Riemann Stieltjes Integration - Definition and Existence of Integral

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Definition

Given a closed interval $I = [a, b]$, a partition of I is any finite strictly increasing sequence of points $P = \{x_0, x_1, \ldots\ldots x_{n-1}, x_n\}$ such that $a = x_0$ and $b = x_n$. The mesh of the partition is defined by

$$meshP = max_{1 \leq j \leq n}(x_j - x_{j-1}).$$

Each partition $P = \{x_0, x_1, \ldots\ldots x_{n-1}, x_n\}$ of I decomposes I into n subintervals $I_j = [x_{j-1,j}]$, j=1,2,. . . ,n, such that

$$I_j \cap I_k \quad = \begin{cases} x_j, & \text{if } k = j + 1 \\ \phi, & \text{if } k \neq j \;\; or \;\; k \neq j + 1 \end{cases}$$

Each such decomposition of I into subintervals is called a subdivision of I.

Definition

Given a function f that is bounded and defined on the closed interval $I = [a, b]$, a function α that is defined and monotonically increasing on I, and a partition $P = \{x_0, x_1, \ldots\ldots x_{n-1}, x_n\}$ of I. Let

$$M_j = sup_{x \in I_j} f(x); \quad m_j = inf_{x \in I_j} f(x), \quad \text{for} \;\; I_j = [x_{j-1}, x_j].$$

Then, upper and lower Riemann Stieltjes sum of f over α with respect to the partition P is defined by

$$U(P, f, \alpha) = \sum_{j=1}^{n} M_j \Delta\alpha_j, \quad L(P, f, \alpha) = \sum_{j=1}^{n} m_j \Delta\alpha_j$$

where $\Delta\alpha_j = (\alpha(x_j) - \alpha(x_{j-1}))$.

Definition

For a partition $P_k = \{x_0, x_1, ...x_{k-1}, x_k\}$ of $I = [a, b]$. If P_n and P_m are partitions of [a,b] having $n + 1$ and $m + 1$ points, respectivly, and $P_n \subset P_m$, then P_m is said to be a refinement of P_n. If the partitions P_n and P_m are chosen independently, then the partition $P_n \cup P_m$ is called a common refinement of P_n and P_m.

- Our next result relates the Riemann sums taken over various partitions of an interval.

Lemma

Suppose f is a real valued bounded function defined on I=[a,b], and a partition $P = \{x_0, x_1, \ldots\ldots x_{n-1}, x_n\}$ *of I. Then*

$$m(\alpha(b)-\alpha(a)) \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M(\alpha(b)-\alpha(a)) \quad \text{and}$$

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

for any refinement P^* *of* P.

The lemma assures that

- lower and upper Riemann Stieltjes sums will remain bounded above by $l(I) sup_{x \in I} f(x)$ and bounded below by $l(I) inf_{x \in I} f(x)$.
- $sup\ \{L(P, f, \alpha); P \in \mathcal{P}\}$ and $inf\ \{U(P, f, \alpha); P \in \mathcal{P}\}$ exists.
- with the refinement of partition lower sum increases while upper sum decreases.

Definition

Suppose that f is a real valued bounded function defined on $I = [a, b]$, $\mathcal{P} = \mathcal{P}[a, b]$ be the set of all partitions of $[a, b]$ and α a monotonically increasing function defined on I. Then the upper and lower Riemann Sieltjes integrals are defined by

$$\overline{\int_a^b} f(x)d\alpha(x) = inf_{\mathcal{P}} U(P, f, \alpha); \quad \underline{\int_a^b} f(x)d\alpha(x) = sup_{\mathcal{P}} L(P, f, \alpha),$$

respectively. If $\overline{\int_a^b} f(x)d\alpha(x) = \underline{\int_a^b} f(x)d\alpha(x)$ then f is said to be Riemann Stieltjes integrable.

