exercises 7.1

27. $\text{LHS} = \dfrac{\cos u \sec u}{\tan u} = \dfrac{\cos u \times \frac{1}{\cos u}}{\frac{\sin u}{\cos u}} = 1 \times \dfrac{\cos u}{\sin u} = \cot u = \text{RHS}$

31.
$$\begin{aligned}\text{LHS} &= \sin B + \cos B \cot B \\ &= \sin B + \frac{\cos B \cos B}{\sin B} \\ &= \frac{\sin^2 B + \cos^2 B}{\sin B} \\ &= \frac{1}{\sin B} \\ &= \csc B \\ &= \text{RHS}\end{aligned}$$

33.
$$\begin{aligned}\text{LHS} &= \cot(-\alpha)\cos(-\alpha) + \sin(-\alpha) \\ &= (-\cot\alpha)(\cos\alpha) - \sin\alpha \\ &= -\left(\frac{\cos\alpha}{\sin\alpha}\right)\cos\alpha - \sin\alpha \\ &= \frac{-\cos^2\alpha - \sin^2\alpha}{\sin\alpha} \\ &= \frac{-\left(\cos^2\alpha + \sin^2\alpha\right)}{\sin\alpha} \\ &= -\frac{1}{\sin\alpha} \\ &= -\csc\alpha \\ &= \text{RHS}\end{aligned}$$

37.
$$\begin{aligned} \text{LHS} &= (1-\cos\beta)(1+\cos\beta) \\ &= 1-\cos^2\beta \\ &= \sin^2\beta \\ &= \frac{1}{\csc^2\beta} \\ &= \text{RHS} \end{aligned}$$

39.
$$\begin{aligned} \text{LHS} &= \frac{(\sin x+\cos x)^2}{\sin^2 x-\cos^2 x} \\ &= \frac{(\sin x+\cos x)(\sin x+\cos x)}{(\sin x-\cos x)(\sin x+\cos x)} \\ &= \frac{\sin x+\cos x}{\sin x-\cos x} \\ \text{RHS} &= \frac{\sin^2 x-\cos^2 x}{(\sin x-\cos x)^2} \\ &= \frac{(\sin x-\cos x)(\sin x+\cos x)}{(\sin x-\cos x)(\sin x-\cos x)} \\ &= \frac{\sin x+\cos x}{\sin x-\cos x} \end{aligned}$$

Thus LHS = RHS, and the identity is verified.

45.
$$\begin{aligned} \text{LHS} &= (\cot x-\csc x)(\cos x+1) \\ &= \left(\frac{\cos x}{\sin x}-\frac{1}{\sin x}\right)(\cos x+1) \\ &= \frac{(\cos x-1)(\cos x+1)}{\sin x} \\ &= \frac{\cos^2 x-1}{\sin x} \\ &= -\frac{\left(1-\cos^2 x\right)}{\sin x} \\ &= -\frac{\sin^2 x}{\sin x} \\ &= -\sin x \\ &= \text{RHS} \end{aligned}$$

47. LHS $= \left(1-\cos^2 x\right)\left(1+\cot^2 x\right)$

$$= \sin^2 x \times \csc^2 x$$

$$= \sin^2 x \times \frac{1}{\sin^2 x}$$

$$= 1$$

$$= \text{RHS}$$

51. LHS $= \dfrac{1-\cos\alpha}{\sin\alpha}$

$$= \frac{(1-\cos\alpha)(1+\cos\alpha)}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{1-\cos^2\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin^2\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \text{RHS}$$

57. LHS $= \dfrac{(\sin t+\cos t)^2}{\sin t\cos t}$

$$= \frac{\sin^2 t+2\sin t\cos t+\cos^2 t}{\sin t\cos t}$$

$$= \frac{1+2\sin t\cos t}{\sin t\cos t}$$

$$= \frac{1}{\sin t\cos t}+\frac{2\sin t\cos t}{\sin t\cos t}$$

$$= \frac{1}{\sin t}\cdot\frac{1}{\cos t}+2$$

$$= \csc t\sec t+2$$

$$= \text{RHS}$$

59.

