UNIVERSITY EXAMINATIONS

UNIVERSITEITEKSAMENS

UNISA university of south africa

MAT3711

May/June 2012

REAL ANALYSIS

Duration 2 Hours

100 Marks

EXAMINATION PANEL AS APPROVED BY THE DEPARTMENT.

This paper consists of 3 pages

Answer ALL questions

QUESTION 1

Let (X, d) be a metric space, $p \in X$ and $r \in \mathbb{R}$ such that $r > 0$

(a) Define each of the following concepts

  (i) The ball with centre p and radius r (1)

  (ii) An interior point of a set $A \subseteq X$ (1)

  (iii) An open subset of X (1)

  (iv) A closed subset of X (1)

  (v) The boundary of A (2)

(b) Prove that every ball in (X, d) is open (7)

(c) Let $S = \{\frac{1}{n} - 3 \quad n \in \mathbb{N}\}$ be viewed as a subset of $\mathbb{R}$ with the usual metric Find the boundary of S (8)
(Give full justification for your answer)

[21]

[TURN OVER]

QUESTION 2

Let (X, d) be a metric space and $\{a_n\}$ be a sequence in X

(a) Define each of the following concepts

(i) $\{a_n\}$ converges in X (1)

(ii) $\{a_n\}$ is a Cauchy sequence (2)

(iii) A complete metric space (2)

(b) Prove that every convergent sequence is a Cauchy sequence (4)

(c) True or false

Every Cauchy sequence is a convergent sequence

Prove or give a counter example (4)

(d) Consider $\mathbb{R}^2$ with its usual norm $||(x,y)|| = \sqrt{x^2+y^2}$, and $\mathbb{R}$ with norm equal to the absolute value Let T $\mathbb{R}^2 \to \mathbb{R}$ be the linear operator defined by

$$T(x,y) = x - 3y$$

for all $(x, y) \in \mathbb{R}^2$

(i) Show that the linear operator T is bounded (4)

(ii) Evaluate $||T||$ (4)

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QUESTION 3

(a) Define each of the following concepts

(i) A *compact* set (3)

(ii) A is *dense* in X if A is a subset of a metric space X (2)

(b) Show that the union of a finite number of compact sets is a compact set (5)

(c) Let A be a subset of a metric space X Show that A is dense in X if and only if $\text{int}(X - A) = \emptyset$ (10)

(d) Let f $\mathbb{R} \to \mathbb{R}$ be a continuous function and define g $\mathbb{R} \to \mathbb{R}^2$ by

$$g(x) = (f(x), x)$$

Use the definition of continuity to show that g is continuous if $\mathbb{R}$ and $\mathbb{R}^2$ have the usual metrics. (10)

[30]

[TURN OVER]

QUESTION 4

(a) Define the Riemann-Stieltjes integral (12)
(Hint. Be sure to define all notations used, for example partition, sub-interval, length of sub-interval, upper Stieltjes integral, lower Stieltjes integral, etc)

(b) Let f and α be functions defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Compute $\int_0^1 f\, d\alpha$ if it exists (8)

(c) The Fundamental Theorem of Calculus states
If f is Riemann-integrable on $[a, b]$ and there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Prove this theorem (8)

[28]

TOTAL: 100 Marks

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Second examiner	Prof I Naidoo
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