UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT3700

October/November 2014

MATHEMATICS III (ENGINEERING)

Duration 2 Hours | 80 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT.

Use of a non-programmable pocket calculator is permissible.

Closed book examination

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This examination question paper consists of 3 pages including this cover page plus

Formulae sheets (page (i) to (v), plus
A table of integrals (page (i) and (ii)) plus
A table of Laplace transforms (page iii)

Please answer all the questions **in numerical order** as far as possible

QUESTION 1

Solve the following differential equations

1 1 $x\dfrac{dy}{dx} = y + 2\sqrt{xy}$ [Hint Put $y = vx$] (5)

1 2 $\left[\sin y - 2xy + x^2\right]dx + \left[x\cos y - x^2\right]dy = 0$

[Hint First show that the equation is exact] (6)

1 3 $\dfrac{dy}{dx} - y = \dfrac{e^x}{x}$, given that y(e)=0 (6)

[17]

QUESTION 2

Find the general solutions of the following differential equations using **D-operator** methods

2 1 $\left(D^2 - 36\right)y = \cosh 3x$ (5)

2 2 $\left(D^2 + 2D + 4\right)y = e^{2x}\sin 2x$ (8)

[13]

QUESTION 3

Solve only for ***y*** in the following set of simultaneous differential equations by using **D-operator** methods

$$(D+1)x - Dy = -1$$
$$(2D-1)x - \left(D - \tfrac{1}{2}\right)y = 1$$

(8)

[8]

QUESTION 4

4 1 Determine the Laplace transform of

4 1 1 $2t\sin 2t$ (1)

4 1 2 $3H(t-2) - \delta(t-4)$ (2)

4 2 Use partial fractions to find the inverse Laplace transform of $\dfrac{5s+2}{(s+1)(s+2)}$ (4)

[7]

QUESTION 5

Determine the unique solution of the following differential equation by using **Laplace transforms:** $y''(t)+2y'(t)+10y(t)=\left(25t^2+16t+2\right)e^{3t}$, if $y(0)=0$ and $y'(0)=0$

(8)

[8]

QUESTION 6

The motion of a mass on a spring is described by the differential equation

$\frac{d^2x}{dt^2}+100x=36\cos 8t$ If $x=0$ and $\frac{dx}{dt}=0$, at $t=0$ find the steady state solution for x(t) and discuss the motion. (11)

[11]

QUESTION 7

If $A=\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$, find an eigenvalue and an eigenvector of $\boldsymbol{A}$ (6)

[6]

QUESTION 8

Given the function defined by $f(t)=\begin{cases} 0 & -\pi < t < -\frac{\pi}{2} \\ 3 & -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t < \pi \end{cases}$, with period 2π

8 1 Sketch the function (2)

8 2 From the graph determine if the function is odd or even. (1)

8 3 Find the Fourier series for *f(t)* (7)

[10]

Full marks = 80

Examiners First Ms L E Greyling

Second Dr J M Manale

External Dr J N Mwambakana

FORMULA SHEET

ALGEBRA

Laws of indices

1 $a^m \times a^n = a^{m+n}$

2 $\dfrac{a^m}{a^n} = a^{m-n}$

3 $(a^m)^n = a^{mn} = (a^n)^m$

4 $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

5 $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$

6 $a^0 = 1$

7 $\sqrt{ab} = \sqrt{a}\sqrt{b}$

8 $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Logarithms

Definitions If $y = a^x$ then $x = \log_a y$

If $y = e^x$ then $x = \ell n\, y$

1 $\log(A \times B) = \log A + \log B$

2 $\log\left(\dfrac{A}{B}\right) = \log A - \log B$

3 $\log A^n = n \log A$

4 $\log_a A = \dfrac{\log_b A}{\log_b a}$

5 $a^{\log_a f} = f$ · $e^{\ell n f} = f$

Factors

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Partial fractions

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Quadratic formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

SERIES

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \ldots$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \ldots$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \ldots$$

COMPLEX NUMBERS

1 $z = a + bj = r(\cos\theta + j\sin\theta) = r\underline{|\theta} = re^{j\theta}$, where $j^2 = -1$

Modulus $r = |z| = \sqrt{(a^2 + b^2)}$

Argument $\theta = \arg z = arc\tan\frac{b}{a}$

2 Addition

$(a + jb) + (c + jd) = (a + c) + j(b + d)$

3 Subtraction

$(a + jb) - (c + jd) = (a - c) + j(b - d)$

4 If $m + jn = p + jq$, then $m = p$ and $n = q$

5 Multiplication $z_1 z_2 = r_1 r_2 \underline{|(\theta_1 + \theta_2)}$

6 Division $\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|(\theta_1 - \theta_2)}$

7 De Moivre's Theorem

$[r\underline{|\theta}]^n = r^n \underline{|n\theta} = r^n(\cos n\theta + j\sin n\theta)$

