Tutorial Letter 201/1/2013

Real Analysis

MAT3711

Semester 1

Department of Mathematical Sciences

Solutions to Assignment 01.

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QUESTION 1

(a) (i) S is not open. For instance, $0 \in S$, but for any real number $r > 0$, $B(0,r) = (-r,r)$, which contains irrational numbers because between any two real numbers there is an irrational number. Thus, $B(0,r) \nsubseteq S$ since every element of S is a rational number.

(ii) For each $n \in \mathbb{N}$, let $x_n = 1 - \dfrac{1}{n}$. Then $\{x_n\}$ is a sequence in S such that $x_n \to 1$. So, by Theorem 2.1.8, $1 \in \overline{S}$. But $1 \notin S$, so $S \neq \overline{S}$. Therefore S is not closed.

(iii) For any $n \in \mathbb{N}$ and $r > 0$, $B\left(1 - \dfrac{1}{n}, r\right) = \left(1 - \dfrac{1}{n} - r, 1 - \dfrac{1}{n} + r\right) \nsubseteq S$ by the same reasoning as in (i). So no point of S is an interior point. Therefore $S^{\circ} = \emptyset$.

(b) Note that, since $S \subseteq \mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q} \subseteq \mathbb{R}\backslash S$, and hence $\mathbb{R} = \overline{\mathbb{R}\backslash\mathbb{Q}} \subseteq \overline{\mathbb{R}\backslash S} \subseteq \mathbb{R}$, so that $\overline{\mathbb{R}\backslash S} = \mathbb{R}$. Next, we show that $\overline{S} = S \cup \{1\}$. As observed in (a)(ii), $1 \in \overline{S}$, and therefore $S \cup \{1\} \subseteq \overline{S}$. Now let p be any real number such that $p \notin S \cup \{1\}$. Clearly, if $p \leq 0$ or $p > 1$, then we can find $r > 0$ such that $(p - r,\ p + r) \cap S = \emptyset$. This shows that $p \notin \overline{S}$, since some neighbourhood of p misses S. On the other hand, if $0 < p < 1$, we can find $m \in \mathbb{N}$ such that $1 - \frac{1}{m} < p < 1 - \frac{1}{m+1}$. Therefore the open interval $\left(1 - \frac{1}{m},\ 1 - \frac{1}{m+1}\right)$ is a neighbourhood of p which misses S; showing that $p \notin \overline{S}$. In all this shows that, for any $x \in \mathbb{R}$, if $x \notin S \cup \{1\}$, then $x \notin \overline{S}$. This implies $\overline{S} \subseteq S \cup \{1\}$, so that $\overline{S} = S \cup \{1\}$. Consequently, $\text{bd}(S) = \overline{S} \cap \overline{\mathbb{R}\backslash S} = (S \cup \{1\}) \cap \mathbb{R} = \{1\} \cup \left\{1 - \frac{1}{n} \,|n \in \mathbb{N}\right\}$.

QUESTION 2

(a) Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$. We have that $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$. Therefore

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

(b) (i) Let $m, n \in Y$. Then

$$d(m,\ n) = \min\{|m - n|, 1\} \leq 1.$$

Thus, $d(a,b) \leq 1$ for all $a, b \in Y$. Therefore Y is bounded.

(ii) Since $\text{diam}(Y) = \sup\{d(a,\ b)\,|a,\ b \in Y\}$, it follows from the above calculation in part (i), that $\text{diam}(Y) \leq 1$. Taking $m = 3$ and $n = -7$ (for instance), we have that

$$\begin{aligned} d(3, -7) &= \min\{|3 - (-7)|, 1\} \\ &= \min\{10, 1\} \\ &= 1. \end{aligned}$$

Therefore $1 \in \{d(a,b) \mid a, b \in Y\}$, whence we deduce that $\sup\{d(a,b) \mid a, b \in Y\} = 1$, that is, $\text{diam}(Y) = 1$.