It is a rather short jump from previous Lemma to upper and lower bounds on the Riemann integrals. They are given by:

Theorem

Suppose that f is a bounded real valued function defined on $I = [a, b]$, α a monotonically increasing function on I, and $m \leq f(x) \leq M$ for all $x \in I$. Then

$$m(\alpha(b)-\alpha(a)) \leq \underline{\int_a^b} f(x)d\alpha(x) \leq \overline{\int_a^b} f(x)d\alpha(x) \leq M(\alpha(b)-\alpha(a)).$$

Furthermore, if f is Riemann Stieltjes integrable on I, then

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f(x)d\alpha(x) \leq M(\alpha(b) - \alpha(a)).$$

It is not worth our while to grind out some tedious processes in order to show that special functions are integrable. Towards this end, we want to seek some properties of functions that would guarantee integrability.

Theorem

Suppose that f is a function that is bounded on an interval $I = [a, b]$ and α is monotonically increasing on I. Then $f \in \mathcal{R}(\alpha)$ on I if and only if for every $\epsilon > 0$ there exists a partition P of I such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon. \tag{1}$$

Proof : Part a.

Let f be a function that is bounded on an interval $I = [a, b]$ and α be monotonically increasing on I. Suppose that for every $\epsilon > 0$ there exists a partition P of I such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

From the definition of the Riemann Stieltjes integral and Lemma 4,

$$0 \leq \overline{\int_a^b} f(x)d\alpha(x) - \underline{\int_a^b} f(x)d\alpha(x) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, it follows that

$$\overline{\int_a^b} f(x)d\alpha(x) - \underline{\int_a^b} f(x)d\alpha(x) = 0 \quad \Rightarrow \quad f \in \mathcal{R}(\alpha).$$

Proof: Part b.

Conversely suppose that $f \in \mathcal{R}(\alpha)$ and let $\epsilon > 0$ is geven. For $\frac{\epsilon}{2} > 0$ definition of supremum and infimum suggests there exists partitions $P_1,\ P_2 \in \mathcal{P}[a,b]$ such that

$$U(P_1,f,\alpha) < \int_a^b f(x)d\alpha(x)+\frac{\epsilon}{2} \& \ L(P_2,f,\alpha) > \int_a^b f(x)d\alpha(x)-\frac{\epsilon}{2}.$$

Let P be the common refinement of P_1 and P_2, then

$$U(P,f,\alpha) \leq U(P_1,f,\alpha) < \int_a^b f(x)d\alpha(x) + \frac{\epsilon}{2}, \quad \text{and}$$

$$L(P,f,\alpha) \geq L(P_2,f,\alpha) > \int_a^b f(x)d\alpha(x) - \frac{\epsilon}{2}.$$

Proof: Part b continues.

Moreover,

$$U(P,f,\alpha) < \int_a^b f(x)d\alpha(x) + \frac{\epsilon}{2}, \text{ and}$$

$$-L(P,f,\alpha) < -\int_a^b f(x)d\alpha(x) + \frac{\epsilon}{2}.$$

Combining above inequalities

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &< \int_a^b f(x)d\alpha(x) - \int_a^b f(x)d\alpha(x) + \epsilon \\ &= \epsilon. \end{aligned}$$

$$\left(\because f \in \mathcal{R}(\alpha) \text{ which implies } \int_a^b f(x)d\alpha(x) = \int_a^b f(x)d\alpha(x).\right)$$

As a fairly immediate consequence of preceding results, we have

Corollary

Suppose that f is bounded on $[a, b]$ and α is monotonically increasing on $[a, b]$.

1. *If (1) holds for some partition $P \in \mathcal{P}[a, b]$ and $\epsilon > 0$, then (1) holds for every refinement P^* of P.*
2. *If (1) holds for some partition $P \in \mathcal{P}[a, b]$ and s_j, t_j are arbitrary points in $I_j = [x_{j-1}, x_j]$, then $\sum_{j=1}^{n} |f(s_j) - f(t_j)|\Delta\alpha_j < \epsilon$.*
3. *If $f \in \mathcal{R}(\alpha)$, equation (1) holds for the partition $P \in \mathcal{P}[a, b]$ and t_j is an arbitrary point in $I_j = [x_{j-1}, x_j]$, then*

$$|\sum_{j=1}^{n} f(t_j)\Delta\alpha_j - \int_a^b f d\alpha| < \epsilon.$$

Proof- Part 1.