8 $z^{\frac{1}{n}}$ has n distinct roots

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\underline{\left|\frac{\theta + k360^\circ}{n}\right.}$ with $k = 0, 1, 2, \ldots, n-1$

9 $re^{j\theta} = r(\cos\theta + j\sin\theta)$

$\Re(re^{j\theta}) = r\cos\theta$ and $\Im(re^{j\theta}) = r\sin\theta$

10 $e^{a+jb} = e^a(\cos b + j\sin b)$

11 $\ell n\, re^{j\theta} = \ell n\, r + j\theta$

GEOMETRY

1 Straight line

$$y = mx + c$$
$$y - y_1 = m(x - x_1)$$

Perpendiculars, then $m_1 = \frac{-1}{m_2}$

2 Angle between two lines

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

3 Circle

$$x^2 + y^2 = r^2$$
$$(x-h)^2 + (y-k)^2 = r^2$$

4 Parabola

$$y = ax^2 + bx + c$$

axis at $x = \frac{-b}{2a}$

5 Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

6 Hyperbola

$$xy = k$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(round } x\text{-axis)}$$
$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(round } y\text{-axis)}$$

MENSURATION

1 Circle (θ in radians)

$$\text{Area} = \pi r^2$$
$$\text{Circumference} = 2\pi r$$
$$\text{Arc length } s = r\theta$$
$$\text{Sector area} = \frac{1}{2}r^2\theta = \frac{1}{2}sr$$
$$\text{Segment area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

2 Ellipse

$$\text{Area} = \pi ab$$
$$\text{Circumference} = \pi(a + b)$$

3 Cylinder

$$\text{Volume} = \pi r^2 h$$
$$\text{Surface area} = 2\pi rh + 2\pi r^2$$

4 Pyramid

$$\text{Volume} = \frac{1}{3}\text{area base} \times \text{height}$$

5 Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$
$$\text{Curved surface} = \pi r\ell$$

6 Sphere

$$A = 4\pi r^2$$
$$V = \frac{4}{3}\pi r^3$$

7 Trapezoidal rule

$$A = h\left(\frac{y_1 + y_n}{2} + y_2 + y_3 + \quad + y_{n-1}\right)$$

8 Simpsons rule

$$A = \frac{s}{3}((F + L) + 4E + 2R)$$

9 Prismoidal rule

$$V = \frac{h}{6}(A_1 + 4A_2 + A_3)$$

HYPERBOLIC FUNCTIONS

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

TRIGONOMETRY

Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = +\cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Compound angle addition and subtraction formulae

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin^2 A = ½(1 - \cos 2A)$

$\cos^2 A = ½(1 + \cos 2A)$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences

$\sin A \cos B = ½(\sin(A + B) + \sin(A - B))$

$\cos A \sin B = ½(\sin(A + B) - \sin(A - B))$

$\cos A \cos B = ½(\cos(A + B) + \cos(A - B))$

$\sin A \sin B = -½(\cos(A + B) - \cos(A - B))$

Sums or differences of sines and cosines into products

$$\sin x + \sin y = 2\sin\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\sin x - \sin y = 2\cos\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

$$\cos x + \cos y = 2\cos\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\cos x - \cos y = -2\sin\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

DIFFERENTIATION

1 $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

2 $\frac{d}{dx} k = 0$

3 $\frac{d}{dx} ax^n = anx^{n-1}$

4 $\frac{d}{dx} f\,g = f\,g' + g\,f'$

5 $\frac{d}{dx} \frac{f}{g} = \frac{g\,f' - f\,g'}{g^2}$

6 $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$

7 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$

8 Parametric equations

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

9 Maximum/minimum

For turning points $f'(x) = 0$

Let $x = a$ be a solution to the above

If $f''(a) > 0$, then a minimum

If $f''(a) < 0$, then a maximum

For points of inflection $f''(x) = 0$

Let $x = b$ be a solution for the above

Test for inflection $f(b - h)$ and $f(b + h)$

Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists

10 $\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$

11 $\frac{d}{dx} \cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$

12 $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1 + [f(x)]^2}$

13 $\frac{d}{dx} \cot^{-1} f(x) = \frac{-f'(x)}{1 + [f(x)]^2}$

14 $\frac{d}{dx} \sec^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

15 $\frac{d}{dx} \operatorname{cosec}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

16 $\frac{d}{dx} \sinh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$

17 $\frac{d}{dx} \cosh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$

18 $\frac{d}{dx} \tanh^{-1} f(x) = \frac{f'(x)}{1 - [f(x)]^2}$

19 $\frac{d}{dx} \coth^{-1} f(x) = \frac{f'(x)}{[f(x)]^2 - 1}$

20 $\frac{d}{dx} \operatorname{sech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{1 - [f(x)]^2}}$

