Tutorial Letter 201/2/2016

Mathematics III (Engineering)

MAT3700

Semester 2

Department of Mathematical sciences

This tutorial letter contains solutions and answers to assignments.

BARCODE

Define tomorrow.

UNISA university of south africa

CONTENTS

Page

I wish you success with the examination.

L E GREYLING
Lecturer
Tel: (011) 471 – 2350 **Fax:** (011) 670 9171 or 086 274 1520
e-mail: lgreylin@unisa.ac.za

1 SOLUTIONS ASSIGNMENT 1

This assignment contributes 10% to your year mark.

1.1 $\frac{dy}{dx}+y\tan x=y^3\sec^4 x$ Bernoulli equation

$$y^{-3}\frac{dy}{dx}+(\tan x)y^{-2}=\sec^4 x$$

Let $z=y^{-2}$ then $\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx}$

$$-2y^{-3}\frac{dy}{dx}-2(\tan x)y^{-2}=-2\sec^4 x$$

then $\frac{dz}{dx}+(-2\tan x)z=-2\sec^4 x$

Thus $P=-2\tan x$ and $Q=-2\sec^4 x$

and $\int Pdx=\int(-2\tan x)dx=-2\ell n\sec x=\ell n\sec^{-2} x$

and $e^{\int Pdx}=e^{\ell n\sec^{-2} x}=\sec^{-2} x$

$$z=\sec^2 x\int\left(-2\sec^4 x\right)\left(\sec^{-2} x\right)dx$$

$$y^{-2}=\sec^2 x\int\left(-2\sec^2 x\right)dx$$ Use rules for exponents (9)

Use no 14 table of integrals

$$\frac{1}{y^2}=\sec^2 x(-2\tan x+C)$$

1.2 $(x+y)dx-xdy=0$ Homogeneous equation

$$\frac{dy}{dx}=\frac{x+y}{x}$$ Put $y = vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{x+vx}{x}$$

$$x\frac{dv}{dx}=1+v-v$$

$$dv=\frac{dx}{x}$$

$$v=\ell n|x|+C$$

$$\frac{y}{x}=\ell n|x|+C \quad \text{(substitute back)}$$

$$y=x\left(\ell n|x|+C\right)$$ (7)

or $(x+y)dx - xdy = 0$ Rewrite as linear equation.

$$\frac{dy}{dx} - \frac{1}{x}y = 1$$

$$e^{\int Pdx} = e^{\int\left(-\frac{1}{x}\right)} = e^{-\ell nx} = e^{\ell nx^{-1}} = x^{-1} = \frac{1}{x}$$

Thus $y = x\left(\int \frac{1}{x}dx\right)$

$$= x(\ell nx + C)$$

or $(x+y)dx - xdy = 0$ Use the integrating factor $\left(-\frac{1}{x^2}\right)$ to obtain an exact equation.

$$-\left(\frac{1}{x} + \frac{y}{x^2}\right)dx + \left(\frac{1}{x}\right)dy = 0$$

of form $M(x,y)dx + N(x,y)dy = 0$

Check: $\frac{\partial M}{\partial y} = -\frac{1}{x^2}$ and $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$

Thus the equation is exact.

$$f(x,y) = \int -\left(\frac{1}{x} + \frac{y}{x^2}\right)dx = -\ell nx + \frac{y}{x} + f(y)$$

and $f(x,y) = \int\left(\frac{1}{x}\right)dy = \frac{y}{x} + f(x)$

Comparing the answers we find $f(x) = -\ell nx$ and $f(y) = 0$

Thus $-\ell nx + \frac{y}{x} = C$

and $\frac{y}{x} = \ell nx + C$

$$y = x(\ell nx + C)$$

1.3 $\cos^2 \frac{dy}{dx} = y + 3$ Separable equation

$$\frac{dy}{y+3} = \frac{dx}{\cos^2 x}$$

$$= \sec^2 x \, dx$$

$$\int \frac{dy}{y+3} = \int \sec^2 x \, dx$$

$\ell n|y+3| = \tan x + C$ (4)

or rewrite as a linear equation: $\frac{dy}{dx} + \left(-\sec^2 x\right)y = 3\sec^2 x$