$$\begin{aligned}
\text{LHS} &= \frac{1+\tan^2 u}{1-\tan^2 u} \\
&= \frac{1+\frac{\sin^2 u}{\cos^2 u}}{1-\frac{\sin^2 u}{\cos^2 u}} \\
&= \frac{\frac{\cos^2 u+\sin^2 u}{\cos^2 u}}{\frac{\cos^2 u-\sin^2 u}{\cos^2 u}} \\
&= \frac{1}{\cos^2 u}\cdot\frac{\cos^2 u}{\cos^2 u-\sin^2 u} \\
&= \frac{1}{\cos^2 u-\sin^2 u} \\
&= \text{RHS}
\end{aligned}$$

63.

$$\begin{aligned}
\text{LHS} &= \sec v-\tan v \\
&= \frac{1}{\cos v}-\frac{\sin v}{\cos v} \\
&= \frac{1-\sin v}{\cos v}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{\sec v+\tan v} \\
&= \frac{1}{\frac{1}{\cos v}+\frac{\sin v}{\cos v}} \\
&= \frac{1}{\frac{1+\sin v}{\cos v}} \\
&= \frac{\cos v}{1+\sin v}
\end{aligned}$$

We have written both the LHS and RHS in terms of sin and cos, and simplified each side as far as possible. Now we need to "introduce something extra", on one of the sides, and see whether it can then be transformed to the other side. Now

$$\begin{aligned}
\text{LHS} &= \frac{1-\sin v}{\cos v} \\
&= \frac{(1-\sin v)(1+\sin v)}{\cos v(1+\sin v)} \\
&= \frac{1-\sin^2 v}{\cos v(1+\sin v)} \\
&= \frac{\cos^2 v}{\cos v(1+\sin v)} \\
&= \frac{\cos v}{1+\sin v} \\
&= \text{RHS}
\end{aligned}$$

69.

$$\begin{aligned}
\text{LHS} &= \tan^2 u - \sin^2 u \\
&= \frac{\sin^2 u}{\cos^2 u} - \sin^2 u \\
&= \frac{\sin^2 u - \sin^2 u \cos^2 u}{\cos^2 u} \\
&= \frac{\sin^2 u \left(1 - \cos^2 u\right)}{\cos^2 u} \\
&= \frac{\sin^2 u \times \sin^2 u}{\cos^2 u} \\
&= \sin^2 u \times \frac{\sin^2 u}{\cos^2 u} \\
&= \sin^2 u \tan^2 u = \text{RHS}
\end{aligned}$$

73.

$$\begin{aligned}
\text{RHS} &= \frac{\sin\theta - \csc\theta}{\cos\theta - \cot\theta} \\
&= \frac{\sin\theta - \frac{1}{\sin\theta}}{\cos\theta - \frac{\cos\theta}{\sin\theta}} \\
&= \frac{\frac{\sin^2\theta - 1}{\sin\theta}}{\frac{\cos\theta\sin\theta - \cos\theta}{\sin\theta}} \\
&= \frac{\sin^2\theta - 1}{\sin\theta} \times \frac{\sin\theta}{\cos\theta(\sin\theta - 1)} \\
&= \frac{(\sin\theta - 1)(\sin\theta + 1)}{\sin\theta} \times \frac{\sin\theta}{\cos\theta(\sin\theta - 1)} \\
&= \frac{\sin\theta + 1}{\cos\theta} \\
&= \frac{(\sin\theta + 1)(\sin\theta - 1)}{\cos\theta(\sin\theta - 1)} \\
&= \frac{\sin^2\theta - 1}{\cos\theta(\sin\theta - 1)} \\
&= \frac{-\left(1 - \sin^2\theta\right)}{\cos\theta(\sin\theta - 1)} \\
&= -\frac{\cos^2\theta}{\cos\theta(\sin\theta - 1)}
\end{aligned}$$

$$= -\frac{\cos\theta}{\sin\theta - 1}$$

$$= \frac{\cos\theta}{1-\sin\theta} = \text{LHS}$$

79.

$$\begin{aligned} \text{LHS} &= (\tan x + \cot x)^2 \\ &= \tan^2 x + 2\tan x \cot x + \cot^2 x \\ &= \tan^2 x + 2 + \cot^2 x \\ &= \left(\tan^2 x + 1\right) + \left(1 + \cot^2 x\right) \\ &= \sec^2 x + \csc^2 x \\ &= \text{RHS} \end{aligned}$$

83.

$$\begin{aligned} \text{LHS} &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \\ &= \frac{(\sin x + \cos x)\left(\sin^2 x - \sin x \cos x + \cos^2 x\right)}{\sin x + \cos x} \\ &= \sin^2 x + \cos^2 x - \sin x \cos x \\ &= 1 - \sin x \cos x \\ &= \text{RHS} \end{aligned}$$

