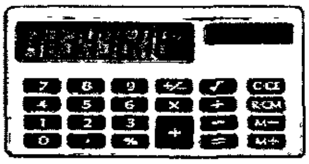
UNIVERSITY EXAMINATIONS



UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT3700

May/June 2014

MATHEMATICS III (ENGINEERING)

Duration 2 Hours

84 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT.

Use of a non-programmable pocket calculator is permissible

Closed book examination.

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This examination question paper consists of 3 pages including this cover page plus

Formulae sheets (page (i) to (v), plus
A table of integrals (page (i) and (ii)) plus
A table of Laplace transforms (page iii)

Please answer all the questions **in numerical order** as far as possible

QUESTION 1

Solve the following differential equations

1.1 $\dfrac{dy}{dx}=\dfrac{y^2}{x^2+9}$ (4)

1.2 $\left(x^2-y^2\right)dy=xydx$ [Hint: Put $y=vx$] (7)

1.3 $\dfrac{dy}{dx}+xy=xe^{-x^2}y^{-3}$ (7)

[18]

QUESTION 2

Find the general solutions of the following differential equations using **D-operator** methods:

2.1 $D\left(D^2-1\right)y=7$ (6)

2.2 $\left(D^2-6D+9\right)y=x^3e^{3x}$ (6)

[12]

QUESTION 3

The conditions in a certain electrical circuit are represented by the following differential equation: $\dfrac{d^2i}{dt^2}+2\dfrac{di}{dt}+2i=85\sin 3t$. Determine the general solution for the current, i, in terms of t, by using **D-operator methods**, given that for t = 0, i = 0 and $\dfrac{di}{dt}=-20$. (9)

[9]

QUESTION 4

Solve for ***y*** in the following set of simultaneous differential equations by using **D-operator** methods:

$$(4D+3)x-Dy=\sin t$$
$$Dx+y=\cos t$$
(8)

[8]

QUESTION 5

5.1 A function f(t) is defined by $f(t)=\begin{cases} t & 0\le t\le 2 \\ 2 & 2\le t \end{cases}$

Write f(t) in terms of the Heaviside unit step function and determine the Laplace transform of f(t) (4)

[TURN OVER]

5 2 Determine $L^{-1}\left\{\dfrac{6}{s^2-4s+13}\right\}$ (3)

[7]

QUESTION 6

Determine the unique solution of the following differential equation by using **Laplace transforms:** $y''(t)-4y'(t)+3y(t)=e^{2t}$, if $y(0)=0$ and $y'(0)=1$

(7)

[7]

QUESTION 7

A vehicle rests on a spring-shock absorber system on each of four wheels The system yields the model $y''(t)+2y'(t)+17y(t)=e^{-t}\cos 4t$, where $e^{-t}\cos 4t$ represents the force resulting from a bumpy road. If $y(0)=0$ and $y'(0)=0$, use **Laplace transforms** to solve the differential equation. (7)

[7]

QUESTION 8

If $A=\begin{bmatrix}3 & 2 & 2\\ 2 & 2 & 0\\ 2 & 0 & 4\end{bmatrix}$, find an eigenvector corresponding to the eigenvalue $\lambda=0$

Also verify that 0 is an eigenvalue of *A*. (6)

[6]

QUESTION 9

Given the function defined by $f(t)=\pi-t,\ 0<t<\pi$, write down the Fourier half range cosine series expansion for *f(t)* (10)

[10]

Full marks = 84

Examiners: First Ms L E Greyling

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FORMULA SHEET

ALGEBRA

Laws of indices

1 $a^m \times a^n = a^{m+n}$

2. $\dfrac{a^m}{a^n} = a^{m-n}$

3. $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$

4 $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

5 $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$

6 $a^0 = 1$

7 $\sqrt{ab} = \sqrt{a}\sqrt{b}$

8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Logarithms

Definitions. If $y = a^x$ then $x = \log_a y$

If $y = e^x$ then $x = \ell n\, y$

1 $\log(A \times B) = \log A + \log B$

2 $\log\left(\dfrac{A}{B}\right) = \log A - \log B$

3 $\log A^n = n \log A$

4 $\log_a A = \dfrac{\log_b A}{\log_b a}$

5 $a^{\log_a f} = f$. $e^{\ell n f} = f$

Factors

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Partial fractions

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Quadratic formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

SERIES

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \ldots$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \ldots$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \ldots$$

COMPLEX NUMBERS

1 $z = a + bj = r(\cos\theta + j\sin\theta) = r\underline{|\theta} = re^{j\theta}$, where $j^2 = -1$

Modulus: $r = |z| = \sqrt{(a^2 + b^2)}$

Argument: $\theta = \arg z = arc\tan\frac{b}{a}$

2 Addition

$(a + jb) + (c + jd) = (a + c) + j(b + d)$

3 Subtraction

$(a + jb) - (c + jd) = (a - c) + j(b - d)$

4 If $m + jn = p + jq$, then $m = p$ and $n = q$

5 Multiplication $z_1 z_2 = r_1 r_2 \underline{|(\theta_1 + \theta_2)}$

6. Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|(\theta_1 - \theta_2)}$

7 De Moivre's Theorem

$[r\underline{|\theta}]^n = r^n\underline{|n\theta} = r^n(\cos n\theta + j\sin n\theta)$

