MEMO STA3702 (MAY/JUNE 2016 EXAMINATION)

Question 1 **[Total marks=30]**

(a)

$$\begin{aligned} f(x|\theta) &= g(x)\exp\{p(\theta)K(x) - q(\theta)\} \\ &= \exp\{p(\theta)K(x) - q(\theta) + \ln[g(x)]\} \\ &= \exp\{p(\theta)K(x) + -q(\theta) + \ln[g(x)]\} \\ &= \exp\{p(\theta)K(x) + q^*(\theta) + g^*(x)\} \end{aligned}$$

where $q^*(\theta) = -q(\theta) \in (-\infty, \infty)$ and $g^*(x) = \ln[g(x)] \in (-\infty, \infty)$. **(4)**

(b) (i)

$$\begin{aligned} f(x|\theta) &= \theta^2 x\exp(-\theta x) \\ &= \exp\{-\theta x + 2\ln\theta + \ln x\} \\ &= \exp\{-\theta x - -2\ln\theta + \ln x\} \\ &= \exp\{p(\theta)K(x) + q^*(\theta) + g^*(x)\} \end{aligned}$$

where $K(x) = -x$, $q^*(\theta) = 2\ln(\theta) = -q(\theta)$, $p(\theta) = \theta$ and $g^*(x) = \ln x = \ln g(x)$. **(4)**

(ii) $\sum_{i=1}^{n} K(X_i) = -\sum_{i=1}^{n} X_i$ since $f(x|\theta)$ belongs to the 1-parameter exponential family. **(3)**

(iii) -$E\left[\sum_{i=1}^{n} X_i\right] = -nE[X] = n\frac{q'(\theta)}{p'(\theta)} = -\frac{2n}{\theta} \equiv E\left[\sum_{i=1}^{n} X_i\right] = \frac{2n}{\theta}$. **(3)**

(iv) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^{2n}\exp\left(-\theta\sum_{i=1}^{n} x_i\right)\prod_{i=1}^{n} x_i \Longrightarrow$$

the log-likelihood function of θ is

$$l(\theta) = \ln L(\theta) = 2n\ln\theta - \theta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\ln x_i \Longrightarrow$$

$$l'(\theta) = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i.$$

(5)

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{2n}{\hat{\theta}} - \sum_{i=1}^{n} x_i \text{ and the solution is } \hat{\theta} = \frac{2}{\bar{x}}.$$

(3)

(v) From part (iii), $E[X] = \frac{2}{\theta} \Longrightarrow \theta = \frac{2}{E[X]} \Longrightarrow$ the MME of θ is $\tilde{\theta} = \frac{2}{\bar{x}}$. **(3)**

(vi) $\frac{1}{2}\bar{X}$ since the estimator is a function of a complete sufficient statistic and $E\left[\frac{1}{2}\bar{X}\right] = \theta$ (see part (v)). **(5)**

Question 2 **[Total marks=30]**

(a) The joint density function is:

$$L(\mathbf{y}|\theta) = \theta^{-n} \exp\left\{-\sum_{i=1}^{n} y_i/\theta\right\} = m_1(\mathbf{y}) \times m_2\left(\sum_{i=1}^{n} y_i, \theta\right)$$

where $m_1(\mathbf{y}) = 1$ and $m_2\left(\sum_{i=1}^{n} y_i, \theta\right) = \theta^{-n}\exp\{-\sum_{i=1}^{n} y_i/\theta\}$. Hence, $\sum_{i=1}^{n} Y_i$ is a sufficient statistic for θ_2 by the factorization theorem. **(3)**

Let $X_1, X_2, ..., X_n$ be a random sample from the same sampled distribution. Then

$$\frac{L(\mathbf{y}|\theta)}{L(\mathbf{x}|\theta)} = \exp\left\{\frac{1}{\theta}\left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right)\right\}$$

which is independent of θ if $(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i) = 0$ equivalently if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. This means $\sum_{i=1}^{n} Y_i$ is a minimal sufficient statistic for θ. **(5)**

(b)

(i) $E[Y] = \theta$ (given) $\Longrightarrow$ the MME of θ is $\tilde{\theta} = \bar{y} = 20/200 = 0.1$. **(3)**