99. (c) $\sec^2 x + \csc^2 x = 1$ is not an identity, since it is not true for all values of x.

$$\begin{aligned} \text{LHS} &= \sec^2 x + \csc^2 x \\ &= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x} \\ &= \frac{1}{\cos^2 x \sin^2 x} \\ &\neq 1 \text{ for all } x \text{ in the domains of the functions.} \end{aligned}$$

If, for example, $x = \frac{\pi}{3}$, then

$$\sec^2\frac{\pi}{3} + \csc^2\frac{\pi}{3} = 2^2 + \left(\frac{2}{\sqrt{3}}\right)^2 = 4 + \frac{4}{3} \neq 1.$$

Since there is at least one value of x for which $\sec^2 x + \csc^2 x = 1$ is not true, the equation is not an identity.

Exercises 7.2

21.
$$\begin{aligned}\sec\left(\tfrac{\pi}{2}-u\right) &= \frac{1}{\cos\left(\frac{\pi}{2}-u\right)}\\ &= \frac{1}{\cos\frac{\pi}{2}\cos u+\sin\frac{\pi}{2}\sin u}\\ &= \frac{1}{0\cdot\cos u+1\cdot\sin u}\\ &= \frac{1}{\sin u}\\ &= \csc u\end{aligned}$$

23.
$$\begin{aligned}\sin\left(x-\frac{\pi}{2}\right) &= \sin x\cos\frac{\pi}{2}-\cos x\sin\frac{\pi}{2}\\ &= \sin x\cdot 0-\cos x\cdot 1\\ &= -\cos x\end{aligned}$$

27.
$$\begin{aligned}\tan(x-\pi) &= \frac{\sin(x-\pi)}{\cos(x-\pi)}\\ &= \frac{\sin x\cos\pi-\cos x\sin\pi}{\cos x\cos\pi+\sin x\sin\pi}\\ &= \frac{\sin x(-1)-\cos x\cdot 0}{\cos x(-1)+\sin x\cdot 0}\\ &= \frac{-\sin x}{-\cos x}\\ &= \tan x\end{aligned}$$

Note that we can also reason as follows:

$$\begin{aligned}\tan(x-\pi) &= \tan(-(\pi-x))\\ &= -\tan(\pi-x) && \text{from Table 6.3.3}\\ &= -(-\tan x) && \text{from Table 6.3.4}\\ &= \tan x\end{aligned}$$

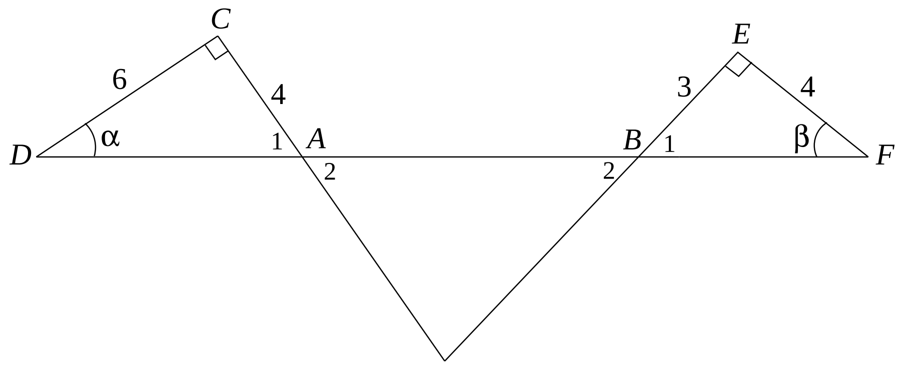
29.
$$\begin{aligned}&\cos\left(x+\frac{\pi}{6}\right)+\sin\left(x-\frac{\pi}{3}\right)\\ &= \cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}+\sin x\cos\frac{\pi}{3}-\cos x\sin\frac{\pi}{3}\\ &= \cos x\frac{\sqrt{3}}{2}-\sin x\cdot\frac{1}{2}+\sin x\cdot\frac{1}{2}-\cos x\frac{\sqrt{3}}{2}\\ &= 0\end{aligned}$$

31.
$$\sin(x+y)-\sin(x-y)$$
$$=(\sin x\cos y+\cos x\sin y)-(\sin x\cos y-\cos x\sin y)$$
$$=\sin x\cos y+\cos x\sin y-\sin x\cos y+\cos x\sin y$$
$$=2\cos x\sin y$$