$$\frac{dy}{dx}+\left(-\sec^2 x\right)y=3\sec^2 x$$

Thus $P=\left(-\sec^2 x\right)$ and $Q=3\sec^2 x$

and $\int Pdx=\int \left(-\sec^2 x\right)dx=-\tan x$

$$y=e^{\tan x}\int\left(3\sec^2 x\right)\left(e^{-\tan x}\right)dx$$

Use no 6 table of integrals

$$=e^{\tan x}\left(-3e^{-\tan x}+k\right)$$

$$=-3+ke^{\tan x}$$

$$y+3=ke^{\tan x}$$

$$\ell n\left|y+3\right|=\tan x+\ell nk$$

$$=\tan x+C$$

TOTAL:20

---

2 SOLUTIONS ASSIGNMENT 2

This assignment is a written assignment based on Study Guide 1 and 2

This assignment contributes 70% to your year mark.

QUESTION 1

1.1 $\left(D^2-5D+6\right)y=e^{-2x}+\sin 2x$

y_{CF} : $m^2-5m+6=0$

$\left(m-3\right)\left(m-2\right)=0$

$m=3$ or $m=2$

$y_{CF}=Ae^{3x}+Be^{2x}$

$$y_{PI}=\frac{1}{D^2-5D+6}\left\{e^{-2x}+\sin 2x\right\}$$

$$=\frac{1}{D^2-5D+6}\left\{e^{-2x}\right\}+\frac{1}{D^2-5D+6}\left\{\sin 2x\right\}$$

$$=\frac{e^{-2x}}{(-2)^2-5\left(-2\right)+6}+\frac{1}{-\left(2\right)^2-5D+6}\left\{\sin 2x\right\}$$

$$=\frac{e^{-2x}}{20}+\frac{1}{-5D+2}\left\{\sin 2x\right\}$$

$$y_{PI} = \frac{e^{-2x}}{20} - \frac{(5D+2)}{(5D-2)(5D+2)}\{\sin 2x\}$$

$$= \frac{e^{-2x}}{20} - \frac{(5D+2)}{(25D^2-4)}\{\sin 2x\}$$

$$= \frac{e^{-2x}}{20} - \frac{(10\cos 2x + 2\sin 2x)}{(-104)}$$

$$= \frac{e^{-2x}}{20} + \frac{(5\cos 2x + \sin 2x)}{52} \qquad (6)$$

$$y_{gen} = Ae^{3x} + Be^{2x} + \frac{e^{-2x}}{20} + \frac{(5\cos 2x + \sin 2x)}{52}$$

1.2 $\left(D^2 + 6D + 9\right)y = e^{-3x}\cosh 3x$

$y_{CF} : (m+3)^2 = 0$

$m = -3$ twice

$y_{CF} = Ae^{-3x} + Bxe^{-3x}$

$$y_{PI} = \frac{1}{(D+3)^2}\left\{e^{-3x}\cosh 3x\right\}$$

$$= e^{-3x}\frac{1}{\left((D-3)+3\right)^2}\{\cosh 3x\}$$

$$= e^{-3x}\frac{1}{D^2}\{\cosh 3x\}$$

$$= e^{-3x}\frac{1}{D}\left\{\frac{1}{3}\sinh 3x\right\}$$

$$= e^{-3x}\left(\frac{1}{9}\cosh 3x\right)$$

$$y_{gen} = Ae^{-3x} + Bxe^{-3x} + \frac{e^{-3x}\cosh 3x}{9}$$

OR

$$y_{PI} = \frac{1}{(D+3)^2}\left\{e^{-3x}\left(\frac{e^{3x}+e^{-3x}}{2}\right)\right\}$$

$$= \frac{1}{2}\frac{1}{(D+3)^2}\left\{1+e^{-6x}\right\}$$

$$= \frac{1}{2}\frac{1}{(D+3)^2}\left\{e^{0x}\right\} + \frac{1}{2}\frac{1}{(D+3)^2}\left\{e^{-6x}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{9}\right) + \frac{1}{2}\left(\frac{e^{-6x}}{9}\right)$$

$$y_{gen} = Ae^{-3x} + Bxe^{-3x} + \frac{1}{18}\left(1+e^{-6x}\right) \qquad (6)$$

[12]

