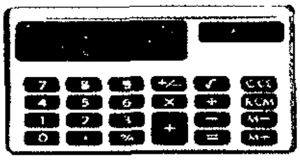
UNIVERSITY EXAMINATIONS



UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT3700

May/June 2016

MATHEMATICS III (ENGINEERING)

Duration 2 Hours

80 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT

Use of a non-programmable pocket calculator is permissible

Closed book examination

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This examination question paper consists of 3 pages including this cover page plus

Formulae sheets (pages 4 to 8) plus
A table of integrals (pages 9 and 10) plus
A table of Laplace transforms (page11)

Please answer all the questions **in numerical order** as far as possible

QUESTION 1

Solve the following differential equations

1.1 $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = y\sin\left(\frac{y}{x}\right) - x$ [Hint Put $y = vx$] (6)

1.2 $\frac{dy}{dx} + y = 2xy^2$ (8)

1.3 $(1+y)^2 = \left(1+x^2\right)\frac{dy}{dx}$ (3)

[17]

QUESTION 2

Find the general solutions of the following differential equations using **D-operator** methods

2.1 $(D-2)^3 y = 60e^{2x}x^2$ (6)

2.2 $\left(D^2 + D + 1\right)y = x + \sin x$ (7)

[13]

QUESTION 3

Solve for ***y*** in the following set of simultaneous differential equations by using **D-operator** methods

$$\begin{aligned}(2D-1)x + (D+1)y &= 5\sin t \\ (3D-1)x + (2D+1)y &= e^t\end{aligned}$$ (9)

[9]

QUESTION 4

4.1 Determine $L\{t\cos t\}$ (1)

4.2 Determine $L^{-1}\left\{e^s\left(\frac{s^2-1}{\left(s^2+1\right)^2}\right)\right\}$ (2)

4.3 Use partial fractions to determine $L^{-1}\left\{\frac{11-3s}{s^2+2s-3}\right\}$ (4)

[7]

[TURN OVER]

QUESTION 5

Determine the unique solution of the following differential equation by using **Laplace transforms:** $y''(x)+6y'(x)+13y(x)=0$, if $y(0)=3$ and $y'(0)=7$ (8)

[8]

QUESTION 6

A particle moves along a line such that the equation $x''(t)+4x'(t)+5x(t)=80\sin 5t$ applies, given the initial conditions $x(0)=0$ and $x'(0)=0$

Determine the unique solution of the displacement x, in terms of the time t (8)

[8]

QUESTION 7

If $A=\begin{bmatrix}-1 & -1 & 1\\ -4 & 2 & 4\\ -1 & 1 & 5\end{bmatrix}$, find an eigenvector corresponding to the eigenvalue $\lambda=2$

Verify that $\lambda=2$ is an eigenvalue of A

(8)

[8]

QUESTION 8

A function *f(x)* is defined over one period by $f(x)=\begin{cases}-x & -\pi<x<0\\ 0 & 0<x<\pi\end{cases}$

8 1 Sketch the function (1)

8 2 Find the Fourier series expansion for *f(x)*, given that $a_n=\dfrac{(-1)^n-1}{n^2}$ (9)

[10]

Full marks = 80

Examiners First Ms L E Greyling
Second Dr J M Manale
External Dr J N Mwambakana

FORMULA SHEET

ALGEBRA

Laws of indices

1 $a^m \times a^n = a^{m+n}$

2 $\dfrac{a^m}{a^n} = a^{m-n}$

3 $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$

4 $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

5 $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$

6 $a^0 = 1$

7 $\sqrt{ab} = \sqrt{a}\sqrt{b}$

8 $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Logarithms

Definitions If $y = a^x$ then $x = \log_a y$

If $y = e^x$ then $x = \ell n\, y$

1 $\log(A \times B) = \log A + \log B$

2 $\log\left(\dfrac{A}{B}\right) = \log A - \log B$

3 $\log A^n = n \log A$

4 $\log_a A = \dfrac{\log_b A}{\log_b a}$

5 $a^{\log_a f} = f$ $e^{\ell n f} = f$

Factors

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Partial fractions

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Quadratic formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

SERIES

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \dots$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \dots$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \dots$$

COMPLEX NUMBERS

1 $z = a + bj = r(\cos\theta + j\sin\theta) = r\angle\theta = re^{j\theta}$,
where $j^2 = -1$

Modulus $r = |z| = \sqrt{(a^2 + b^2)}$

Argument $\theta = \arg z = arc\tan\frac{b}{a}$

2 Addition
$(a + jb) + (c + jd) = (a + c) + j(b + d)$

3 Subtraction
$(a + jb) - (c + jd) = (a - c) + j(b - d)$

4 If $m + jn = p + jq$, then $m = p$ and $n = q$

5 Multiplication $z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$

6 Division $\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$

7 De Moivre's Theorem
$[r\angle\theta]^n = r^n\angle n\theta = r^n(\cos n\theta + j\sin n\theta)$

8 $z^{\frac{1}{n}}$ has n distinct roots
$z^{\frac{1}{n}} = r^{\frac{1}{n}}\angle\frac{\theta + k360°}{n}$ with $k = 0, 1, 2, \dots, n-1$

