Tutorial letter 202/2/2018

LINEAR ALGEBRA

MAT3701

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions to Assignment 02.

BARCODE

ASSIGNMENT 02
Solution
UNIQUE ASSIGNMENT NUMBER: 711122

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the exam. It is therefore good practice to work through a complete set of four assignments for a given year, and for this reason the assignments and worked solutions of previous years are posted on myUnisa under Additional Resources – see also the letter **MAT3701 Exam Preparation** under Additional Resources.

Worked solutions to all the questions for this assignment will be sent to all students and posted on myUnisa shortly after the due date. Your answers to the assignment questions should be fully motivated.

For this assignment, questions 1,2, 7 and 8 will be marked.

Question 1

Let $<,>$ denote the standard inner product on C^n, i.e. $< x, y >= y^*x$ where x and y are column vectors in C^n. Let $A \in M_{n\times n}(C)$ be a non-singular matrix. Show that $<,>_1$: $C^n \times C^n \to C$ defined by $< x, y >_1 = < Ax, Ay >$ is an inner product on C^n over C. **[20]**

Solution

Let x, y and z be column vectors in C^n and let $c \in C$.

IP1

$$\begin{aligned}\langle x+z, y\rangle_1 &= \langle A(x+z), Ay\rangle, \quad \text{definition of } <,>_1\\ &= \langle Ax + Az, Ay\rangle\\ &= \langle Ax, Ay\rangle + \langle Az, Ay\rangle, \quad \text{since } <,> \text{ is an inner product}\\ &= \langle x, y\rangle_1 + \langle z, y\rangle_1, \quad \text{definition of } <,>_1\end{aligned}$$

Thus IP1 is satisfied.

IP2

$$\begin{aligned}\langle cx, y\rangle_1 &= \langle A(cx), Ay\rangle, \quad \text{definition of } <,>_1\\ &= \langle cAx, Ay\rangle\\ &= c\langle Ax, Ay\rangle, \quad \text{since } <,> \text{ is an inner product}\\ &= c\langle x, y\rangle_1, \quad \text{definition of } <,>_1\end{aligned}$$

Thus IP2 is satisfied.

IP3

$$\begin{aligned}\overline{\langle x,y\rangle_1} &= \overline{\langle Ax,Ay\rangle}, \quad \text{definition of } <,>_1 \\ &= \langle Ay,Ax\rangle, \quad \text{since } <,> \text{ is an inner product} \\ &= \langle y,x\rangle_1, \quad \text{definition of } <,>_1\end{aligned}$$

Thus IP3 is satisfied.

IP4

$$\langle x,x\rangle_1 = \langle Ax,Ax\rangle \geq 0, \quad \text{since } <,> \text{ is an inner product}$$

and

$$\begin{aligned}\langle x,x\rangle_1 = 0 &\Leftrightarrow \langle Ax,Ax\rangle = 0, \quad \text{definition of } <,>_1 \\ &\Leftrightarrow Ax = 0, \quad \text{since } <,> \text{ is an inner product} \\ &\Leftrightarrow x = 0, \quad \text{since } A \text{ is non-singular}\end{aligned}$$

Thus IP4 is satisfied, and therefore $<,>$ is an inner product on C^n.

Question 2

Given that the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 6 \\ 3x_1 + x_2 &= 0 \end{aligned}$$

is inconsistent, find a least squares approximate solution in R^2 (see SG: Section 12.6, p 120). **[20]**

Solution

Use the method described in Section 12.6 of the Study Guide. Denote the coefficient matrix and constant column of the system by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

respectively. Then

$$A^*A = \begin{bmatrix} 11 & 3 \\ 3 & 3 \end{bmatrix} \quad \text{and} \quad A^*y = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

The least squares approximate solution x_0 is obtained from $A^*Ax_0 = A^*y$, and yields

$$x_0 = \frac{1}{2}\begin{bmatrix} 3 \\ -7 \end{bmatrix} \text{ (show).}$$

Question 3

This question was not marked. It is still important to work through the solutions and compare them with your own attempts.

Let $T : C^3 \to C^3$ be the linear operator defined by

$$T(z_1, z_2, z_3) = (z_2 + z_3, iz_1 + z_2 + z_3,\ z_2 + z_3)$$

(a) Find a formula for $T^*(z_1, z_2, z_3)$.

(b) Find a basis for $N(T)$.

(c) Find a basis for $R(T^*)$.

(d) Show that $R(T^*) = N(T)^\perp$.