QUESTION 2

Solve for ***x*** *and* ***y*** in the following set of simultaneous differential equations by using **D-operator** methods:

$$(D-3)x \quad + \quad y = -1$$
$$-x \ + (D-1)y = 4e^t$$

Answer:

Use Cramer's rule:

$$\begin{vmatrix} (D-3) & 1 \\ -1 & (D-1) \end{vmatrix} x = \begin{vmatrix} (-1) & 1 \\ 4e^t & D-1 \end{vmatrix}$$

$$\left[(D-3)(D-1)+1\right]x = \left[(D-1)(-1)-(1)\left(4e^t\right)\right]$$

$$\left(D^2-4D+4\right)x = 1-4e^t$$

$$x_{CF} : m^2-4m+4=0$$
$$(m-2)(m-2)=0$$
$$m=2 \text{ twice}$$
$$x_{CF} = Ae^{2t}+Bte^{2t}$$

$$x_{PI} = \frac{1}{D^2-4D+4}\left\{1-4e^t\right\}$$
$$= \frac{1}{D^2-4D+4}\left\{e^{0t}\right\} - 4\frac{1}{D^2-4D+4}\left\{e^t\right\}$$
$$= \frac{1}{4} - 4e^t$$

$$x_{gen} = (A+Bt)e^{2t} + \frac{1}{4} - 4e^t$$

And

$$\begin{vmatrix} (D-3) & 1 \\ -1 & (D-1) \end{vmatrix} y = \begin{vmatrix} (D-3) & (-1) \\ -1 & 4e^t \end{vmatrix}$$

$$\left(D^2-4D+4\right)y = \left[(D-3)\left(4e^t\right)-(1)\right]$$
$$= -8e^t - 1$$

$$y_{CF} : m^2-4m+4=0$$
$$(m-2)(m-2)=0$$
$$m=2 \text{ twice}$$
$$y_{CF} = Ce^{2t}+Dte^{2t}$$

$$y_{PI} = \frac{1}{D^2-4D+4}\left\{-1-8e^t\right\}$$
$$= \frac{-1}{D^2-4D+4}\left\{e^{0t}\right\} - 8\frac{1}{D^2-4D+4}\left\{e^t\right\}$$
$$= \frac{-1}{4} - 8e^t$$

$$y_{gen} = (C+Dt)e^{2t} - \frac{1}{4} - 8e^t$$

[12]

QUESTION 3

3.1 Determine $L\left\{\left(t^2-1\right)H(t-1)\right\}$ See page 299 of Text book (4)

Answer:

$$L\left\{\left(t^2-1\right)H(t-1)\right\}=L\left\{\left[(t-1+1)^2-1\right]H(t-1)\right\}$$
$$=L\left\{\left[(t-1)^2-2(t-1)+1-1\right]H(t-1)\right\}$$
$$=L\left\{\left[(t-1)^2-2(t-1)\right]H(t-1)\right\}$$
$$=L\left\{(t-1)^2H(t-1)\right\}-2L\left\{(t-1)H(t-1)\right\}$$

The given function has now been rewritten so that we can use the table to read off the answer.

$$=\frac{2e^{-s}}{s^3}+\frac{2e^{-s}}{s^2}$$

3.2 Determine $L^{-1}\left\{\frac{1}{s^2+6s+8}\right\}$ (4)

Answer:

$$L^{-1}\left\{\frac{1}{s^2+6s+8}\right\}=L^{-1}\left\{\frac{1}{\left(s^2+6s+9\right)-9+8}\right\}$$
$$=L^{-1}\left\{\frac{1}{(s+3)^2-1}\right\}$$
$$=e^{-3t}\sinh t$$

or

$$=L^{-1}\left\{\frac{1}{(s+2)(s+4)}\right\}$$

Use partial fractions to find

$$=L^{-1}\left\{\frac{\left(\frac{1}{2}\right)}{(s+2)}\right\}-L^{-1}\left\{\frac{-\left(\frac{1}{2}\right)}{(s+4)}\right\}$$
$$=\frac{e^{-2t}}{2}-\frac{e^{-4t}}{2}$$

[8]

QUESTION 4

Determine the unique solution of the following differential equation by using **Laplace transforms:** $y''(t)+4y'(t)+4y(t)=4e^{-2t}$, if $y(0)=-1$ and $y'(0)=4$.

