QUESTION 1

Let z and w be complex numbers with $z \neq w$ and $z\bar{z} = |z|^2 = 1$. Show that we then have that

$$\left|\frac{z-w}{\bar{z}w-1}\right| = 1.$$

Solution:
Note that $|\bar{z}| = |z| = 1$. Hence

$$\begin{aligned}
\left|\frac{z-w}{\bar{z}w-1}\right| &= \frac{|\bar{z}|\,|z-w|}{|\bar{z}w-1|} \qquad (\text{since } |\bar{z}| = 1) \\
&= \frac{|\bar{z}z-\bar{z}w|}{|\bar{z}w-1|} \\
&= \frac{|1-\bar{z}w|}{|\bar{z}w-1|} \qquad (\text{since } 1 = |z|^2 = z\bar{z}) \\
&= 1.
\end{aligned}$$

QUESTION 2

Let $z \to e^z$ be the function defined by $e^z = e^x(\cos y + i\sin y)$ where $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. Now let

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

and

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

Show that the equation $\tan z = i$ and $\tan z = -i$ where $\tan z = \dfrac{\sin z}{\cos z}$ have no solution.
(**Hint:** $e^w \neq 0$ for all $w \in \mathbb{C}$.)

Solution:
If it were possible that $\tan z = i$, we would have

$$\begin{aligned}
\frac{1}{2i}\left(e^{iz} - e^{iz}\right) &= \sin z = i\cos z = \frac{i}{2}\left(e^{iz} + e^{-iz}\right) \\
\Rightarrow \quad e^{iz} - e^{-iz} &= -\left(e^{iz} + e^{-iz}\right) \\
\Rightarrow \quad 2e^{iz} &= 0.
\end{aligned}$$

But this is impossible since $e^{iz} \neq 0$ for all z. Similarly $\tan z = -i$ has no solution.

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QUESTION 3

3.1 Let f be a complex function defined on an open set G. Let

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are the so-called real and imaginary parts of f. Show that if f is differentiable at $z_0 = x_0 + iy_0 \in G$, then the first partial derivatives of u and v exist at (x_0, y_0) and satisfy

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0). \end{aligned}$$

3.2 Discuss the differentiability of the function

$$f(z) = x^2 + iy^2$$

where $z = x + iy$ and explain why f is nowhere holomorphic.

Solution:

3.1

3.2 Here $u = x^2$ and $v = y^2$. Thus

$$\begin{aligned} u_x &= 2x \qquad v_y = 2y \\ u_y &= 0 \qquad v_x = 0 \end{aligned}$$

The Cauchy–Riemann equations can only be satisfied on the line $x = y$. Thus in any open region we can always find points not on this line and hence points where f is not differentiable. By definition f is therefore nowhere holomorphic, since for any neighbourhood of any point f is never differentiable in all of that neighbourhood.

QUESTION 4

Let C denote the positively oriented boundary of the square with vertices

$$2 + 2i, \; 2 - 2i, \; -2 - 2i \quad \text{and} \quad -2 + 2i.$$

Now evaluate each of the following:

4.1 $\int_C \frac{\cos z}{z(z^2+8)}dz$

4.2 $\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-x_0)^2}dz$ where x_0 is a real number with $-2 < x_0 < 2$.

Solution:

4.1 The only singularities of $\frac{\cos z}{z(z^2+8)}$ are at $z = 0$ and $z = \pm 2\sqrt{2}i$. Of these only $z = 0$ lies inside the square. Hence $\frac{\cos z}{z^2+8}$ is holomorphic inside and on C. Thus

$$\begin{aligned}\int_C \frac{\cos z}{z(z^2+8)}dz &= \int_C \frac{\cos z/\left(z^2+8\right)}{z}dz \\ &= 2\pi i\left(\frac{\cos 0}{8}\right) = \frac{\pi i}{4}.\end{aligned}$$

4.2 $\tan\left(\frac{z}{2}\right)$ has singularities where $\cos(z/2) = 0$, i.e. where $\frac{z}{2} = (2k+1)\frac{\pi}{2}$. But all the points

$$z = (2k+1)\pi \quad (k \in \mathbb{Z})$$

lie outside the curve C (since $\pi > 2$) and hence $\tan\frac{z}{2}$ is holomorphic inside and on C. Thus

$$\begin{aligned}\int_C \frac{\tan\left(\frac{z}{2}\right)}{(z-x_0)}dz &= 2\pi i\frac{d}{dz}\left(\tan\frac{z}{2}\right)\Big|_{z=x_0} \\ &= i\pi\sec^2 x_0.\end{aligned}$$

QUESTION 5

Let C be a positively oriented closed contour, let f be holomorphic inside and on C, and let z_0 be a point in the interior of C. Explain with full justification why it then follows that

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz.$$

Solution:
If f is holomorphic on a domain it has derivatives of all orders and hence f is then also holomorphic on that domain. We may therefore apply Cauchy's integral formulae to both f and f' to get

$$\int_C \frac{f(z)}{(z-z_0)^2}dz = \frac{2\pi i}{1!}f'(z_0) = 2\pi i f'(z_0) = \int_C \frac{f'(z)}{z-z_0}dz$$

as required. (Here we have applied Theorem 5.4 with $n = 1$ to f and Theorem 5.1 to f'.)