Solution

(a) Use Theorem 6.10. Let β be the standard basis for C^3, i.e.

$$\beta = \{(1,0,0), (0,1,0), (0,0,1)\}.$$

Then

$$[T]_\beta = \begin{bmatrix} 0 & 1 & 1 \\ i & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{(check)}.$$

Since β is an orthonormal basis for C^3,

$$\begin{aligned} [T^*]_\beta &= [T]_\beta^* \\ &= \begin{bmatrix} 0 & -i & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} [T^*(z_1, z_2, z_3)]_\beta &= [T^*]_\beta [(z_1, z_2, z_3)]_\beta \\ &= \begin{bmatrix} 0 & -i & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ &= \begin{bmatrix} -iz_2 \\ z_1 + z_2 + z_3 \\ z_1 + z_2 + z_3 \end{bmatrix} \end{aligned}$$

and therefore

$$T^*(z_1, z_2, z_3) = (-iz_2,\ z_1 + z_2 + z_3,\ z_1 + z_2 + z_3).$$

(b)

$$
\begin{aligned}
(z_1, z_2, z_3) \in N(T) &\Leftrightarrow T(z_1, z_2, z_3) = (0,0,0) \\
&\Leftrightarrow \begin{cases} z_2 + z_3 = 0 \\ iz_1 + z_2 + z_3 = 0 \\ z_2 + z_3 = 0 \end{cases}
\end{aligned}
$$

Writing the system in matrix form and row reduce yields

$$
\begin{bmatrix} 0 & 1 & 1 \\ i & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}
$$

hence

$$
z_1 = 0, z_3 = t, \ z_2 = -t, \text{where } t \in C
$$

and therefore

$$
\begin{aligned}
N(T) &= \{t(0,-1,1) : t \in C\} \\
&= \operatorname{span}\{(0,-1,1)\}
\end{aligned}
$$

with basis

$$
\{(0,-1,1)\}. \qquad ...(1)
$$

(c)

$$
\begin{aligned}
R(T^*) &= \operatorname{span}\{T^*(1,0,0), T^*(0,1,0), T^*(0,0,1)\} \\
&= \operatorname{span}\{(0,1,1), (-i,1,1), (0,1,1)\}.
\end{aligned}
$$

Writing the vectors in matrix form and row reduce yields

$$
\begin{bmatrix} 0 & 1 & 1 \\ -i & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ iR_2
$$

The nonzero rows from a basis for $R(T^*)$, namely

$$
\{(1,0,0), \ (0,1,1)\}. \qquad ...(2)
$$

(d) From (1)

$$
\begin{aligned}
(z_1, z_2, z_3) \in N(T)^{\perp} &\Leftrightarrow \langle (z_1, z_2, z_3), (0,-1,1) \rangle = 0 \\
&\Leftrightarrow -z_2 + z_3 = 0 \\
&\Leftrightarrow z_1 = t, z_2 = s, z_3 = s, \text{ where } s, t \in C
\end{aligned}
$$

so

$$
\begin{aligned}
N(T)^{\perp} &= \{(t,s,s) : s,t \in C\} \\
&= \{t(1,0,0) + s(0,1,1) : s,t \in C\} \\
&= \operatorname{span}\{(1,0,0), (0,1,1)\}
\end{aligned}
$$

and therefore $R(T^*) = N(T)^{\perp}$ from (2), as desired.

Question 4

This question was not marked. It is still important to work through the solutions and compare them with your own attempts.

Let V be a finite-dimensional inner product space and let $y \in V$ be a unit vector. Define $T : V \to V$ by $T(x) = x - \langle x, y\rangle\, y$ for all $x \in V$.

(a) Show that T is linear.

(b) Show that T is a projection.

(c) Show that T is a self-adjoint.

(d) Briefly explain why T is an orthogonal projection.

(e) Describe the subspace onto which T projects.

Solution

(a) T is a linear operator since for $x, z \in V$ and $a \in F$,

$$\begin{aligned} T(x+z) &= x + z - \langle x+z, y\rangle y \\ &= x + z - (\langle x, y\rangle + \langle z, y\rangle)y, \quad \text{property of the inner product} \\ &= (x - \langle x, y\rangle y) + (z - \langle z, y\rangle y) \\ &= T(x) + T(z) \end{aligned}$$

and

$$T(ax) = ax - \langle ax, y\rangle y = ax - a\langle x, y\rangle y = a(x - \langle x, y\rangle y) = aT(x).$$

(b)

$$\begin{aligned} T^2(x) &= T(x - \langle x, y\rangle\, y) \\ &= T(x) - \langle x, y\rangle\, T(y) \\ &= T(x) - \langle x, y\rangle\, (y - < y, y > y) \\ &= T(x) - \langle x, y\rangle\, (y - y), \quad \text{since } y \text{ is a unit vector} \\ &= T(x) \end{aligned}$$

Thus T is a projection.

