UNIVERSITY EXAMINATIONS

UNIVERSITEITEKSAMENS

UNISA university of south africa

MAT3711

October/November 2014

REAL ANALYSIS

Duration · 2 Hours

100 Marks

EXAMINATION PANEL AS APPROVED BY THE DEPARTMENT

This paper consists of 3 pages

Answer ALL questions

QUESTION 1

(a) Let (X, d) be a metric space, $p \in X$ and r be a real number with $r > 0$ Define each of the following

- (i) The ball with centre p and radius r. (3)
- (ii) An interior point of a set $S \subseteq X$ (3)
- (iii) An open subset of X (3)
- (iv) The closure of a set $S \subseteq X$ (3)

(b) Let A be a subset of a metric space, and suppose the boundary of A is empty Show that A is both open and closed (10)

(c) Give an example (with reasons) of a metric space which is not complete (3)

[25]

QUESTION 2

(a) Let X be a metric space and $S \subseteq X$

- (i) What is meant by an open cover of S? (3)
- (ii) What does it mean to say S is compact? (2)

(b) Prove the theorem which states, "Every sequentially compact metric space is totally bounded and complete." (10)

[TURN OVER]

(c) Let $\{f_n\}$ be a sequence of functions mapping $[0, \infty)$ into $\mathbb{R}$ defined by

$$f_n(x) = x\left(1 + \frac{1}{n}\right)$$

for each $n \in \mathbb{N}$ and $x \in [0, \infty)$

(i) Show that there is no function to which $\{f_n\}$ converges uniformly on $[0, \infty)$ (5)

(ii) Let A be a bounded subset of $[0, \infty)$ Now show that $\{f_n\}$ converges uniformly on A to some function $f \; [0, \infty) \to \mathbb{R}$ (5)

[25]

QUESTION 3

(a) Let X, Y be metric spaces and $f \; X \to Y$ be a function Prove that f is continuous if and only if $f^{-1}[G]$ is open in X for each open subset G of Y (10)

(b) Let $\{a_n\}$ be a sequence of real numbers which is bounded above and let u_n be defined by

$$u_n = \sup\{a_n | k \geq n\}$$

for each $n \in \mathbb{N}$ Then $\{u_n\}$ is an increasing sequence. (Do not prove this)

(i) Show that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} u_n$ (10)

(ii) Find u_k, and hence $\limsup_{n\to\infty} a_n$, if

$$a_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ odd} \\ 1 - \frac{1}{n} & \text{for } n \text{ even} \end{cases}$$

(5)

[25]

[TURN OVER]

QUESTION 4

(a) Equip $\mathbb{R}$ with the norm equal to absolute value, and $\mathbb{R}^2$ with the norm $||(x,y)|| = \sqrt{x^2+y^2}$
Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the linear operator defined by

$$T(x,y) = x + 2y$$

for all $(x,y) \in \mathbb{R}^2$

(i) Show that T is a bounded linear operator. (*Do* <u>*not*</u> *show linearity, show only boundedness.*) (5)

(ii) Compute $||T||$. (3)

(b) Let $f : [1,2] \to [1,2]$ be the function defined by

$$f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

(i) Show that f is a contraction on $[1,2]$ (5)

(ii) Without finding the fixed point, explain fully (giving reasons) why we know that f has a unique fixed point in $[1,2]$ (5)

(c) Let f and α be functions defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \qquad \text{and} \qquad \alpha(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

(i) Calculate $\underline{\int_0^1} f\, d\alpha$ and $\overline{\int_0^1} f\, d\alpha$ (5)

(ii) Does $\int_0^1 f\, d\alpha$ exist? Give reasons (2)

[25]

TOTAL: 100 Marks

First examiner: Prof SJ Johnston
External examiner Prof S Currie (University of the Witwatersrand)