MEMO STA3702 (MAY/JUNE 2015 EXAMINATION)

Question 1 **[Total marks= 25]**

(a)

$$\begin{aligned} f(x|\theta) &= \frac{1}{3!\theta^4}x^3\exp\left(-\frac{x}{\theta}\right) \\ &= \exp\left(-\frac{x}{\theta}-4\ln(\theta)-\ln(3!)+3\ln(x)\right) \\ &= \exp\left(-\eta x-4\ln(\eta^{-1})-\ln(3!)+3\ln(x)\right) \\ &= \exp\{\eta t(x)-\psi(\eta)+g(x)\} \end{aligned}$$

where $\eta = \frac{1}{\theta}$, $t(x) = -x$, $\psi(\eta) = -4\ln(\eta)$ and $g(x) = 3\ln(x) - \ln(3!)$. **(4)**

(b) $\sum_{i=1}^n t(X_i) = -\sum_{i=1}^n X_i$ since $f(x|\theta)$ belongs to regular exponential family. **(3)**

(c) $l(\theta) = -\frac{1}{\theta}\sum_{i=1}^n x_i - 4n\ln(\theta) - n\ln(3!) + 3\sum_{i=1}^n \ln(x_i)$ **(4)**

(d) $E[X] = -\psi'(\eta) = 4\eta^{-1} = 4\theta$ and $l''(\theta) = -\frac{2}{\theta^3}\sum_{i=1}^n x_i + \frac{4n}{\theta^2} \Longrightarrow$

$$I_n(\theta) = -E[l''(\theta)] = \frac{2}{\theta^3}\sum_{i=1}^n E[X_i] - \frac{4n}{\theta^2} = \frac{8n\theta}{\theta^3} - \frac{4n}{\theta^2} = \frac{4n}{\theta^2}.$$

(4)

(e) The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{1}{\hat{\theta}^2}\sum_{i=1}^n x_i - \frac{4n}{\hat{\theta}}$$

The solution is $\hat{\theta} = \frac{1}{4n}\sum_{i=1}^n x_i$ which depends on the sample through complete sufficient statistic **(4)**

(f) $CRLB = \frac{[\theta']^2}{I_n(\theta)} = \frac{\theta^2}{4n}$. **(3)**

(g) That $\hat{\theta}$ is a function of a complete sufficient statistic and $E[\hat{\theta}] = \frac{4n\theta}{4n} = \theta$ implies that $\hat{\theta}$ is a MVUE of θ. **(3)**

Question 2 **[Total marks=25]**

(a) Note

$$\begin{aligned} f(y|\theta) &= \frac{2}{\theta}y\exp\left(-\frac{y^2}{\theta}\right) \text{ if } y>0 \text{ and } \theta>0 \\ &= \exp\left\{-\frac{y^2}{\theta}+\ln\left(\frac{2}{\theta}y\right)\right\} \\ &= \exp\left\{-\frac{y^2}{\theta}-\ln(\theta)+\ln(2y)\right\} \\ &= \exp\left\{-\eta y^2-\ln(\eta^{-1})+\ln(2y)\right\} = f(y|\eta) \end{aligned}$$

which is in the form of the *pdf* in Question 1 with $t(y) = -y^2$, $\theta = 1/\eta$ and $\psi(\eta) = -\ln(\eta)$. Hence complete sufficient statistic for θ is $\sum_{i=1}^n t(Y_i) = -\sum_{i=1}^n Y_i^2$. **(5)**

(b) $\sum_{i=1}^n Y_i^2$ is a minimal sufficient statistic for θ because it is a complete sufficient statistic for θ. **(2)**

(c) From Question 1,

$$E[-\sum_{i=1}^n Y_i^2] = -nE[Y^2] = \frac{\partial - \ln\eta}{\partial\eta} = -n/\eta = -n\theta \equiv E[\sum_{i=1}^n Y_i^2] = n\theta,$$

and

$$Var[-\sum_{i=1}^{n} Y_i^2] = Var[\sum_{i=1}^{n} Y_i^2] = nVar[Y^2] = \frac{\partial^2 - \ln\eta}{\partial\eta^2} = n/\eta^2 = n\theta^2.$$

(2+2=4)

(d) $l(\eta) = \ln L(\eta|\mathbf{y}) = -\eta\sum_{i=1}^{n} y_i^2 + n\ln(\eta) + \sum_{i=1}^{n}\ln(2y_i) \Longrightarrow l'(\eta) = -\sum_{i=1}^{n} y_i^2 + \frac{n}{\eta} \Longrightarrow$ the MLE of η is $\hat{\eta}$ which solves the equation

$$l'(\hat{\eta}) = -\sum_{i=1}^{n} y_i^2 + \frac{n}{\hat{\eta}} = 0.$$

The solution is $\hat{\eta} = \frac{n}{\sum_{i=1}^{n} y_i^2} \Longrightarrow$ the MLE of θ is

$$\hat{\theta} = 1/\hat{\eta} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

by the invariance property of $MLEs$.
From above and part (a), $\hat{\theta}$ is a function of a complete sufficient statistic, and from part(c) $E[\hat{\theta}] = \theta$ ($\hat{\theta}$ is an unbiased estimator of θ). Therefore, $\hat{\theta}$ also the $MVUE$ of θ. **(4+3=7)**