QUESTION 6

Show that if a function is both holomorphic and bounded on $\mathbb{C}$, it must be constant.

Solution:

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QUESTION 7

Suppose that both f and g are holomorphic in the disc

$$\{z \in \mathbb{C} \mid |z| < 1\}$$

and that $f \cdot g = 0$. Show that at least one of f or g must then be identically zero on the disc.
(**Hint:** Use the identity theorem.)

Solution:
If both f and g have finitely many zeros in some closed subdisc, say $|z| \leq \frac{1}{2}$, then so will $f \cdot g$. Thus since $fg = 0$ on all of the disc $|z| < 1$, at least one of f and g (say f) must have infinitely many zeros in the subdisc $|z| \leq \frac{1}{2}$. But then these zeros have a cluster point in $|z| \leq \frac{1}{2}$. In particular they will have a cluster point inside $|z| < 1$. Thus by the identity theorem (Theorem 5.14), f itself is zero.

QUESTION 8

Give two Laurent expansions in powers of z of the function $\dfrac{1}{z^2(1-z)}$ and specify the region of validity of each.

Solution:
If $|z| < 1$ then

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

whereas if $|z| > 1$ $\left(\text{i.e. } \frac{1}{|z|} < 1\right)$ then

$$\frac{1}{1-z} = \left(-\frac{1}{z}\right)\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right)$$

Thus

$$\begin{aligned} \frac{1}{z^2(1-z)} &= \frac{1}{z^2}\left(1 + z + z^2 + \ldots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + \ldots \quad \text{if } 0 < |z| < 1 \end{aligned}$$

and

$$\begin{aligned}\frac{1}{z^2(1-z)} &= \frac{1}{z^2}\left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\dots\right)\\ &= -\frac{1}{z^3}-\frac{1}{z^4}-\frac{1}{z^5}-\dots \quad \text{if } |z|>1.\end{aligned}$$

QUESTION 9

9.1 Locate only those singularities of the function

$$z \longrightarrow \frac{2z^2-1}{(z^2+4)(z^2+1)}$$

which lie in the upper half–plane (the region $\operatorname{Im}(z) > 0$) and determine the residue at each of these singularities.

9.2 Evaluate

$$\int_{\infty}^{\infty} \frac{2x^2-1}{(x^2+4)(x^2+1)}dx$$

using the method of residues. Justify your assertions.

Solution:

9.1 The singularities of

$$\frac{2z^2-1}{(z^2+4)(z^2+1)}$$

are at $z^2 = -4$ and $z^2 = -1$, i.e. at $z = \pm 2i$ and $z = \pm i$. Of these $z = 2i$ and $z = i$ lie in the upper halfplane. Both of these are simple poles and so

$$\begin{aligned}\operatorname{Res}\{f(z);\ 2i\} &= \lim_{z\to 2i}\frac{(z-2i)(2z^2-1)}{(z-2i)(z+2i)(z^2+1)}\\ &= -\frac{9}{4i(-3)} = -\frac{3}{4}i\\ \operatorname{Res}\{f(z);\ i\} &= \lim_{z\to i}\frac{(z-i)(2z^3-1)}{(z-i)(z+i)(z^2+4)}\\ &= -\frac{3}{2i(3)} = \frac{1}{2}i.\end{aligned}$$

9.2 We integrate around the region bounded by $[-R, R]$ and C_R: $z(t) = R\,e^{it}$, $0 \le t \le \pi$. For $R > 0$ big enough both $2i$ and i lie inside this region and so by the residue theorem

$$\int_{-R}^{R} \frac{2x^2-1}{(x^2+4)(x^2+1)}dx + \int_{C_R} \frac{2z^2-1}{(z^2+4)(z^2+1)}dz$$
$$= 2\pi i\left(-\frac{3}{4}i + \frac{1}{2}i\right) = \frac{\pi}{2}.$$

But since the length of C_R is $R\pi$ we get

$$\left|\int_{C_R} \frac{2z^2-1}{(z+4)(z^2+1)}dz\right| \le \frac{2R^2+1}{(R^2-4)(R^2-1)}\pi R.$$

Now as $R \to \infty$, the RHS tends to 0 and so

$$\int_{-\infty}^{\infty} \frac{2x^2-1}{(x^2+4)(x^2+1)}dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{2x^2-1}{(x^2+4)(x^2+1)}dx = \frac{\pi}{2}.$$