

**MAT3700**

May/June 2013

MATHEMATICS III (ENGINEERING)

Duration 2 Hours

84 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT**Use of a non-programmable pocket calculator is permissible****Closed book examination.****This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue.****This examination question paper consists of 3 pages including this cover page plus****Formulae sheets (page (i) to (v), plus****A table of integrals (page (i) and (ii)) plus****A table of Laplace transforms (page iii)****Answer all the questions in numerical order.**

QUESTION 1

Solve the following differential equations

$$1 \ 1 \quad x(y-3)dy = 4ydx \quad (4)$$

$$1 \ 2 \quad (x^3 + y^3)dx - 3xy^2dy = 0 \quad [\text{Hint Put } y = vx] \quad (7)$$

$$1 \ 3 \quad \frac{dy}{dx} + y \cot x = 5e^{\cos x} \quad (5)$$

[16]

QUESTION 2

Find the general solutions of the following differential equations using D-operator methods

$$2 \ 1 \quad (D^2 + 9)y = \cos(2x + 3) \quad (5)$$

$$2 \ 2 \quad D(D^2 + 3D + 2)y = x^2 + 4x + 8 \quad (9)$$

[14]

QUESTION 3

The equation of motion of a body is given by $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = e^{2t} \cos t$, where y is the distance and t is the time

Determine a general solution for y in terms of t , by using D-operator methods. (9)

[9]

QUESTION 4

Solve for x in the following set of simultaneous differential equations by using D-operator methods

$$\begin{aligned} (D + 2)x + 3y &= 0 \\ 3x + (D + 2)y &= 2e^{2t} \end{aligned} \quad (7)$$

[7]

QUESTION 5

Determine the following

$$5 \ 1 \quad L\{t^2H(t-1)\} \quad (3)$$

$$5 \ 2 \quad L^{-1}\left\{\frac{4s+5}{s^2+9}\right\} \quad (2)$$

[5]

[TURN OVER]

QUESTION 6

Determine the unique solution of the following differential equation by using Laplace transforms: $y''(t) + 2y'(t) - 3y(t) = e^{-3t}$, if $y(0) = 0$ and $y'(0) = 0$

$$\left[\text{Given } \frac{1}{(s+3)^2(s-1)} = \frac{-1}{16(s+3)} - \frac{1}{4(s+3)^2} + \frac{1}{16(s-1)} \right] \quad (7)$$

[7]

QUESTION 7

A cantilever beam, clamped at $x = 0$ and free at $x = b$, carries a uniform load k per unit

length. Given the differential equation $\frac{d^4y}{dx^4} = \frac{k}{EI}$, $0 < x < b$ and boundary conditions

$y(0) = 0, y'(0) = 0, y''(b) = 0$ and $y'''(b) = 0$, show that the deflection of the beam is given by

$$y(x) = \frac{k}{24EI} (x^4 - 4bx^3 + 6b^2x^2) \text{ by using Laplace transforms.} \quad (10)$$

[10]

QUESTION 8

$$\text{If } A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \text{ determine the eigenvalues of } A \text{ and an eigenvector for } A \quad (6)$$

[6]

QUESTION 9

Given the function defined by $f(t) = \begin{cases} -\pi & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$, with a period of 2π .

9.1 Determine a_0 and a_n to write down the Fourier series for $f(t)$ (8)

9.2 Write down the Fourier series for $f(t)$ given that $b_n = \frac{1-2(-1)^n}{n}$ (2)

[10]