(c) For all $x, z \in V$

$$\begin{aligned} \langle T^*(x), z\rangle &= \langle x, T(z)\rangle \\ &= \langle x, z - \langle z, y\rangle\, y\rangle \\ &= \langle x, z\rangle + \langle x, -\langle z, y\rangle\, y\rangle \\ &= \langle x, z\rangle - \overline{\langle z, y\rangle}\langle x, y\rangle \\ &= \langle x, z\rangle - \langle y, z\rangle\langle x, y\rangle \\ &= \langle x, z\rangle + \langle -\langle x, y\rangle y, z\rangle \\ &= \langle x - \langle x, y\rangle\, y, z\rangle \end{aligned}$$

thus

$$T^*(x) = x - \langle x, y \rangle \, y = T(x), \text{ i.e. } T^* = T.$$

(d) This follows form (c), since according to Friedberg: Theorem 6.24 a projection is orthogonal if and only if it is self-adjoint.

(e) T projects onto $R(T)$, and in this case we can show that $R(T) = \{y\}^\perp$ as follows: For all $x \in V$,

$$\begin{aligned} < T(x), y > &= < x- < x, y > y, y > \\ &= < x, y > - < x, y >< y, y > \\ &= < x, y > - < x, y > \quad \text{since } y \text{ is a unit vector} \\ &= 0 \end{aligned}$$

therefore $R(T) \subseteq \{y\}^\perp$. Conversely

$$x \in \{y\}^\perp \Rightarrow T(x) = x- < x, y > y = x$$

hence $x \in R(T)$ and therefore $\{y\}^\perp \subseteq R(T)$. Thus it follows that $R(T) = \{y\}^\perp$.

Question 5

This question was not marked.
It is still important to work through the solutions and compare them with your own attempts.

Let T be a linear operator on a finite-dimensional inner product space V and let W be a T- invariant subspace of V. Show that:

(a) $W^\perp$ is T^*- invariant.

(b) If T is invertible, then W is T^{-1} - invariant.

(c) If T is unitary, then $W^\perp$ is T- invariant.

Solution

(a)

$$\langle T^*(W^\perp), W \rangle = \langle W^\perp, T(W) \rangle = 0 \text{ since } T(W) \subseteq W.$$

Thus $T^*(W^\perp) \subseteq W^\perp$, and hence $W^\perp$ is T^*- invariant.

(b) Since W is T-invariant, T maps W into W. Since T is also one-to-one and W finite dimensional, it follows that T maps W onto W, i.e. $T(W) = W$. Therefore

$$T^{-1}(W) = T^{-1}(T(W)) = (T^{-1}T)(W) = I_V(W) = W$$

and hence W is T^{-1} - invariant.

(c) Since $T^* = T^{-1}$, it follows from (a) that $W^\perp$ is T^{-1}-invariant, and thus it follows from (b) that $W^\perp$ is T- invariant.

Question 6

This question was not marked. It is still important to work through the solutions and compare them with your own attempts.

Let $f : R^2 \to R^2$ be a rigid motion and $v_0 \in R^2$ a fixed vector. Define $(f^{-1} + v_0)(v) = f^{-1}(v) + v_0$ for all $v \in R^2$.

(a) Show that $f^{-1} + v_0$ is a rigid motion.

(b) Suppose f is defined by

$$f(a, b) = (1 + b, 3 + a)$$

Express f in the form $f = g \circ T$ where g is a translation and T an orthogonal operator on R^2. Find the vector v_0 by which g translates. If T is a rotation, find the angle θ through which it rotates. If T is a reflection about a line L through the origin, find the equation of L and the angle α it makes with the positive x-axis.

Solution

(a) For all $v, w \in R^2$

$$\begin{aligned}
\|(f^{-1} + v_0)(v) - (f^{-1} + v_0)(w)\| &= \|(f^{-1}(v) + v_0) - (f^{-1}(w) + v_0)\| \\
&= \|f^{-1}(v) - f^{-1}(w)\| \\
&= \|f(f^{-1}(v)) - f(f^{-1}(w))\| \quad \text{since } f \text{ is a rigid motion} \\
&= \|(f \circ f^{-1})(v) - (f \circ f^{-1})(w)\| \\
&= \|v - w\|
\end{aligned}$$

therefore f^{-1} is a rigid motion.

(b)

$$\begin{aligned}
f(a, b) &= (1, 3) + (b, a) \\
&= (g \circ T)(a, b)
\end{aligned}$$

where

$$g(a, b) = (1, 3) + (a, b) \qquad \text{and} \qquad T(a, b) = (b, a).$$

Hence g translates by $v_0 = (1, 3)$. Let β be the standard ordered basis for R^2, then

$$[T]_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

T is orthogonal since

$$[T]_\beta^* [T]_\beta = [T]_\beta [T]_\beta^* = I_2.$$

Finally, since

$$\det\left([T]_\beta\right) = -1,$$

T is a reflection about a line L through the origin, and since

$$[T]_\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$

it follows that L makes an angle of $\alpha = \frac{\pi}{4}$ with the positive x-axis, and that its equation is given by $y = \tan(\frac{\pi}{4})\ x = x$.