21 $\frac{d}{dx} \operatorname{cosech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 + 1}}$

22 Increments $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{\partial z}{\partial w}\delta w$

23 Rate of change

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$

INTEGRATION

1 By parts $\int u\,dv = uv - \int v\,du$

2 $\int_a^b f(x)\,dx = F(b) - F(a)$

3 Mean value $= \frac{1}{b-a}\int_a^b y\,dx$

4 $(R.M.S.)^2 = \frac{1}{b-a}\int_a^b y^2\,dx$

TABLE OF INTEGRALS

1 $\int x^n dx = \frac{x^{(n+1)}}{n+1} + c, \quad n \neq -1$	2 $\int (u+v)dx = \int udx + \int vdx$
3 $\int audx = a\int udx$, a a constant	4 $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$
5 $\int \frac{f'(x)}{f(x)} dx = \ell n\lvert f(x)\rvert + c$	6 $\int f'(x).e^{f(x)} dx = e^{f(x)} + c$
7 $\int f'(x).a^{f(x)} dx = \frac{a^{f(x)}}{\ell na} + c$	8 $\int f'(x).\sin f(x)dx = -\cos f(x) + c$
9 $\int f'(x).\cos f(x)dx = \sin f(x) + c$	10 $\int f'(x).\tan f(x)dx = \ell n \sec f(x) + c$

11 $\int f'(x).\cot f(x)dx = \ell n \sin f(x) + c$

12 $\int f'(x).\sec f(x)dx = \ell n[\sec f(x) + \tan f(x)] + c$

13 $\int f'(x).\mathrm{cosec} f(x)dx = \ell n[\mathrm{cosec} f(x) - \cot f(x)] + c$

14 $\int f'(x).\sec^2 f(x)dx = \tan f(x) + c$

15 $\int f'(x).\mathrm{cosec}^2 f(x)dx = -\cot f(x) + c$

16 $\int f'(x).\sec f(x) \tan f(x)dx = \sec f(x) + c$

17 $\int f'(x).\mathrm{cosec} f(x) \cot f(x)dx = -\mathrm{cosec} f(x) + c$

18 $\int f'(x).\sinh f(x)dx = \cosh f(x) + c$

19 $\int f'(x).\cosh f(x)dx = \sinh f(x) + c$

20 $\int f'(x).\tanh f(x)dx = \ell n \cosh f(x) + c$

21 $\int f'(x).\coth f(x)dx = \ell n \sinh f(x) + c$

22 $\int f'(x) \mathrm{sech}^2 f(x)dx = \tanh f(x) + c$

[TURN OVER]

23 $\int f'(x)\operatorname{cosech}^2 f(x)dx = -\coth f(x) + c$

24 $\int f'(x)\operatorname{sech} f(x)\tanh f(x)dx = -\operatorname{sech} f(x) + c$

25 $\int f'(x)\operatorname{cosech} f(x)\coth f(x)dx = -\operatorname{cosech} f(x) + c$

26 $\int \frac{f'(x)}{[f(x)]^2 - a^2}dx = -\frac{1}{a}\operatorname{arccoth}\left(\frac{f(x)}{a}\right) + c$

27 $\int \frac{f'(x)}{a^2 - [f(x)]^2}dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{f(x)}{a}\right) + c$

28 $\int \frac{f'(x)}{[f(x)]^2 + a^2}dx = \frac{1}{a}\arctan\left(\frac{f(x)}{a}\right) + c$

29 $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \arcsin\left(\frac{f(x)}{a}\right) + c$

30 $\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}dx = \operatorname{arcsinh}\left(\frac{f(x)}{a}\right) + c$

31 $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}dx = \operatorname{arccosh}\left(\frac{f(x)}{a}\right) + c$

32 $\int f'(x)\sqrt{a^2 - [f(x)]^2}\,dx = \frac{a^2}{2}\arcsin\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{a^2 - [f(x)]^2} + c$

33 $\int f'(x)\sqrt{[f(x)]^2 + a^2}\,dx = \frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 + a^2} + c$

34 $\int f'(x)\sqrt{[f(x)]^2 - a^2}\,dx = -\frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 - a^2} + c$

TABLE OF LAPLACE TRANSFORMS

Study Guide 2 page 20 and page 51	
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3,$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}, \quad n = 1, 2, 3,$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t \cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt} \sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt} \cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt} \sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt} \cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c)F(t-c)$	$e^{-cs} f(s)$
$\delta(t-a)$	e^{-as}
Study Guide 2 page 55	
$\mathcal{L}\{f(t)\} = F(s)$	
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	
$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	