(e) By the invariance property of $MLE's$, the MLE of the variance of the complete sufficient statistic found in part(a) is $n\hat{\theta}^2$. **(3)**

(f) From part (d), the MLE of θ is $\hat{\theta} = \frac{100}{200} = 0.5$. **(1)**

$se(\hat{\theta}) = -\frac{1}{\sqrt{l''(\hat{\theta})}} = \sqrt{\frac{\hat{\theta}^2}{n}} = 0.5/\sqrt{200} = 0.035355.$ **(1+3=4)**

Question 3 **[Total marks=25]**

(a) $MME:\ E[X] = n\theta \Longrightarrow E[X/n] = \theta \Longrightarrow MME$ of θ is x/n. **(3)**
$MLE: L(\theta) = \frac{n!}{(n-x)!x!}\theta^x(1-\theta)^{n-x}$ and
$l(\theta) = \ln L(\theta) = \ln\left(\frac{n!}{(n-x)!x!}\right) + x\ln(\theta) + (n-x)\ln[(1-\theta)] \Longrightarrow l'(\theta) = \frac{x}{\theta} - \frac{n-x}{1-\theta)} \Longrightarrow$ the MLE of θ is $\hat{\theta}$ which solves $\frac{x}{\hat{\theta}} - \frac{n-x}{1-\hat{\theta})} = 0 \equiv x - n\hat{\theta} = 0$ and the solution is $\hat{\theta} = x/x$ which is also the MME of θ. **(6)**

(b) $l''(\theta) = -\frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2} \Longrightarrow I_1(\theta) = -E[l''(\theta)] = \frac{E[X]}{\theta^2} + \frac{n-E[x]}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n}{\theta(1-\theta)}$. **(4)**

(c) $Var(\hat{\theta}) = \frac{1}{n^2}Var(X) = \frac{n\theta(1-theta)}{n^2} = \frac{\theta(1-\theta)}{n}$. **(4)**

(d) $g(\theta) = E(\hat{\theta}) = E(X/n) = \theta \Longrightarrow g'(\theta) = 1 \Longrightarrow CRLB = \frac{[g'(\theta)]^2}{I_1(\theta)} = \frac{\theta(1-\theta)}{n} = Var(\hat{\theta})$. **(4)**

(e) Since $E(\hat{\theta}) = \theta$ and $Var(\hat{\theta}) = CRLB$, this means $\hat{\theta}$ is a $MVUE$ of θ. **(4)**

Question 4 **[Total marks=25]**

(a)

$$\begin{aligned} MSE(T) &= E[(T-\theta)^2] = E[(T-E[T]+E[T]-\theta)^2] \\ &= E[(T-E[T])^2 + 2(T-E[T])(E[T]-\theta) + (E[T]-\theta)^2] \\ &= E[(T-E[T])^2] + 2(E[T]-\theta)E[(T-E[T])] + E[(E[T]-\theta)^2] \\ &= Var[T] + 0 + (E[T]-\theta)^2 \end{aligned}$$

(5)

(b) If T is any unbiased estimator of θ then $E[T] - \theta = 0 \Longrightarrow MSE(T) = Var(T)$. **(2)**

(c) $E[cT_1 + (1-c)T_2] = cE[T_1] + (1-c)E[T_2] = c\theta + (1-c)\theta = \theta$. **(3)**

(d)

$$\begin{aligned} Var[T] &= c^2 Var[T_1] + (1-c)^2 Var[T_2] = 2c^2 Var[T_2] + (1-c)^2 Var[T_2] \\ &= 3c^2 - 2c + 1) Var[T_2] \end{aligned}$$

(2)

$$0 = \frac{\partial V[T]}{\partial c} = (6c-2) Var[T_2] \Longrightarrow c = \frac{2}{6} = \frac{1}{3}$$

minimizes the variance of the estimator $T = cT_1 + (1-c)T_2$ of θ. **(3)**

(e) $re(T_2, T_1) = 2$ and $re(T_2, T) = \frac{2}{3}$. T is the best estimator because it has the smallest variance.

(1+3+2=6)

(f) $0.95 = P(0.1 \leq \frac{T}{\theta} \leq 10) = P(0.1 \leq \frac{\theta}{T} \leq 10) = P(0.1T \leq \theta \leq 10T)$ which means the 95% confidence interval for θ is $[0.1T; 10T]$. **(4)**

TOTAL [100]