Full marks = 84

Examiners First Ms L E Greyling

Second Dr J M Manale

External Dr J N Mwambakana

FORMULA SHEET

ALGEBRA		Factors
<u>Laws of indices</u>		$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
1	$a^m \times a^n = a^{m+n}$	
2	$\frac{a^m}{a^n} = a^{m-n}$	
3	$(a^m)^n = a^{mn} = (a^n)^m$	
4	$a^{\frac{m}{n}} = \sqrt[n]{a^m}$	
5	$a^{-n} = \frac{1}{a^n}$ and $a^n = \frac{1}{a^{-n}}$	
6	$a^0 = 1$	
7	$\sqrt{ab} = \sqrt{a}\sqrt{b}$	
8	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	
<u>Logarithms</u>		
Definitions If $y = a^x$ then $x = \log_a y$		
If $y = e^x$ then $x = \ln y$		
1	$\log(A \times B) = \log A + \log B$	
2	$\log\left(\frac{A}{B}\right) = \log A - \log B$	
3	$\log A^n = n \log A$	
4	$\log_a A = \frac{\log_b A}{\log_b a}$	
5	$a^{\log_a f} = f$ $\therefore e^{\ln f} = f$	
<u>Determinants</u>		Quadratic formula
$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ $= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$	If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	

SERIES**Binomial Theorem**

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 +$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 +$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} +$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} +$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) +$$

COMPLEX NUMBERS

1 $z = a + bj = r(\cos \theta + j \sin \theta) = r|\underline{\theta}| = re^{j\theta}$,
where $j^2 = -1$

Modulus $r = |z| = \sqrt{(a^2 + b^2)}$

Argument $\theta = \arg z = \arctan \frac{b}{a}$

2 Addition

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

3 Subtraction

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

4 If $m + jn = p + jq$, then $m = p$ and $n = q$

5 Multiplication $z_1 z_2 = r_1 r_2 |\underline{(\theta_1 + \theta_2})|$

6 Division $\frac{z_1}{z_2} = \frac{r_1}{r_2} |\underline{(\theta_1 - \theta_2})|$

7 De Moivre's Theorem

$$[r|\underline{\theta}]^n = r^n |\underline{n\theta}| = r^n (\cos n\theta + j \sin n\theta)$$

8 $z^{\frac{1}{n}}$ has n distinct roots

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left| \frac{\theta + k360^\circ}{n} \right| \text{ with } k = 0, 1, 2, \dots, n-1$$

9 $re^{j\theta} = r(\cos \theta + j \sin \theta)$

$\therefore \Re(re^{j\theta}) = r \cos \theta$ and $\Im(re^{j\theta}) = r \sin \theta$

10 $e^{a+jb} = e^a (\cos b + j \sin b)$

11 $\ln re^{j\theta} = \ln r + j\theta$

GEOMETRY		MENSURATION
1 Straight line	$y = mx + c$ $y - y_1 = m(x - x_1)$	1 Circle (θ in radians) Area = πr^2 Circumference = $2\pi r$ Arc length $s = r\theta$
	Perpendiculars, then $m_1 = \frac{-1}{m_2}$	Sector area = $\frac{1}{2}r^2\theta = \frac{1}{2}sr$ Segment area = $\frac{1}{2}r^2(\theta - \sin \theta)$
2 Angle between two lines	$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$	2 Ellipse Area = πab Circumference = $\pi(a + b)$
3 Circle	$x^2 + y^2 = r^2$ $(x - h)^2 + (y - k)^2 = r^2$	3. Cylinder. Volume = $\pi r^2 h$ Surface area = $2\pi rh + 2\pi r^2$
4 Parabola	$y = ax^2 + bx + c$ axis at $x = \frac{-b}{2a}$	4 Pyramid Volume = $\frac{1}{3}$ area base \times height
5 Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	5 Cone Volume = $\frac{1}{3}\pi r^2 h$ Curved surface = $\pi r l$
6 Hyperbola	$xy = k$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (round x-axis) $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (round y-axis)	6 Sphere $A = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$
		7 Trapezoidal rule $A = h \left(\frac{y_1 + y_n}{2} + y_2 + y_3 + \dots + y_{n-1} \right)$
		8 Simpsons rule $A = \frac{s}{3} ((F + L) + 4E + 2R)$
		9 Prismoidal rule $V = \frac{h}{6} (A_1 + 4A_2 + A_3)$

HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{Definitions} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\begin{aligned}\cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2 \cosh^2 x - 1 \\ &= 1 + 2 \sinh^2 x\end{aligned}$$

TRIGONOMETRY*Identities*

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = +\cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

Compound angle addition and subtraction formulae

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 2 \cos^2 A - 1$$

$$= 1 - 2 \sin^2 A$$

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences

$$\sin A \cos B = \frac{1}{2}(\sin(A + B) + \sin(A - B))$$

$$\cos A \sin B = \frac{1}{2}(\sin(A + B) - \sin(A - B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A + B) + \cos(A - B))$$

$$\sin A \sin B = -\frac{1}{2}(\cos(A + B) - \cos(A - B))$$

Sums or differences of sines and cosines into products

$$\sin x + \sin y = 2 \sin \left[\frac{x+y}{2} \right] \cos \left[\frac{x-y}{2} \right]$$

$$\sin x - \sin y = 2 \cos \left[\frac{x+y}{2} \right] \sin \left[\frac{x-y}{2} \right]$$

$$\cos x + \cos y = 2 \cos \left[\frac{x+y}{2} \right] \cos \left[\frac{x-y}{2} \right]$$

$$\cos x - \cos y = -2 \sin \left[\frac{x+y}{2} \right] \sin \left[\frac{x-y}{2} \right]$$

DIFFERENTIATION

1 $\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

2 $\frac{d}{dx} k = 0$

3 $\frac{d}{dx} ax^n = anx^{n-1}$

4 $\frac{d}{dx} f g = f g' + g f'$

5 $\frac{d}{dx} \frac{f}{g} = \frac{g f' - f g'}{g^2}$

6 $\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1} f'(x)$

7 $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$

8 Parametric equations

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \right)$$

9 Maximum/minimum

For turning points $f'(x) = 0$ Let $x = a$ be a solution for the aboveIf $f'(a) > 0$, then a minimumIf $f'(a) < 0$, then a maximumFor points of inflection $f''(x) = 0$ Let $x = b$ be a solution for the aboveTest for inflection $f(b-h)$ and $f(b+h)$ Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists

10 $\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$

11 $\frac{d}{dx} \cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

12 $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1+[f(x)]^2}$

13 $\frac{d}{dx} \cot^{-1} f(x) = \frac{-f'(x)}{1+[f(x)]^2}$

14 $\frac{d}{dx} \sec^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

15 $\frac{d}{dx} \operatorname{cosec}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$

16 $\frac{d}{dx} \sinh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$

17 $\frac{d}{dx} \cosh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$

18 $\frac{d}{dx} \tanh^{-1} f(x) = \frac{f'(x)}{1-[f(x)]^2}$

19 $\frac{d}{dx} \coth^{-1} f(x) = \frac{f'(x)}{[f(x)]^2 - 1}$

20 $\frac{d}{dx} \operatorname{sech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$

21 $\frac{d}{dx} \operatorname{cosech}^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 + 1}}$

22 Increments $\delta z = \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y + \frac{\partial z}{\partial w} \delta w$