9 $re^{j\theta} = r(\cos\theta + j\sin\theta)$
$\Re(re^{j\theta}) = r\cos\theta$ and $\Im(re^{j\theta}) = r\sin\theta$

10 $e^{a+jb} = e^a(\cos b + j\sin b)$

11 $\ell n\, re^{j\theta} = \ell n\, r + j\theta$

GEOMETRY

1 Straight line $y = mx + c$

$y - y_1 = m(x - x_1)$

Perpendiculars, then $m_1 = \frac{-1}{m_2}$

2 Angle between two lines

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

3 Circle

$x^2 + y^2 = r^2$

$(x - h)^2 + (y - k)^2 = r^2$

4 Parabola

$y = ax^2 + bx + c$

axis at $x = \frac{-b}{2a}$

5 Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

6 Hyperbola

$xy = k$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (round x - axis)

$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (round y - axis)

MENSURATION

1 Circle (θ in radians)

$\text{Area} = \pi r^2$

$\text{Circumference} = 2\pi r$

$\text{Arc length } s = r\theta$

$\text{Sector area} = \frac{1}{2} r^2\theta = \frac{1}{2} sr$

$\text{Segment area} = \frac{1}{2} r^2(\theta - \sin\theta)$

2 Ellipse

$\text{Area} = \pi ab$

$\text{Circumference} = \pi(a + b)$

3 Cylinder

$\text{Volume} = \pi r^2 h$

$\text{Surface area} = 2\pi rh + 2\pi r^2$

4 Pyramid

$\text{Volume} = \frac{1}{3} \text{area base} \times \text{height}$

5 Cone

$\text{Volume} = \frac{1}{3}\pi r^2 h$

$\text{Curved surface} = \pi r \ell$

6 Sphere

$A = 4\pi r^2$

$V = \frac{4}{3}\pi r^3$

7 Trapezoidal rule

$A = h\left(\frac{y_1 + y_n}{2} + y_2 + y_3 + \quad + y_{n-1}\right)$

8 Simpsons rule

$A = \frac{s}{3}((F + L) + 4E + 2R)$

9 Prismoidal rule

$V = \frac{h}{6}(A_1 + 4A_2 + A_3)$

HYPERBOLIC FUNCTIONS

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

TRIGONOMETRY

Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = +\cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Compound angle addition and subtraction formulae

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin^2 A = ½(1 - \cos 2A)$

$\cos^2 A = ½(1 + \cos 2A)$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences

$\sin A \cos B = ½(\sin(A + B) + \sin(A - B))$

$\cos A \sin B = ½(\sin(A + B) - \sin(A - B))$

$\cos A \cos B = ½(\cos(A + B) + \cos(A - B))$

$\sin A \sin B = -½(\cos(A + B) - \cos(A - B))$

Sums or differences of sines and cosines into products

$$\sin x + \sin y = 2\sin\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\sin x - \sin y = 2\cos\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

$$\cos x + \cos y = 2\cos\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\cos x - \cos y = -2\sin\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

DIFFERENTIATION

1 $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

2 $\frac{d}{dx} k = 0$

3 $\frac{d}{dx} ax^n = anx^{n-1}$

4 $\frac{d}{dx} f\,g = f\,g' + g\,f'$

5 $\frac{d}{dx} \frac{f}{g} = \frac{g\,f' - f\,g'}{g^2}$

6 $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}\, f'(x)$

7 $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

8 Parametric equations

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

9 Maximum/minimum

For turning points $f'(x) = 0$
Let $x = a$ be a solution for the above
If $f'(a) > 0$, then a minimum
If $f'(a) < 0$, then a maximum
For points of inflection $f''(x) = 0$
Let $x = b$ be a solution for the above
Test for inflection $f(b - h)$ and $f(b + h)$
Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists

10 $\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$

11 $\frac{d}{dx} \cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$

12 $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1 + [f(x)]^2}$

13 $\frac{d}{dx} \cot^{-1} f(x) = \frac{-f'(x)}{1 + [f(x)]^2}$

14 $\frac{d}{dx} \sec^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

15 $\frac{d}{dx} \operatorname{cosec}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

16 $\frac{d}{dx} \sinh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$

17 $\frac{d}{dx} \cosh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$

18 $\frac{d}{dx} \tanh^{-1} f(x) = \frac{f'(x)}{1 - [f(x)]^2}$

19 $\frac{d}{dx} \coth^{-1} f(x) = \frac{f'(x)}{[f(x)]^2 - 1}$

20 $\frac{d}{dx} \operatorname{sech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{1 - [f(x)]^2}}$