Question 7

Let $V = P_2(R)$ denote the inner product space over R with inner product defined by

$$\langle g, h\rangle = g\,(-1)\,h\,(-1) + g(0)h\,(0) + g\,(1)\,h\,(1)\,.$$

Let $T : V \to V$ be the orthogonal projection on

$$W = \text{span}\,(S) \;\text{ where } S = \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\ x\right\}$$

(a) Show that S is orthonormal. (10)

(b) Find a formula for $T(ax^2 + bx + c)$ expressed as a single polynomial in terms of $a,\ b$ and c.(10)

(c) Use T to express x^2 as $x^2 = f + g$ where $f \in W$ and $g \in W^\perp$. (5)

(d) Find the polynomial in W closest to x^2. (5)

[30]

Solution

(a)

$$\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{1}{3}(1(-1)\cdot 1(-1) + 1(0)\cdot 1(0) + 1(1)\cdot 1(1)) = \frac{1}{3}(1\cdot 1 + 1\cdot 1 + 1\cdot 1) = 1$$

$$\left\langle \frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{2}}x \right\rangle = \frac{1}{2}((-1)\cdot(-1) + 0\cdot 0 + 1\cdot 1) = \frac{1}{2}(1+0+1) = 1$$

$$\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x \right\rangle = \frac{1}{\sqrt{6}}(1\cdot(-1) + 1\cdot 0 + 1\cdot 1) = \frac{1}{\sqrt{6}}(-1+0+1) = 0$$

Thus S is an orthonormal subset of V.

(b)

$$\begin{aligned}
&T(ax^2+bx+c)\\
=&< ax^2+bx+c, \frac{1}{\sqrt{3}} > \frac{1}{\sqrt{3}} \quad + < ax^2+bx+c, \frac{1}{\sqrt{2}}x > \frac{1}{\sqrt{2}}x\\
=&\frac{1}{3}\{[a(-1)^2+b(-1)+c]\cdot 1+[a(0)^2+b(0)+c]\cdot 1+[a(1)^2+b(1)+c]\cdot 1\}\cdot 1\\
+&\frac{1}{2}\{[a(-1)^2+b(-1)+c]\cdot(-1)+[a(0)^2+b(0)+c]\cdot 0+[a(1)^2+b(1)+c]\cdot 1\}\cdot x\\
=&\frac{1}{3}(2a+3c)+bx
\end{aligned}$$

(c)

$$\begin{aligned}
x^2 &= T(x^2)+(x^2-T(x^2)) \quad \text{where} \quad T(x^2)\in W \text{ and } \quad (x^2-T(x^2))\in W^{\perp}\\
&= \frac{2}{3}+(x^2-\frac{2}{3}) \quad \text{from } (b), \quad \text{where} \quad \frac{2}{3}\in W \text{ and } \quad (x^2-\frac{2}{3})\in W^{\perp}
\end{aligned}$$

(d) It follows from (c) that $\frac{2}{3}$ is the polynomial in W closest to x^2.

Question 8

It is given that $A \in M_{3\times 3}(C)$ is a normal matrix with eigenvalues i and $-i$ and corresponding eigenspaces

$$E_i = \text{span}\left\{\frac{1}{3}(-2,1,2), \frac{1}{3}(2,2,1)\right\}$$

and

$$E_{-i} = \text{span}\left\{\frac{1}{3}(1,-2,2)\right\}.$$

(a) Find the spectral decomposition of A. (25)

(b) Find A. (5)

[30]

Solution

(a)

$$P_i = \begin{bmatrix} \frac{-2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{-2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & \frac{2}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

and

$$P_{-i} = \begin{bmatrix} \frac{1}{3} \\ \frac{-2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & \frac{-2}{9} & \frac{2}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{-4}{9} \\ \frac{2}{9} & \frac{-4}{9} & \frac{4}{9} \end{bmatrix}.$$

The spectral decomposition is

$$\begin{aligned} A &= i \cdot P_i - i \cdot P_{-i} \\ &= i \cdot \begin{bmatrix} \frac{8}{9} & \frac{2}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{5}{9} & \frac{4}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{5}{9} \end{bmatrix} - i \cdot \begin{bmatrix} \frac{1}{9} & \frac{-2}{9} & \frac{2}{9} \\ \frac{-2}{9} & \frac{4}{9} & \frac{-4}{9} \\ \frac{2}{9} & \frac{-4}{9} & \frac{4}{9} \end{bmatrix} \end{aligned}$$

(b)

$$A = \begin{bmatrix} \frac{7i}{9} & \frac{4i}{9} & \frac{-4i}{9} \\ \frac{4i}{9} & \frac{i}{9} & \frac{8i}{9} \\ \frac{-4i}{9} & \frac{8i}{9} & \frac{i}{9} \end{bmatrix}$$