(ii) The likelihood function of θ is

$$L(\theta) = \theta^{-n} \exp\left\{-\sum_{i=1}^{n} y_i/\theta\right\}$$

and the log-likelihood function of θ is

$$l(\theta) = -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} y_i.$$

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = -\frac{n}{\hat{\theta}} + \frac{1}{\hat{\theta}^2}\sum_{i=1}^{n} y_i.$$

The solution is $\hat{\theta} = \bar{y} = 0.1$ **(9)**

(iii) $l''(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} y_i = \frac{n}{\theta^2} - \frac{2n}{\theta^3}\bar{y}$. Hence $I(\mathbf{y}) = -l''(\hat{\theta}) = -\frac{n}{\hat{\theta}^2} + \frac{2n}{\hat{\theta}^3}\bar{y} = \frac{n}{\hat{\theta}^2} = 100n = 20000$. **(4)**

(iv) Since $\hat{\theta} = \tilde{\theta}$, the standard errors of both estimates are equal to

$$se = \sqrt{1/I(\mathbf{y})} = \sqrt{1/20000} = 0.007071.$$

(6)

Question 3 **[Total marks=25]**

(a) $E[X] = 0 \times \frac{2}{3}\theta + 1 \times \frac{1}{3}\theta + 2 \times \frac{2}{3}(1-\theta) + 3 \times \frac{1}{3}(1-\theta) = \frac{7}{3} - 2\theta \Longrightarrow \theta = \frac{1}{6}(7 - 3E[X]) \Longrightarrow$ the MME of θ is

$$\tilde{\theta} = \frac{1}{6}(7 - 3\bar{x}) = \frac{1}{6}(7 - 3 \times 1.4) = \frac{7}{15} = 0.4667.$$

(5)

(b) The likelihood function of θ is:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} P(X = x_i) \\ &= \left(\frac{2}{3}\theta\right)^2 \times \left(\frac{1}{3}\theta\right)^2 \times \left[\frac{2}{3}(1-\theta)\right]^6 = \frac{2^8}{3^{10}}\theta^4(1-\theta)^6 \end{aligned}$$

The log-likelihood function is:

$$l(\theta) = \ln L(\theta) = constant + 4\ln\theta + 6\ln(1-\theta) \Longrightarrow l'(\theta) = \frac{4}{\theta} - \frac{6}{1-\theta}.$$

The MLE of θ is $\hat{\theta}$ which solves the equation:

$$0 = l'(\hat{\theta}) = \frac{4}{\hat{\theta}} - \frac{6}{1-\hat{\theta}}.$$

The solution is $\hat{\theta} = \frac{4}{10} = 0.4$. **(8)**

(c) $l''(\theta) = -\frac{4}{\theta^2} - \frac{6}{(1-\theta)^2}$. Hence:

(i) $Var[\tilde{\theta}] \approx \dfrac{1}{-l''(0.4667)} = \dfrac{1}{18.3673 + 21.09375} = \dfrac{1}{39.461097} = 0.02534,\ se(\tilde{\theta}) \approx \sqrt{0.02534} = 0.1592.$ **(5)**

(ii) $Var[\hat{\theta}] \approx \dfrac{1}{-l''(0.4)} = \dfrac{1}{25 + 16.6667} = \dfrac{1}{41.6667} = 0.024,\ se(\hat{\theta}) \approx \sqrt{0.024} = 0.15492.$ **(3)**

(d) Based on the standard errors of the estimates, the MLE is preferred as it has a smaller se. **(4)**

Question 4 **[Total marks=15]**

(a)

(i) The level of significance of the test is

$$\alpha = P\left(X_1 \geq \frac{1}{2}|\theta = 1\right) = \int_{1/2}^{1} dx = [x]_{1/2}^{1} = 1/2.$$

(4)

(ii) The probability of the type II error is

$$\beta = P\left(X_1 \leq \frac{1}{2}|\theta = 2\right) = \int_{0}^{1/2} 2x = \left[x^2\right]_0^{1/2} = 1/4.$$

(4)

Hence the power of the test is

$$Power = 1 - \beta = 3/4.$$

(2)

(b) $0.1 = P\left(X_1 \geq c|\theta = 1\right) = \int_c^1 dx = [x]_c^1 = 1 - c$. This means

$$0.1 = 1 - c \Longrightarrow c = 1 - 0.1 = 0.9.$$

(5)

TOTAL [100]