Answer:

$$y''(t) \quad + \; 4y'(t) \quad +4y(t) = 4e^{-2t}$$

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{4}{(s+2)}$$

$$s^2Y(s) + s - 4 + 4sY(s) + 4 + 4Y(s) = \frac{4}{(s+2)}$$

$$\left(s^2 + 4s + 4\right)Y(s) = \frac{4}{(s+2)} - s$$

$$(s+2)^2 Y(s) = \frac{4}{(s+2)} - s$$

$$Y(s) = \frac{4}{(s+2)^3} - \frac{s}{(s+2)^2}$$

Use partial fractions: $\dfrac{s}{(s+2)^2} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2}$

$$s = A(s+2) + B$$
$$= As + 2A + B$$

Thus $1 = A$

$$0 = 2A + B$$
$$B = -2$$

$$Y(s) = \frac{4}{(s+2)^3} - \frac{1}{(s+2)} + \frac{2}{(s+2)^2}$$

$$y(t) = 2t^2e^{-2t} - e^{-2t} + 2te^{-2t}$$

[11]

QUESTION 5

For a certain electrical circuit the applicable differential equation is: $100\dfrac{d^2i}{dt^2} + 200\dfrac{di}{dt} + \dfrac{i}{0{,}005} = 0$,

with initial conditions $i(0) = 0$ and $i'(0) = 1$.

Determine the unique solution for the current, *i* in terms of the time, *t.* (7)

Answer:

Using D-operator methods:

$$\left(100D^2 + 200D + 200\right)i = 0$$

$$i_{CF} : m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{-4}}{2}$$
$$= -1 + j$$

$$i_{CF} = e^{-t}\left(A\cos t + B\sin t\right)$$

Use the given initial conditions to find the values of the constants A and B:

$$\text{Given } i(0) = 0 : 0 = A\cos 0 + B\sin 0$$
$$A = 0$$

$$i'(0) = \frac{di}{dt} = -e^{-t}\left(A\cos t + B\sin t\right) + e^{-t}\left(-A\sin t + B\cos t\right)$$
$$\text{Given } i'(0) = 1 : 1 = B\cos 0$$
$$1 = B$$
$$i(t) = e^{-t}\sin t$$

Using Laplace transforms:

$$100\frac{d^2 i}{dt^2} + 200\frac{di}{dt} + \frac{i}{0{,}005} = 0, \text{ thus } \frac{d^2 i}{dt^2} + 2\frac{di}{dt} + 2i = 0$$

$$\left(s^2 I(s) - si(0) - i'(0)\right) + 2\left(sI(s) - y(0)\right) + 2I(s) = 0$$
$$\left(s^2 I(s) - 0 - 1\right) + 2\left(sI(s) - 0\right) + 2I(s) = 0$$
$$\left(s^2 + 2s + 2\right)I(s) = 1$$
$$I(s) = \frac{1}{\left(s^2 + 2s + 2\right)}$$
$$= \frac{1}{(s+1)^2 + 1}$$
$$i(t) = e^{-t}\sin t$$

[7]

TOTAL = 50

3 SOLUTIONS ASSIGNMENT 3

This assignment contributes 20% to your year mark.

QUESTION 1

7.1 If $B = \begin{bmatrix} -6 & 5 \\ 4 & 2 \end{bmatrix}$, find the eigenvalues of B. (4)

Answer: For eigenvalues put $\begin{bmatrix} -6-\lambda & 5 \\ 4 & 2-\lambda \end{bmatrix} = 0$

$$(-6-\lambda)(2-\lambda)-(5)(4)=0$$
$$\lambda^2+4\lambda-12-20=0$$
Thus $$\lambda^2+4\lambda-32=0$$
$$(\lambda+8)(\lambda-4)=0$$
$$\lambda=-8 \text{ or } \lambda=4$$

7.2 If $A=\begin{bmatrix}3 & 2 & 2\\ 0 & 2 & 1\\ 0 & 0 & 4\end{bmatrix}$, find an eigenvector corresponding to the eigenvalue $\lambda=2$.