For any refinement P^* of P, Lemma 4 gives

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P,f,\alpha).$$

From this it is easy to observe that

$$\begin{aligned} U(P^*,f,\alpha) - L(P^*,f,\alpha) &\leq U(P,f,\alpha) - L(P,f,\alpha) \\ &< \epsilon, \quad \text{from (1)}. \end{aligned}$$

$\square$

Proof- Part 2.

Suppose that $M_j = sup_{x \in I_j} f(x)$, $m_j = inf_{x \in I_j} f(x)$ and s_j, t_j are arbitrary points in I_j, $j = 1, 2, \ldots, n$. Then $f(s_j)$, $f(t_j) \in [m_j, M_j]$ and hence

$$|f(s_j) - f(t_j)| \leq M_j - m_j,$$

$$\begin{aligned} \text{i.e. } \sum_{j=1}^{n} |f(s_j) - f(t_j)| \Delta\alpha_j &\leq \sum_{j=1}^{n} M_j \Delta\alpha_j - \sum_{j=1}^{n} m_j \Delta\alpha_j, \\ &= U(P, f, \alpha) - L(P, f, \alpha), \\ &< \epsilon, \quad \text{from (1)}. \end{aligned}$$

$\square$

Proof- Part 3.

From the definition of Riemann Stieltjes integral and Lemma 4,

$$L(P,f,\alpha) \leq \int_a^b f(x)d\alpha(x) \leq U(P,f,\alpha). \tag{2}$$

Moreover, for m_j, M_j are as defined earlier and $j = 1, 2, \ldots, n$, $t_j \in [x_{j-1}, x_j]$ therefore $f(t_j) \in [m_j, M_j]$. From this it is easy to construct the inequality

$$L(P,f,\alpha) \leq \sum_{j=1}^{n} f(t_j)\Delta\alpha_j \leq U(P,f,\alpha). \tag{3}$$

From inequalities (2) and (3) it can be concluded that $|\sum_{j=1}^{n} f(t_j)\Delta\alpha_j - \int_a^b f d\alpha| \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$ □

So far we have gone through results which will be useful to us whenever we have a way of closing the gap between functional values on the same intervals. Next two results give us two "big" classes of integrable functions in the sense of Riemann Stieltjes integration.

Theorem

If f is a function that is continuous on the interval $I = [a, b]$, then f is Riemann Stieltjes integrable on [a,b].

Proof.

Let α be monotonically increasing on I and f be continuous on I. Suppose that $\epsilon > 0$ is given. Then there exists an $\eta > 0$ such that

$$[\alpha(b) - \alpha(a)]\eta < \epsilon.$$

Clearly $I = [a, b]$ is compact and therefore f is uniformly continuous in $[a, b]$. Hence, there exists a $\delta > 0$ such that

$$\forall x,\ t \in I, \quad |x - t| < \delta \quad \Rightarrow \quad |f(x) - f(t)| < \eta.$$

$\square$

Proof Continues.

Let $P = \{x_0, x_1, ...x_{n-1}, x_n\}$ be the partition of I for which

$$mesh\ P < \delta \quad i.e., \quad \Delta x_j = (x_j - x_{j-1}) < \delta \text{ for any } j.$$

Since f is uniformly continuous this implies that $M_j - m_j < \eta$ for any i. Consider

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^{n} (M_j - m_j)\Delta\alpha_j \\ &< \eta \sum_{j=1}^{n} \Delta\alpha_j \\ &< \epsilon. \end{aligned}$$

□

Proof Continues.

In view of the Integrability Criterion, $f \in \mathcal{R}(\alpha)$. Because α was arbitrary, we conclude that f is Riemann Stieltjes Integrable (with respect to any monotonically increasing function on [a,b]. $\square$

As an immediate consequence of the above theorem, we have

Corollary

If f is a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I, then $f \in \mathcal{R}(\alpha)$.

So we can summarize the results as follows;

1. Bounded and continous function f can be integrated with respect to any monotonic incresing function α.
2. Bounded and monotonic function f can be integrated with repsect to any monotonic incresing and continous function α.

1. Walter Rudin : Principles of Mathematical Analysis, McGraw Hill Pulishers.
2. T. Apostol, Mathematical Analysis, Narosa Publication.
3. A. Kaushik, Lecture Notes, Directorate of Distance Education, Kurukshetra University Kurukshetra.

Thank You !