UNISA university of south africa

MAT3711

May/June 2013

REAL ANALYSIS

Duration 2 Hours 100 Marks

EXAMINATION PANEL AS APPROVED BY THE DEPARTMENT

This paper consists of 3 pages

Answer ALL questions

QUESTION 1

Let (X, d) be a metric space, $p \in X$ and $r \in \mathbb{R}$ such that $r > 0$

(a) Define each of the following concepts

- (i) The ball with centre p and radius r (1)
- (ii) An interior point of a set $A \subseteq X$ (1)
- (iii) An open subset of X (1)
- (iv) A closed subset of X (1)
- (v) A neighbourhood of p (1)
- (vi) A bounded set $A \subseteq X$ (1)
- (vii) The diameter of a set $A \subseteq X$ (2)

(b) Let X be the set of real numbers and let d be the metric on X given by

$$d(x, y) = \min\{|x - y|, 2\}$$

where $x, y \in X$ Let $A \subseteq X$ be the set of integers

- (i) Is A bounded in (X, d)? Give reasons for your answer (3)
- (ii) Calculate the diameter of A (5)

(c) Let (X, d) be a metric space and $S \subseteq X$ be equipped with the subspace metric, i e (S, d_S) is the metric space where d_S is the restriction of d to $S \times S$ Prove that a set $A \subseteq S$ is open in S if and only if $A = S \cap U$ for some set $U \subseteq X$ which is open in X (11)

[27]

[TURN OVER]

QUESTION 2

(a) Let $\langle\ ,\ \rangle$ be an inner product Show that if $\langle x,y\rangle = \langle x,z\rangle$ for all x, then $y = z$ (5)

(b) Prove the Cauchy-Bunyakowski-Schwarz inequality

If V is an inner product space, then for all $x, y \in V$

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\ \ \sqrt{\langle y,y\rangle}$$

(8)

(c) Let $[a,b]$ be a compact interval in $\mathbb{R}$ A function $f\ \ [a,b] \to \mathbb{C}$ is continuous if and only if its real and imaginary parts are continuous real-valued functions, i e if for $f(t) = u(t) + iv(t)$ where $u\ \ [a,b] \to \mathbb{R}$ and $v\ \ [a,b] \to \mathbb{R}$, u and v are continuous For such a function, the integral $\int_a^b f$ is defined by

$$\int_a^b f = \int_a^b u + i\int_a^b v$$

Let V be the set of all continuous complex-valued functions on $[a,b]$ Define $\langle\ ,\ \rangle$ on V by

$$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\ dt$$

Show that $(V, \langle\ ,\ \rangle)$ is an inner product space (10)

[23]

QUESTION 3

(a) Let (X,d) be a metric space and $S \subseteq M \subseteq X$

(i) What is meant by an open cover of S? (2)

(ii) What does it mean to say that S is compact? (1)

(iii) Show that S is compact in (X,d) if and only if it is compact in (M, d_M) (12)

(b) Prove the theorem which states, "Every sequentially compact metric space is totally bounded and complete " (12)

(c) Explain why $\mathbb{R}$ is not compact (3)

[30]

[TURN OVER]

QUESTION 4

(a) Define the Riemann-Stieltjes integral (12)

(Hint Be sure to define all notations used, for example partition, sub-interval, length of sub-interval, upper Stieltjes integral, lower Stieltjes integral, etc)

(b) Let f and α be functions defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

(i) Compute $\underline{\int_0^1} f \, d\alpha$ and $\overline{\int_0^1} f \, d\alpha$ (6)

(ii) Does $\int_0^1 f \, d\alpha$ exist? Give reasons (2)

[20]

TOTAL: 100 Marks

First examiner	Prof SJ Johnston
Second examiner	Dr JRA Gray
External examiner	Dr S Currie (University of the Witwatersrand)