21 $\frac{d}{dx} \operatorname{cosech}^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 + 1}}$

22 Increments $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{\partial z}{\partial w}\delta w$

23 Rate of change

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$

INTEGRATION

1 By parts $\int u\,dv = uv - \int v\,du$

2 $\int_a^b f(x)dx = F(b) - F(a)$

3 Mean value $= \frac{1}{b-a}\int_a^b y\,dx$

4 $(\text{R M S})^2 = \frac{1}{b-a}\int_a^b y^2\,dx$

TABLE OF INTEGRALS

1 $\int x^n dx = \frac{x^{(n+1)}}{n+1} + c, \quad n \neq -1$

2 $\int (u+v)dx = \int udx + \int vdx$

3 $\int audx = a\int udx$, a a constant

4 $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$

5 $\int \frac{f'(x)}{f(x)} dx = \ell n|f(x)| + c$

6 $\int f'(x) e^{f(x)} dx = e^{f(x)} + c$

7 $\int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ell na} + c$

8 $\int f'(x) \sin f(x)dx = -\cos f(x) + c$

9 $\int f'(x) \cos f(x)dx = \sin f(x) + c$

10 $\int f'(x) \tan f(x)dx = \ell n \sec f(x) + c$

11 $\int f'(x) \cot f(x)dx = \ell n \sin f(x) + c$

12 $\int f'(x) \sec f(x)dx = \ell n [\sec f(x) + \tan f(x)] + c$

13 $\int f'(x) \operatorname{cosec} f(x)dx = \ell n [\operatorname{cosec} f(x) - \cot f(x)] + c$

14 $\int f'(x) \sec^2 f(x)dx = \tan f(x) + c$

15 $\int f'(x) \operatorname{cosec}^2 f(x)dx = -\cot f(x) + c$

16 $\int f'(x) \sec f(x) \tan f(x)dx = \sec f(x) + c$

17 $\int f'(x) \operatorname{cosec} f(x) \cot f(x)dx = -\operatorname{cosec} f(x) + c$

18 $\int f'(x) \sinh f(x)dx = \cosh f(x) + c$

19 $\int f'(x) \cosh f(x)dx = \sinh f(x) + c$

20 $\int f'(x) \tanh f(x)dx = \ell n \cosh f(x) + c$

21 $\int f'(x) \coth f(x)dx = \ell n \sinh f(x) + c$

22 $\int f'(x) \operatorname{sech}^2 f(x)dx = \tanh f(x) + c$

23 $\int f'(x)\operatorname{cosech}^2 f(x)dx = -\coth f(x) + c$

24 $\int f'(x)\operatorname{sech} f(x)\tanh f(x)dx = -\operatorname{sech} f(x) + c$

25 $\int f'(x)\operatorname{cosech} f(x)\coth f(x)dx = -\operatorname{cosech} f(x) + c$

26 $\int \frac{f'(x)}{[f(x)]^2 - a^2}dx = -\frac{1}{a}\operatorname{arc}\coth\left(\frac{f(x)}{a}\right) + c$

27 $\int \frac{f'(x)}{a^2 - [f(x)]^2}dx = \frac{1}{a}\operatorname{arc}\tanh\left(\frac{f(x)}{a}\right) + c$

28 $\int \frac{f'(x)}{[f(x)]^2 + a^2}dx = \frac{1}{a}\operatorname{arc}\tan\left(\frac{f(x)}{a}\right) + c$

29 $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \operatorname{arc}\sin\left(\frac{f(x)}{a}\right) + c$

30 $\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}dx = \operatorname{arc}\sinh\left(\frac{f(x)}{a}\right) + c$

31 $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}dx = \operatorname{arc}\cosh\left(\frac{f(x)}{a}\right) + c$

32 $\int f'(x)\sqrt{a^2 - [f(x)]^2}\,dx = \frac{a^2}{2}\operatorname{arc}\sin\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{a^2 - [f(x)]^2} + c$

33 $\int f'(x)\sqrt{[f(x)]^2 + a^2}\,dx = \frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 + a^2} + c$

34 $\int f'(x)\sqrt{[f(x)]^2 - a^2}\,dx = -\frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 - a^2} + c$

TABLE OF LAPLACE TRANSFORMS

Study Guide 2 page 20 and page 51	
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3,$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}, \quad n = 1\ 2, 3,$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t \cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt} \sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt} \cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt} \sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt} \cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c).F(t-c)$	$e^{-cs} f(s)$
$\delta(t-a)$	e^{-as}
Study Guide 2 page 55	
$\mathcal{L}\{f(t)\} = F(s)$	
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	
$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	