UNISA university of south africa

MAT3711

October/November 2012

REAL ANALYSIS

Duration : 2 Hours 100 Marks

EXAMINATION PANEL AS APPROVED BY THE DEPARTMENT.

This paper consists of 3 pages.

Answer ALL questions.

QUESTION 1

Let (X, d) be a metric space, $p \in X$ and $r \in \mathbb{R}$ such that $r > 0$.

(a) Define each of the following concepts.

- (i) The ball with centre p and radius r. (1)
- (ii) An interior point of a set $A \subseteq X$. (1)
- (iii) An open subset of X. (1)
- (iv) A closed subset of X. (1)
- (v) A neighbourhood of p. (1)

(b) Consider $\mathbb{R}$ with its usual metric and let $S \subseteq \mathbb{R}$ be defined by

$$S = \left\{2 - \frac{1}{n} : n \in \mathbb{N}\right\}.$$

Give full reasons for your answers to the following questions.

- (i) Is S open in $\mathbb{R}$? (5)
- (ii) Is S closed in $\mathbb{R}$? (5)

(c) Let A be a subset of a metric space X. Show that A is dense in X if and only if A intersects every non-empty open set. (10)

[25]

[TURN OVER]

QUESTION 2

Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X.

(a) Define each of the following concepts.

(i) $\{x_n\}$ converges in X. (1)

(ii) $\{x_n\}$ is a Cauchy sequence. (2)

(b) Prove the following theorem: (4)

Theorem: Every convergent sequence is a Cauchy sequence.

(c) Give an example that illustrates why the converse of the Theorem in (b) is not true. (3)

[10]

QUESTION 3

Let (X, d) be a metric space.

(a) Define each of the following concepts.

(i) A connected metric space. (Hint: Define disconnected first.) (3)

(ii) A separation of X. (2)

(b) Prove that if A and B are a separation of a metric space (X, d), and if H is any connected subset of X, then $H \subseteq A$ or $H \subseteq B$. (5)

(c) Consider the set $X = [1, \infty)$ which has the usual metric d, that is, $d(x, y) = |x - y|$. Let λ be a real number with $1 < \lambda < 2$. Let $f : X \to X$ be defined by

$$f(x) = \frac{\lambda + x}{1 + x}.$$

(i) Explain why the metric space (X, d) is complete. (4)

(ii) Show that f is a contraction on X. (7)

(iii) From (i) and (ii), how do we know that f has a unique fixed point? (1)

(iv) Find the fixed point of f. (3)

[25]

[TURN OVER]

QUESTION 4

(a) Let $T : V \to W$ be a bounded linear operator.

(i) How is the operator norm $||T||$ defined? (2)

(ii) For which vectors $v \in V$ does the inequality $||Tv|| \leq ||T||\ ||v||$ hold? (2)

(iii) Show that T is continuous. (8)

(b) Consider $\mathbb{R}^2$ with its usual norm $||(x,y)|| = \sqrt{x^2+y^2}$, and $\mathbb{R}$ with norm equal to the absolute value. Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the linear operator defined by

$$T(x,y) = x + 2y$$

for all $(x,y) \in \mathbb{R}^2$.

(i) Show that T is a bounded linear operator. (Show only boundedness.) (4)

(ii) Evaluate $||T||$. (4)

[20]

QUESTION 5

(a) Define the Riemann-Stieltjes integral. (12)
(Hint: Be sure to define all notations used, for example: partition, sub-interval, length of sub-interval, upper Stieltjes integral, lower Stieltjes integral, etc.)

(b) Let f and α be functions defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{and} \quad \alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Compute $\int_0^1 f\, d\alpha$ if it exists. (8)

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TOTAL: 100 Marks

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