Tutorial letter 201/1/2018

REAL ANALYSIS

MAT3711

Semester 1

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 01.

BARCODE

ASSIGNMENT 01
Solution
Total Marks: 40
UNIQUE ASSIGNMENT NUMBER: 791595

Question 1: 20 Marks

(1.1) Let $A_1, A_2, \cdots, A_n$ be countable sets. Prove that the product (8)

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) \mid a_i \in A_i\}$$

is countable.
Solution: *To show that*

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) \mid a_i \in A_i\}$$

is countable, we first show that $A_1 \times A_2 \times \cdots \times A_n$ is countable and we then continue by induction to conclude. Let $\mathbb{N}$ be the set of natural numbers.
Choose surjective functions $f\colon \mathbb{N} \to A_1$ and $g\colon \mathbb{N} \to A_2$. Then the function $h\colon \mathbb{N} \times \mathbb{N} \to A_1 \times A_2$ given by $h(1,2) = f(1) \circ g(2) = (a_1, a_2)$ is surjective, so that $A_1 \times A_2$ is countable. We continue by induction. Suppose $A_1 \times A_2 \times \cdots \times A_{n-1}$ is countable since each A_i is countable. We note that there is a bijective correspondence

$$j\colon A_1 \times A_2 \times \cdots \times A_n \to (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n.$$

defined by

$$j(a_1, a_2, \cdots, a_n) = ((a_1, a_2, \cdots, a_{n-1}), a_n).$$

Since $A_1 \times A_2 \times \cdots \times A_{n-1}$ is countable by induction, and A_n is countable by hypothesis, the product of the two sets is countable. We therefore conclude that $A_1 \times A_2 \times \cdots \times A_n$ is countable.

(1.2) Let A be a subset of a metric space X.

(a) Prove that $A^0, (X \setminus A)^0$ and $\mathrm{bd}(A)$ are pairwise disjoint sets whose union is X. (5)

Solution: *In other words, every point of X satisfies exactly one of the following properties: it is interior to A, interior to $X\setminus A$, or on the common boundary $bd(A) = bd(X\setminus A)$. Since $X \setminus A^0 = \overline{(X \setminus A)}$ by Activity 03.1 (f(i)) on page 39, the assertion is exactly the statement*

$$\overline{X \setminus A} = bd(A) \cup (X \setminus A)^0$$

with $bd(A)$ disjoint from $(X \setminus A)^0$. But by definition of boundary for $X \setminus A$ we have a disjoint union decomposition

$$\overline{X \setminus A} = bd(X \setminus A) \cup (X \setminus A)^0.$$

Therefore, we only need to show that $bd(A) = bd(X \setminus A)$. Now $bd(A) = \overline{A} \cap \overline{X \setminus A}$. Then since the right side is unaffected by replacing A with $X \setminus A$ everywhere (because $X \setminus (X \setminus A) = A$), it follows that $bd(A) = bd(X \setminus A)$.

(b) Prove that $\overline{A} = A^0 \cup \mathrm{bd}(A)$. (7)

Solution: *Suppose, $x \in \overline{A}$. So x is an adherent point of S. If for some $r > 0$, the open ball $B(x, r)$ is contained in A, then $x \in A^0$. If not, then for all positive integers n, the open ball $B(x, \frac{1}{n})$ intersects the complement of A, so that $x \in bd(A)$. Conversely, it is immediate that any point in $bd(A)$ or A^0 belongs to A. We therefore conclude that $\overline{A} = A^0 \cup bd(A)$.*

Question 2: 20 Marks

(2.1) Show that the intersection of an arbitrary collection of compact sets is compact. (10)

Solution:
Let $\{K_i \mid i \in I\}$ be a family of compact sets, and let $K = \bigcap K_i$ denote their intersection. We will show that K is compact by showing that it is closed and bounded. Each K_i is bounded (because it is compact) and $K \subseteq K_i$ (for all i) so K must be bounded, as any bound for K_i will also be a bound for K. The set K is also closed because the intersection of closed sets is a closed set.

(2.2) Show that a closed subset of a complete metric space is compact if and only if it is totally bounded. (10)

Solution:
Let A be a closed subset of a complete metric space (X, d). If A is compact then by Theorem 4.02.13, and Theorem 4.02.10, A is totally bounded (and also complete).
Conversely, suppose A is a totally bounded closed subset of a complete metric space (X, d) then A is complete, by Theorem 4.02.10. So, by Theorem 4.02.13, we have that A is compact.