35.
$$\text{RHS} = \frac{\sin(x-y)}{\cos x\cos y}$$
$$=\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}$$
$$=\frac{\sin x\cos y}{\cos x\cos y}-\frac{\cos x\sin y}{\cos x\cos y}$$
$$=\frac{\sin x}{\cos x}-\frac{\sin y}{\cos y}$$
$$=\tan x-\tan y$$
$$=\text{LHS}$$

37.
$$\text{LHS} = \frac{\sin(x+y)-\sin(x-y)}{\cos(x+y)+\cos(x-y)}$$
$$=\frac{(\sin x\cos y+\cos x\sin y)-(\sin x\cos y-\cos x\sin y)}{(\cos x\cos y-\sin x\sin y)+(\cos x\cos y+\sin x\sin y)}$$
$$=\frac{\sin x\cos y+\cos x\sin y-\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y+\cos x\cos y+\sin x\sin y}$$
$$=\frac{2\cos x\sin y}{2\cos x\cos y}$$
$$=\frac{\sin y}{\cos y}$$
$$=\tan y$$
$$=\text{RHS}$$

49.



$A_2 + \gamma + B_2 = 180^o$

But $A_2 = A_1$ and $B_2 = B_1$ (vertically opposite angles).

By the angle sum of a triangle

$$A_1 = 90^o - \alpha \text{ and } B_1 = 90^o - \beta,$$

thus

$$180^o - (\alpha + \beta) + \gamma = 180^o$$

i.e. $$90^o - \alpha + \gamma + 90^o - \beta = 180^o$$

i.e. $$-(\alpha + \beta) + \gamma = 180^o$$

i.e. $$\alpha + \beta = \gamma.$$

$$\tan\gamma = \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Now $\sin\alpha = \frac{CA}{DA}$ and $\cos\alpha = \frac{CD}{DA}$.

By Pythagoras

$$DA^2 = 4^2 + 6^2 = 52, \text{ i.e. } DA = \sqrt{52}.$$

Thus

$$\sin\alpha = \frac{4}{\sqrt{52}} \text{ and } \cos\alpha = \frac{6}{\sqrt{52}}$$

also,

$$\sin\beta = \frac{BE}{BF} \text{ and } \cos\beta = \frac{FE}{BF}.$$

By Pythagoras $BF^2 = 3^2 + 4^2 = 25$, i.e. $BF = 5$.

Thus $\sin\beta = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$.

Thus

$$\begin{aligned}\tan\gamma &= \frac{\frac{4}{\sqrt{52}}\cdot\frac{4}{5}+\frac{6}{\sqrt{52}}\cdot\frac{3}{5}}{\frac{6}{\sqrt{52}}\cdot\frac{4}{5}-\frac{4}{\sqrt{52}}\cdot\frac{3}{5}} \\ &= \frac{\frac{16+18}{5\sqrt{52}}}{\frac{24-12}{5\sqrt{52}}} \\ &= \frac{34}{5\sqrt{52}}\times\frac{5\sqrt{52}}{12}=\frac{17}{6}.\end{aligned}$$

Exercises 7.3

61. $$\begin{aligned}\text{LHS} &= (\sin x+\cos x)^2 \\ &= \sin^2 x+2\sin x\cos x+\cos^2 x \\ &= \left(\sin^2 x+\cos^2 x\right)+2\sin x\cos x \\ &= 1+\sin 2x \\ &= \text{RHS}\end{aligned}$$

63. $$\begin{aligned}\text{LHS} &= \frac{\sin 4x}{\sin x} \\ &= \frac{2\sin 2x\cos 2x}{\sin x} \\ &= \frac{4\sin x\cos x\cos 2x}{\sin x} \\ &= 4\cos x\cos 2x \\ &= \text{RHS}\end{aligned}$$