8 $z^{\frac{1}{n}}$ has n distinct roots:

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\underline{\left|\frac{\theta + k360^\circ}{n}\right.}$ with $k = 0, 1, 2, \ldots, n-1$

9. $re^{j\theta} = r(\cos\theta + j\sin\theta)$

$\therefore \Re(re^{j\theta}) = r\cos\theta$ and $\Im(re^{j\theta}) = r\sin\theta$

10 $e^{a+jb} = e^a(\cos b + j\sin b)$

11. $\ell n\, re^{j\theta} = \ell n\, r + j\theta$

GEOMETRY

1. Straight line: $y = mx + c$
   $y - y_1 = m(x - x_1)$
   Perpendiculars, then $m_1 = \frac{-1}{m_2}$
2. Angle between two lines:
   $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$
3. Circle:
   $x^2 + y^2 = r^2$
   $(x-h)^2 + (y-k)^2 = r^2$
4. Parabola.
   $y = ax^2 + bx + c$
   axis at $x = \frac{-b}{2a}$
5. Ellipse:
   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
6. Hyperbola
   $xy = k$
   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (round x - axis)
   $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (round y - axis)

MENSURATION

1. Circle (θ in radians)
   $\text{Area} = \pi r^2$
   $\text{Circumference} = 2\pi r$
   Arc length $s = r\theta$
   $\text{Sector area} = \frac{1}{2} r^2\theta = \frac{1}{2} sr$
   $\text{Segment area} = \frac{1}{2} r^2 (\theta - \sin\theta)$
2. Ellipse:
   $\text{Area} = \pi ab$
   $\text{Circumference} = \pi(a + b)$
3. Cylinder
   $\text{Volume} = \pi r^2 h$
   $\text{Surface area} = 2\pi rh + 2\pi r^2$
4. Pyramid.
   $\text{Volume} = \frac{1}{3}$ area base $\times$ height
5. Cone:
   $\text{Volume} = \frac{1}{3}\pi r^2 h$
   $\text{Curved surface} = \pi r \ell$
6. Sphere:
   $A = 4\pi r^2$
   $V = \frac{4}{3}\pi r^3$
7. Trapezoidal rule:
   $A = h\left(\frac{y_1 + y_n}{2} + y_2 + y_3 + \ldots + y_{n-1}\right)$
8. Simpsons rule:
   $A = \frac{s}{3}\left((F + L) + 4E + 2R\right)$
9. Prismoidal rule
   $V = \frac{h}{6}\left(A_1 + 4A_2 + A_3\right)$

HYPERBOLIC FUNCTIONS

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

TRIGONOMETRY

Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = +\cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Compound angle addition and subtraction formulae.

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin^2 A = ½(1 - \cos 2A)$

$\cos^2 A = ½(1 + \cos 2A)$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences.

$\sin A \cos B = ½(\sin(A + B) + \sin(A - B))$

$\cos A \sin B = ½(\sin(A + B) - \sin(A - B))$

$\cos A \cos B = ½(\cos(A + B) + \cos(A - B))$

$\sin A \sin B = -½(\cos(A + B) - \cos(A - B))$

Sums or differences of sines and cosines into products

$$\sin x + \sin y = 2\sin\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\sin x - \sin y = 2\cos\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

$$\cos x + \cos y = 2\cos\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\cos x - \cos y = -2\sin\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

DIFFERENTIATION

1. $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

2. $\frac{d}{dx} k = 0$

3. $\frac{d}{dx} ax^n = anx^{n-1}$

4. $\frac{d}{dx} f\,g = f\,g' + g.f'$

5. $\frac{d}{dx} \frac{f}{g} = \frac{g\,f' - f\,g'}{g^2}$

6. $\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1}.f'(x)$

7 $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$

8 Parametric equations

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

9 Maximum/minimum

For turning points: $f'(x) = 0$

Let $x = a$ be a solution for the above

If $f'(a) > 0$, then a minimum

If $f'(a) < 0$, then a maximum

For points of inflection: $f''(x) = 0$

Let $x = b$ be a solution for the above

Test for inflection: $f(b - h)$ and $f(b + h)$

Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists

10. $\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$

11. $\frac{d}{dx} \cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

12 $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1+[f(x)]^2}$

13. $\frac{d}{dx} \cot^{-1} f(x) = \frac{-f'(x)}{1+[f(x)]^2}$

14 $\frac{d}{dx} \sec^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

15. $\frac{d}{dx} \text{cosec}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

