UNISA university of south africa

# MAT3711

October/November 2013

## REAL ANALYSIS

Duration : 2 Hours 100 Marks

EXAMINATION PANEL AS APPROVED BY THE DEPARTMENT.



This paper consists of 3 pages.
Answer ALL questions

### QUESTION 1

Let $(X, d)$ be a metric space, $p \in X$ and $r \in \mathbb{R}$ such that $r > 0$

(a) Define each of the following concepts

- (i) The ball with centre $p$ and radius $r$. (1)
- (ii) An interior point of a set $A \subseteq X$. (1)
- (iii) An open subset of $X$. (1)
- (iv) A closed subset of $X$. (1)
- (v) A neighbourhood of $p$. (1)
- (vi) A Cauchy sequence. (2)
- (vii) A complete metric space. (2)

(b) Let $(X,\ d)$ be a metric space and let $\rho : X \times X \to \mathbb{R}$ be the metric on $X$ given by

$$\rho(x,\ y) = \min\{d(x,\ y),\ 1\}$$

- (i) Show that every Cauchy sequence in $(X, \rho)$ is a Cauchy sequence in $(X, d)$. (8)
- (ii) Show that if $(X,\ d)$ is complete, then $(X,\ \rho)$ is complete. (6)

**[23]**

[TURN OVER]

**QUESTION 2**

(a) Let $\langle\ ,\ \rangle$ be an inner product. Show that if $\langle x, y\rangle = \langle x, z\rangle$ for all $x$, then $y = z$. (5)

(b) Prove the Cauchy-Bunyakowski-Schwarz inequality:

If $V$ is an inner product space, then for all $x, y \in V$

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\ \sqrt{\langle y, y\rangle}.$$

(8)

(c) Let $[a, b]$ be a compact interval in $\mathbb{R}$. A function $f : [a, b] \to \mathbb{C}$ is continuous if and only if its real and imaginary parts are continuous real-valued functions, i.e. if for $f(t) = u(t) + iv(t)$ where $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$, $u$ and $v$ are continuous. For such a function, the integral $\int_a^b f$ is defined by

$$\int_a^b f = \int_a^b u + i \int_a^b v.$$

Let $V$ be the set of all continuous complex-valued functions on $[a, b]$. Define $\langle\ ,\ \rangle$ on $V$ by

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}\ dt$$

Show that $(V, \langle\ ,\ \rangle)$ is an inner product space (10)

**[23]**

**QUESTION 3**

(a) Define each of the following concepts

(i) A contraction mapping. (3)

(ii) A compact set. (3)

(b) Let $(X,\ d)$ be a complete metric space, and suppose $T : X \to X$ is a function such that $T^2$ is a contraction where $T^2$ is the function $T^2 : X \to X$ given by $T^2(x) = T(T(x))$. Show that $T$ has a unique fixed point in $X$. (12)

(c) Let $Y = [0, 1)$ with its usual metric Prove that $Y$ is not compact. (10)

(d) Prove the following theorem:

Let $(X, d)$ be a metric space, $K \subseteq X$ be compact and $p \in X \backslash K$. Then there exist disjoint open sets $U$ and $V$ such that $K \subseteq U$ and $p \in V$.

(6)

**[34]**

**[TURN OVER]**

## QUESTION 4

(a) Define the Riemann-Stieltjes integral (12)
(Hint: Be sure to define all notations used, for example partition, sub-interval, length of sub-interval, upper Stieltjes integral, lower Stieltjes integral, etc.)

(b) Let $f$ and $\alpha$ be functions defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

(i) Compute $\underline{\int_0^1} f\, d\alpha$ and $\overline{\int_0^1} f\, d\alpha$. (6)

(ii) Does $\int_0^1 f\, d\alpha$ exist? Give reasons. (2)

**[20]**

**TOTAL: 100 Marks**

First examiner: Prof SJ Johnston
Second examiner: Dr JRA Gray
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