23 Rate of change

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$$

INTEGRATION

1 By parts $\int u dv = uv - \int v du$

2 $\int_a^b f(x) dx = F(b) - F(a)$

3 Mean value $= \frac{1}{b-a} \int_a^b y dx$

4 $(RMS)^2 = \frac{1}{b-a} \int_a^b y^2 dx$

TABLE OF INTEGRALS

1	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$	2	$\int (u+v)dx = \int udx + \int vdx$
3	$\int a u dx = a \int u dx, \quad a \text{ a constant}$	4	$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$
5	$\int \frac{f''(x)}{f(x)} dx = \ln f(x) + c$	6	$\int f'(x)e^{f(x)} dx = e^{f(x)} + c$
7	$\int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ln a} + c$	8	$\int f'(x) \sin f(x) dx = -\cos f(x) + c$
9	$\int f'(x) \cos f(x) dx = \sin f(x) + c$	10	$\int f'(x) \tan f(x) dx = \ln \sec f(x) + c$
11	$\int f'(x) \cot f(x) dx = \ln \sin f(x) + c$		
12	$\int f'(x) \sec f(x) dx = \ln [\sec f(x) + \tan f(x)] + c$		
13	$\int f'(x) \operatorname{cosec} f(x) dx = \ln [\operatorname{cosec} f(x) - \cot f(x)] + c$		
14	$\int f'(x) \sec^2 f(x) dx = \tan f(x) + c$		
15	$\int f'(x) \operatorname{cosec}^2 f(x) dx = -\cot f(x) + c$		
16	$\int f'(x) \sec f(x) \tan f(x) dx = \sec f(x) + c$		
17	$\int f'(x) \operatorname{cosec} f(x) \cot f(x) dx = -\operatorname{cosec} f(x) + c$		
18	$\int f'(x) \sinh f(x) dx = \cosh f(x) + c$		
19	$\int f'(x) \cosh f(x) dx = \sinh f(x) + c$		
20	$\int f'(x) \tanh f(x) dx = \ln \cosh f(x) + c$		
21	$\int f'(x) \coth f(x) dx = \ln \sinh f(x) + c$		
22	$\int f'(x) \operatorname{sech}^2 f(x) dx = \tanh f(x) + c$		

$$23 \int f'(x) \operatorname{cosech}^2 f(x) dx = -\coth f(x) + c$$

$$24 \int f'(x) \operatorname{sech} f(x) \tanh f(x) dx = -\operatorname{sech} f(x) + c$$

$$25 \int f'(x) \operatorname{cosech} f(x) \coth f(x) dx = -\operatorname{cosech} f(x) + c$$

$$26 \int \frac{f'(x)}{[f(x)]^2 - a^2} dx = -\frac{1}{a} \operatorname{arccoth} \left(\frac{f(x)}{a} \right) + c$$

$$27 \int \frac{f'(x)}{a^2 - [f(x)]^2} dx = \frac{1}{a} \operatorname{arctanh} \left(\frac{f(x)}{a} \right) + c$$

$$28 \int \frac{f'(x)}{[f(x)]^2 + a^2} dx = \frac{1}{a} \operatorname{arctan} \left(\frac{f(x)}{a} \right) + c$$

$$29 \int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \operatorname{arcsin} \left(\frac{f(x)}{a} \right) + c$$

$$30 \int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}} dx = \operatorname{arsinh} \left(\frac{f(x)}{a} \right) + c$$

$$31 \int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}} dx = \operatorname{arccosh} \left(\frac{f(x)}{a} \right) + c$$

$$32 \int f'(x) \sqrt{a^2 - [f(x)]^2} dx = \frac{a^2}{2} \operatorname{arcsin} \left(\frac{f(x)}{a} \right) + \frac{f(x)}{2} \sqrt{a^2 - [f(x)]^2} + c$$

$$33 \int f'(x) \sqrt{[f(x)]^2 + a^2} dx = \frac{a^2}{2} \ln \left| f(x) + \sqrt{[f(x)]^2 + a^2} \right| + \frac{f(x)}{2} \sqrt{[f(x)]^2 + a^2} + c$$

$$34 \int f'(x) \sqrt{[f(x)]^2 - a^2} dx = -\frac{a^2}{2} \ln \left| f(x) + \sqrt{[f(x)]^2 - a^2} \right| + \frac{f(x)}{2} \sqrt{[f(x)]^2 - a^2} + c$$

TABLE OF LAPLACE TRANSFORMS

Study Guide 2 page 20 and page 51	
$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}, \quad n=1, 2, 3,$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}, \quad n=1, 2, 3,$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t \cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt} \sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt} \cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt} \sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt} \cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c)F(t-c)$	$e^{-cs} f(s)$
$\delta(t-a)$	e^{-as}
Study Guide 2 page 55	
$\mathcal{L}\{f(t)\} = F(s)$	
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	
$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	