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UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT3711

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REAL ANALYSIS

Duration 2 Hours

100 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT

Closed book examination

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This paper consists of 3 pages
Answer ALL questions

[TURN OVER]

QUESTION 1

Let (X, d) be a metric space, $a \in X$ and $r \in \mathbb{R}$ such that $r > 0$

(a) Define each of the following concepts

- (i) The closed ball with centre a and radius r (1)
- (ii) A neighbourhood of a (1)
- (iii) An interior point of a set $S \subseteq X$ (1)
- (iv) An open subset of X (1)
- (v) A closed subset of X (1)
- (vi) Adherent points (2)
- (vii) A bounded set $A \subseteq X$ (2)
- (viii) A diameter of a set $A \subseteq X$ (2)
- (ix) A Cauchy sequence (2)

(b) Prove that X and $\emptyset$ are closed sets (3)

(c) Let $\{A_\alpha \mid \alpha \in \Gamma\}$ be any collection of open subsets of X Prove that $\bigcup_{\alpha \in \Gamma} A_\alpha$ is open in X (5)

(d) Prove that if x_0 is an element of X, then $\{x_0\}$ is closed (5)

(e) Let A and B be subsets of $\mathbb{R}$ each bounded above Define the subset C of $\mathbb{R}$ by

$$C = \{x + y \mid x \in A \text{ and } y \in B\}$$

Prove that $\sup C = \sup A + \sup B$ (9)

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QUESTION 2

Let (X, d) be a metric space

(a) Define each of the following concepts

- (i) A complete metric space (2)
- (ii) A compact subset $Y \subseteq X$ (3)

(b) Let $\{x_n\}$ be a sequence in (X, d) Prove that $x_n \to p$ if and only if for every neighbourhood U of p, $\{x_n\}$ is eventually in U (5)

(c) Prove that every Cauchy sequence is bounded (5)

(d) Let $K \subseteq X$ be compact and $C \subseteq X$ be closed Use the definition of compactness to show that $K \cap C$ is compact (10)

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QUESTION 3

(a) Consider the set $X = [1, \infty)$ endowed with usual metric d, that is $d(x, y) = |x - y|$ Let λ be a real number with $0 < \lambda < 2$ Let $f : X \to X$ be defined by

$$f(x) = \frac{\lambda + x}{1 + x}$$

- (i) The metric space (X, d) is complete Give full reasons why this is so (4)
- (ii) Show that f is a contraction on X (6)
- (iii) From (i) and (ii) how do we know that f has a unique fixed point? (1)
- (iv) Find the fixed point of f (2)

(b) Let $T : V \to W$ be a bounded linear operator

- (i) How is the operator norm $\|T\|$ defined? (2)
- (ii) For which vectors $v \in V$ does the inequality

$$\|Tv\| \leq \|T\|\|v\|$$

hold? (2)

(c) Consider $\mathbb{R}^2$ with its usual norm $\|(x, y)\| = \sqrt{x^2 + y^2}$, and $\mathbb{R}$ with norm equal to the absolute value Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the linear operator defined by

$$T(x, y) = x + 3y$$

for all $(x, y) \in \mathbb{R}^2$

- (i) Show that the linear operator T is bounded Do not show linearity (4)
- (ii) Evaluate $\|T\|$ (4)

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QUESTION 4

(a) Define the Riemann-Stieltjes integral (10)

(b) Let f and α be functions defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \qquad \alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Compute $\int_0^1 f \, d\alpha$ if it exists (5)

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TOTAL: 100 Marks

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