16 $\frac{d}{dx} \sinh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$

17. $\frac{d}{dx} \cosh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$

18 $\frac{d}{dx} \tanh^{-1} f(x) = \frac{f'(x)}{1-[f(x)]^2}$

19. $\frac{d}{dx} \coth^{-1} f(x) = \frac{f'(x)}{[f(x)]^2 - 1}$

20. $\frac{d}{dx} \text{sech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$

21. $\frac{d}{dx} \text{cosech}^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 + 1}}$

22 Increments. $\delta z = \frac{\partial z}{\partial x}.\delta x + \frac{\partial z}{\partial y}\delta y + \frac{\partial z}{\partial w}.\delta w$

23. Rate of change:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$

INTEGRATION

1 By parts: $\int u\,dv = uv - \int v\,du$

2. $\int_a^b f(x)dx = F(b) - F(a)$

3 Mean value $= \frac{1}{b-a}\int_a^b y\,dx$

4. $(\text{R M S.})^2 = \frac{1}{b-a}\int_a^b y^2\,dx$

TABLE OF INTEGRALS

1 $\int x^n\,dx = \frac{x^{(n+1)}}{n+1} + c, \quad n \neq -1$

2 $\int (u+v)dx = \int udx + \int vdx$

3 $\int audx = a\int udx$, a a constant

4 $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$

5. $\int \frac{f'(x)}{f(x)}dx = \ell n|f(x)| + c$

6 $\int f'(x)\,e^{f(x)}dx = e^{f(x)} + c$

7. $\int f'(x).a^{f(x)}dx = \frac{a^{f(x)}}{\ell na} + c$

8 $\int f'(x)\sin f(x)dx = -\cos f(x) + c$

9 $\int f'(x).\cos f(x)dx = \sin f(x) + c$

10 $\int f'(x).\tan f(x)dx = \ell n \sec f(x) + c$

11 $\int f'(x)\cot f(x)dx = \ell n \sin f(x) + c$

12 $\int f'(x)\sec f(x)dx = \ell n[\sec f(x) + \tan f(x)] + c$

13 $\int f'(x)\text{cosec} f(x)dx = \ell n[\text{cosec} f(x) - \cot f(x)] + c$

14 $\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$

15 $\int f'(x)\text{cosec}^2 f(x)dx = -\cot f(x) + c$

16 $\int f'(x)\sec f(x)\tan f(x)dx = \sec f(x) + c$

17 $\int f'(x).\text{cosec} f(x)\cot f(x)dx = -\text{cosec} f(x) + c$

18 $\int f'(x)\sinh f(x)dx = \cosh f(x) + c$

19 $\int f'(x)\cosh f(x)dx = \sinh f(x) + c$

20 $\int f'(x)\tanh f(x)dx = \ell n \cosh f(x) + c$

21 $\int f'(x).\coth f(x)dx = \ell n \sinh f(x) + c$

22 $\int f'(x).\text{sech}^2 f(x)dx = \tanh f(x) + c$

23 $\int f'(x)\operatorname{cosech}^2 f(x)dx = -\coth f(x) + c$

24 $\int f'(x)\operatorname{sech} f(x) \tanh f(x) dx = -\operatorname{sech} f(x) + c$

25 $\int f'(x)\operatorname{cosech} f(x).\coth f(x)dx = -\operatorname{cosech} f(x) + c$

26 $\int \frac{f'(x)}{[f(x)]^2 - a^2}dx = -\frac{1}{a} arc\coth\left(\frac{f(x)}{a}\right) + c$

27 $\int \frac{f'(x)}{a^2 - [f(x)]^2}dx = \frac{1}{a} arc\tanh\left(\frac{f(x)}{a}\right) + c$

28. $\int \frac{f'(x)}{[f(x)]^2 + a^2}dx = \frac{1}{a} arc\tan\left(\frac{f(x)}{a}\right) + c$

29. $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = arc\sin\left(\frac{f(x)}{a}\right) + c$

30. $\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}dx = arc\sinh\left(\frac{f(x)}{a}\right) + c$

31 $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}dx = arc\cosh\left(\frac{f(x)}{a}\right) + c$

32. $\int f'(x)\sqrt{a^2 - [f(x)]^2}\,dx = \frac{a^2}{2} arc\sin\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{a^2 - [f(x)]^2} + c$

33. $\int f'(x)\sqrt{[f(x)]^2 + a^2}\,dx = \frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 + a^2} + c$

34 $\int f'(x)\sqrt{[f(x)]^2 - a^2}\,dx = -\frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 - a^2} + c$

TABLE OF LAPLACE TRANSFORMS

Study Guide 2 page 20 and page 51	
$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}$, $n=1,2,3,$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}$, $n=1,2,3,$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t\sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t\cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt}\cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt}\sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt}\cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c)F(t-c)$	$e^{-cs}f(s)$
$\delta(t-a)$	e^{-as}

Study Guide 2 page 55
$\mathcal{L}\{f(t)\}=F(s)$
$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$
$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$
$\mathcal{L}\{f'''(t)\}=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$