(4)

Answer:

$$\begin{bmatrix}3-2 & 2 & 2\\ 0 & 2-2 & 1\\ 0 & 0 & 4-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=[0]$$

thus $$\begin{bmatrix}1 & 2 & 2\\ 0 & 0 & 1\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=[0]$$

$$x_1+2x_2+2x_3=0 \qquad (1)$$
$$x_3 \quad =0 \qquad (2)$$

We can now write down the relationships between x_1 and x_2

From equation $(1): x_1=-2x_2 \qquad (3)$

To write down an eigenvector we must make a choice for a value.

Choose x_2=1 we get from equation (3), $x_1=-2$ and from equation (2) $x_3=0$

Thus an eigenvector for λ=0 is $\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}$

Note that we don’t have to work out all the eigenvalues as it is not asked.

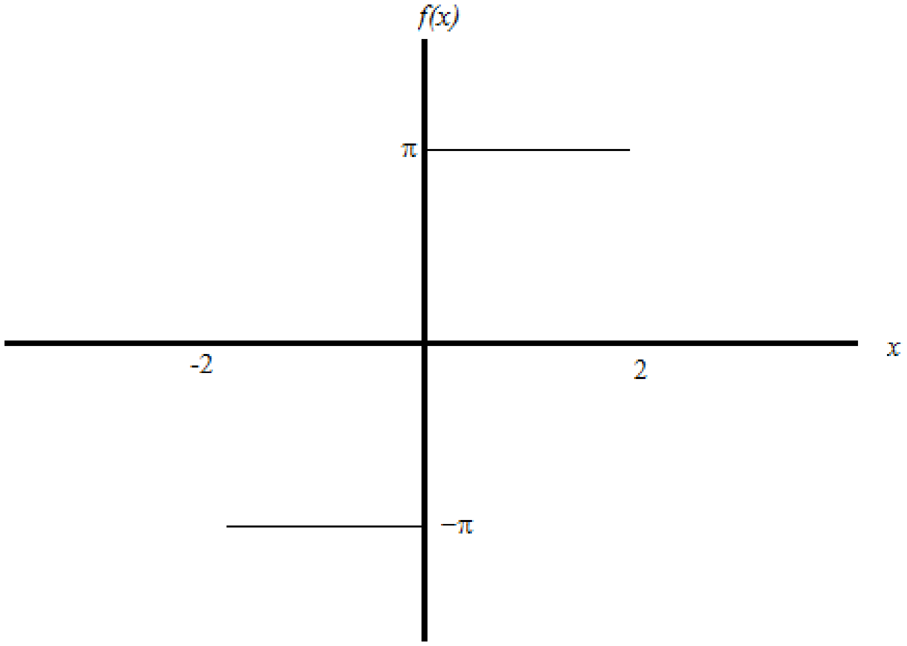
[8]

QUESTION 8

A function *f(x)* is defined over one period by $f(x)=\begin{cases}-\pi & -2<x<0\\ \pi & 0<x<2\end{cases}$

Answer:

8.1 Sketch the given function:



(3)

8.2 Odd function because $f(-x) = -f(x)$ (1)

8.3 Because it is an even function $a_0 = a_n = 0$

The period = 4, therefore L = 2.

Note: As the two integrals from - 2 to 0 and from 0 to 2 are equal we can save time by multiplying one of them by 2.

$$b_n = \tfrac{2}{L}\int_0^L f(x)\sin\left(\tfrac{n\pi x}{L}\right)dx$$

$$= \tfrac{2}{2}\int_0^2 (\pi)\sin\left(\tfrac{n\pi x}{2}\right)dx$$

(π) is a number and can be taken out of the integral

$$b_n = (\pi)\int_0^2 \sin\left(\tfrac{n\pi x}{2}\right)dx$$

$$=(\pi)\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2$$

$$= -\frac{2}{n}\left[\cos(n\pi) - (\cos 0)\right]$$

$$= -\frac{2}{n}\left[(-1)^n - 1\right] = \frac{2}{n}\left[1 - (-1)^n\right] = \begin{cases} \dfrac{4}{n} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

Thus $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\left(\tfrac{n\pi x}{L}\right)\right) = \sum_{n=1}^{\infty} \frac{2}{n}\left(1-(-1)^n\right)\sin\left(\tfrac{n\pi x}{2}\right)$ **[12]**

$$= 4\sin\left(\tfrac{\pi x}{2}\right) + \frac{4}{3}\sin\left(\tfrac{3\pi x}{2}\right) + \ldots.$$