67. $$\begin{aligned}\text{RHS} &= \frac{3\tan x-\tan^3 x}{1-3\tan^2 x} \\ &= \frac{\tan x\left(3-\tan^2 x\right)}{1-3\tan^2 x} \\ &= \frac{\frac{\sin x}{\cos x}\left(3-\frac{\sin^2 x}{\cos^2 x}\right)}{1-3\frac{\sin^2 x}{\cos^2 x}} \\ &= \frac{\frac{\sin x}{\cos x}\left(\frac{3\cos^2 x-\sin^2 x}{\cos^2 x}\right)}{\frac{\cos^2 x-3\sin^2 x}{\cos^2 x}} \\ &= \frac{\sin x}{\cos x}\left(\frac{3\left(1-\sin^2 x\right)-\sin^2 x}{\cos^2 x}\right)\frac{\cos^2 x}{1-\sin^2 x-3\sin^2 x} \\ &= \frac{\sin x}{\cos x}\left(\frac{3-4\sin^2 x}{1-4\sin^2 x}\right)\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \tan 3x \\
&= \frac{\sin(2x+x)}{\cos(2x+x)} \\
&= \frac{\sin 2x \cos x + \cos 2x \sin x}{\cos 2x \cos x - \sin 2x \sin x} \\
&= \frac{2\sin x \cos x \cos x + \left(\cos^2 x - \sin^2 x\right)\sin x}{\left(\cos^2 x - \sin^2 x\right)\cos x - 2\sin x \cos x \sin x} \\
&= \frac{2\sin x \cos^2 x + \left(1 - \sin^2 x - \sin^2 x\right)\sin x}{\left(1 - \sin^2 x - \sin^2 x\right)\cos x - 2\sin^2 x \cos x} \\
&= \frac{2\sin x\left(1 - \sin^2 x\right) + \left(1 - 2\sin^2 x\right)\sin x}{\cos x\left(1 - 2\sin^2 x - 2\sin^2 x\right)} \\
&= \frac{2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x}{\cos x\left(1 - 4\sin^2 x\right)} \\
&= \frac{3\sin x - 4\sin^3 x}{\cos x\left(1 - 4\sin^2 x\right)} \\
&= \frac{\sin x\left(3 - 4\sin^2 x\right)}{\cos x\left(1 - 4\sin^2 x\right)}
\end{aligned}$$

Thus LHS = RHS, and the identity is valid.

A slightly shorter method.

$$\begin{aligned}
\text{RHS} &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \\
&= \frac{\tan x\left(3 - \tan^2 x\right)}{1 - 3\tan^2 x} \\
&= \frac{\tan x\left(3 - \left(\sec^2 x - 1\right)\right)}{1 - 3\left(\sec^2 x - 1\right)} \\
&= \frac{\tan x\left(4 - \sec^2 x\right)}{4 - 3\sec^2 x}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \tan 3x \\
&= \frac{\sin(2x+x)}{\cos(2x+x)} \\
&= \frac{\sin 2x \cos x + \cos 2x \sin x}{\cos 2x \cos x - \sin 2x \sin x} \\
&= \frac{(2\sin x \cos x)\cos x + \left(2\cos^2 x - 1\right)\sin x}{(2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x} \\
&= \frac{4\sin x \cos^2 x - \sin x}{(2\cos^2 x - 1)\cos x - 2(1-\cos^2 x)\cos x} \\
&= \frac{\sin x\left(4\cos^2 x - 1\right)}{\cos x\,(4\cos^2 x - 3)} = \frac{\tan x\left(4 - \sec^2 x\right)}{4 - 3\sec^2 x}
\end{aligned}$$

Thus LHS = RHS, and the identity is valid. In the two alternative proofs above, we apply the Addition Rules for Sine and Cosine. The proof given in SRW is much shorter because the Addition Rule for Tangent is applied. In the exam, though, it might be easier to rely on recollecting the first two rules only.

73.

$$\begin{aligned}
\text{LHS} &= \frac{\sin 10x}{\sin 9x + \sin x} \\
&= \frac{2\sin 5x \cos 5x}{2\sin\frac{9x+x}{2}\cos\frac{9x-x}{2}} \\
&= \frac{2\sin 5x \cos 5x}{2\sin 5x \cos 4x} \\
&= \frac{\cos 5x}{\cos 4x} = \text{RHS}
\end{aligned}$$

Additional Exercises 7.5.1

1. $0 \le x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < x < 2\pi$
2. $\frac{\pi}{2} < x < \frac{3\pi}{2}$
   (Remember that there are no values of x such that $\sin x > 1$.)
3. $\frac{\pi}{3} < x < \frac{5\pi}{3}$
4. $-\frac{\pi}{2} < x < 0$ (Hint: Write $\frac{\sin x}{\cos x} - \sin x < 0$, and factorise.)
5. $\frac{\pi}{6} < x < \frac{5\pi}{6}$ or $\pi < x < 2\pi$