Tutorial Letter 201/1/2015

REAL ANALYSIS

MAT3711

Semester 1

Department of Mathematical Sciences

This tutorial letter contains the solutions to the Assignment 01.

BAR CODE

ASSIGNMENT 01
Solution
Total Marks: 43
UNIQUE ASSIGNMENT NUMBER: 599134

Question 1: 23 Marks

(1.1) Let $S = \left\{2 - \frac{1}{n} \,|n \in \mathbb{N}\right\}$ be viewed as a subset of $\mathbb{R}$ with the usual metric. Give full reasons for your answers to the following questions:

(a) Is S open? (5)

Solution: *S is not open. For instance $1 \in S$, but for any $r > 0$, $B(1, r) = (1-r, 1+r)$, which contains irrational numbers because between any two real numbers there is an irrational number. Thus $B(1, r) \not\subseteq S$ since every element of S is a rational number.*

(b) Is S closed? (5)

Solution: *For each $n \in \mathbb{N}$, let $x_n = 2 - \frac{1}{n}$. Then $\{x_n\}$ is a sequence in S such that $x_n \to 2$. So, by Theorem 2.1.8, $2 \in \overline{S}$. But $2 \notin S$, so $S \neq \overline{S}$. Therefore S is not closed.*

(c) Is the interior of S nonempty? (5)

Solution: *For any $p \in S$ and any $r > 0$, $B(p, r) = (p - r, p + r) \not\subseteq S$ by the same argument as in (i). So no point of S is an interior point of S. Thus $S^0 = \emptyset$.*

(1.2) Find the boundary of the set S above. Show how you arrive at your answer, and give reasons for your statements. (8)

Solution: *For any $p \in S$, every neighbourhood of p meets S (at p, for instance) and also meets $\mathbb{R}\backslash S$ since, as observed above, every ball around p contains irrational numbers. Thus, $S \subseteq bd(S)$. Next, every neighbourhood of 2 contains 2 which is not in S, and also contains $2 - \frac{1}{m}$ for some $m \in \mathbb{N}$ since $2 - \frac{1}{n} \to 2$ as $n \to \infty$. This shows that $2 \in bd(S)$. Consequently, $S \cup 2 \subseteq bd(S)$. Now note that if $q \notin S \cup 2$, then we can find $\epsilon > 0$ such that $(q - \epsilon, q + \epsilon)$ does not contain an element of S. This shows that if $q \notin S \cup 2$, then $q \notin bd(S)$. Consequently, $bd(S) \subseteq S \cup 2$, and hence $bd(S) = S \cup 2$.*

Question 2: 20 Marks

(2.1) Let A be a subset of a metric space X. Show that A is dense in X if and only if $\text{int}(X\backslash A) = \emptyset$. (10)

Solution:

($\Rightarrow$) *Assume A is dense. If $int\,(X \setminus A)$ were not empty, then in view of $int\,(X \setminus A)$ being an open set, we would have $A \cap int\,(X \setminus A) \neq \emptyset$, by Theorem 1.3.30, since A is dense. But $int\,(X \setminus A) \subseteq X \setminus A$, so we would have $A \cap (X \setminus A) \neq \emptyset$, which is false. Therefore $int\,(X \setminus A) = \emptyset$.*

($\Leftarrow$) *Let $U \subseteq X$ be any nonempty open set. By Theorem 1.3.30, it suffices to show that $U \cap A \neq \emptyset$. Suppose, by way of contradiction, that $U \cap A = \emptyset$. Then $U \subseteq X \setminus A$, and hence*

$$\emptyset \neq U = int\,(U) \subseteq int\,(X \setminus A)\,,$$

which contradicts the current hypothesis that $int\,(X \setminus A) = \emptyset$.

(2.2) Let $(X,\ d)$ be a metric space. Show that a subset A of X is closed if and only if for each $x \in X \backslash A,\ d(x, A) \neq 0$. (10)

Solution:

($\Rightarrow$) *Assume A is closed. Let $x \in X \setminus A$. Suppose, by way of contradiction, that $d(x, A) = 0$. Thus*

$$\sup \{d(x, a) \mid a \in A\} = 0.$$

So, for any $n \in \mathbb{N}$, we can find $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Then the sequence $\{a_n\}$ in A has the property that $a_n \to x$. Thus, $x \in \overline{A}$, by Theorem 2.1.8. But A is closed, so $\overline{A} = A$, and hence $x \in A$, contrary to the fact that $x \in X \setminus A$.

($\Leftarrow$) *Assume that $d(x, A) \neq 0$ for each $x \in X \setminus A$. Let $p \in \overline{A}$. By Theorem 2.1.8, there is a sequence $\{a_n\}$ in A such that $a_n \to p$. Hence $\lim_{n\to\infty} d(a_n, p) = 0$. Thus*

$$\sup \{d(z, p) \mid z \in A\} = 0$$

and hence p is not in $X \setminus A$. So $p \in A$, and hence $\overline{A} \subseteq A \subseteq \overline{A}$, implying $A = \overline{A}$, whence A is closed.