UNIVERSITY EXAMINATIONS

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MATHEMATICS III (ENGINEERING)

Duration 2 Hours

84 Marks

EXAMINATION PANEL AS APPOINTED BY THE DEPARTMENT.

Use of a non-programmable pocket calculator is permissible

Closed book examination.

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This examination question paper consists of 3 pages including this cover page plus

Formulae sheets (page (i) to (v), plus
A table of integrals (page (i) and (ii)) plus
A table of Laplace transforms (page iii)

Answer all the questions **in numerical order.**

QUESTION 1

Solve the following differential equations:

1.1 $2x(y+1)dx - ydy = 0$, given that if $x=0$ then $y=-2$. (6)

1.2 $x\frac{dy}{dx} - y = x\tan\left(\frac{y}{x}\right)$ [Hint: Put $y = vx$] (5)

1.3 $\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$ (6)

[17]

QUESTION 2

Find the general solutions of the following differential equations using **D-operator** methods:

2.1 $(D^2-9)y = 3e^x + x - \sin 4x$ (7)

2.2 $(D^2-4D+4)y = e^{2x}(x^2+1)$ (7)

[14]

QUESTION 3

The conditions in a certain electrical circuit are represented by the following differential equation: $10\frac{d^2i}{dt^2} + 60\frac{di}{dt} + \frac{i}{0{,}004} = 240\cos 5t$. Determine the general and the steady state solutions for the current, i in terms of t, by using **D-operator methods.** (8)

[8]

QUESTION 4

Solve for **only** x in the following set of simultaneous differential equations by using **D-operator** methods:

$$\begin{aligned} (D^2-1)y + \quad 5Dx &= t \\ 2Dy - (D^2-4)x &= 2 \end{aligned}$$ (8)

[8]

QUESTION 5

Determine the following:

5.1 $L\{2t^4 + H(t-3) + 4\cos 5t\}$ (3)

5.2 $L^{-1}\left\{\frac{2}{(s+2)^3}\right\}$ (3)

[6]

[TURN OVER]

QUESTION 6

Determine the unique solution of the following differential equation by using

Laplace transforms: $y''(t)+6y'(t)+13y(t)=0$, if $y(0)=3$ and $y'(0)=7$. (7)

[7]

QUESTION 7

The motion of a mass-spring system, with no friction, is given by the equation

$$\frac{d^2y}{dt^2}+y=2\delta(t-2\pi)-2\delta(t-4\pi)$$

with an impulsive force at $t=2\pi$ and an equal and opposite force at $t=4\pi$.
If $y(0)=0$ and $y'(0)=0$, use Laplace transform methods to solve for y. (8)

[8]

QUESTION 8

If $A=\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$, determine the eigenvalues of A and an eigenvector for A. (6)

[6]

QUESTION 9

Given the function defined by $f(x)=x, 0 \le x \le 2$, find the half-range Fourier sine series for *f(x)* Sketch the function within and outside of this range. (10)

[10]

Full marks = 84

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FORMULA SHEET

ALGEBRA

Laws of indices

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn} = (a^n)^m$
4. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
5. $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$
6. $a^0 = 1$
7. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Logarithms

Definitions. If $y = a^x$ then $x = \log_a y$

If $y = e^x$ then $x = \ell n\, y$

1. $\log(A \times B) = \log A + \log B$
2. $\log\left(\dfrac{A}{B}\right) = \log A - \log B$
3. $\log A^n = n \log A$
4. $\log_a A = \dfrac{\log_b A}{\log_b a}$
5. $a^{\log_a f} = f \quad \therefore \quad e^{\ell n f} = f$

Factors

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Partial fractions

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

Quadratic formula

If $\quad ax^2 + bx + c = 0$

then $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

SERIES

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \dots$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \dots$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \dots$$

COMPLEX NUMBERS

1. $z = a + bj = r(\cos\theta + j\sin\theta) = r\underline{|\theta} = re^{j\theta}$, where $j^2 = -1$

   Modulus: $r = |z| = \sqrt{(a^2 + b^2)}$

   Argument: $\theta = \arg z = arc\tan\frac{b}{a}$
2. Addition:

   $(a + jb) + (c + jd) = (a + c) + j(b + d)$
3. Subtraction:

   $(a + jb) - (c + jd) = (a - c) + j(b - d)$
4. If $m + jn = p + jq$, then $m = p$ and $n = q$
5. Multiplication: $z_1 z_2 = r_1 r_2 \underline{|(\theta_1 + \theta_2)}$
6. Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|(\theta_1 - \theta_2)}$
7. De Moivre's Theorem

   $[r\underline{|\theta}]^n = r^n \underline{|n\theta} = r^n(\cos n\theta + j\sin n\theta)$
8. $z^{\frac{1}{n}}$ has n distinct roots:

   $z^{\frac{1}{n}} = r^{\frac{1}{n}}\underline{\left|\frac{\theta + k360^\circ}{n}\right.}$ with $k = 0, 1, 2, \dots, n-1$
9. $re^{j\theta} = r(\cos\theta + j\sin\theta)$

   $\therefore \Re(re^{j\theta}) = r\cos\theta$ and $\Im(re^{j\theta}) = r\sin\theta$
10. $e^{a+jb} = e^a(\cos b + j\sin b)$
11. $\ell n\, re^{j\theta} = \ell n\, r + j\theta$

GEOMETRY

1. Straight line: $y = mx + c$
   $y - y_1 = m(x - x_1)$

   Perpendiculars, then $m_1 = \frac{-1}{m_2}$

2. Angle between two lines.

   $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

3. Circle:

   $x^2 + y^2 = r^2$

   $(x-h)^2 + (y-k)^2 = r^2$

4. Parabola:

   $y = ax^2 + bx + c$

   axis at $x = \frac{-b}{2a}$

5. Ellipse:

   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

6. Hyperbola:

   $xy = k$

   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (round x - axis)

   $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (round y - axis)

MENSURATION

1 Circle (θ in radians)

   $\text{Area} = \pi r^2$

   $\text{Circumference} = 2\pi r$

   $\text{Arc length } s = r\theta$

   $\text{Sector area} = \frac{1}{2}r^2\theta = \frac{1}{2}sr$

   $\text{Segment area} = \frac{1}{2}r^2(\theta - \sin\theta)$

2. Ellipse.

   $\text{Area} = \pi ab$

   $\text{Circumference} = \pi(a + b)$

3. Cylinder:

   $\text{Volume} = \pi r^2 h$

   $\text{Surface area} = 2\pi rh + 2\pi r^2$

4. Pyramid:

   $\text{Volume} = \frac{1}{3}\text{area base} \times \text{height}$

5. Cone:

   $\text{Volume} = \frac{1}{3}\pi r^2 h$

   $\text{Curved surface} = \pi r\ell$

6. Sphere:

   $A = 4\pi r^2$

   $V = \frac{4}{3}\pi r^3$

7. Trapezoidal rule:

   $A = h\left(\frac{y_1 + y_n}{2} + y_2 + y_3 + \ldots + y_{n-1}\right)$

8. Simpsons rule:

   $A = \frac{s}{3}\left((F + L) + 4E + 2R\right)$

9. Prismoidal rule

   $V = \frac{h}{6}(A_1 + 4A_2 + A_3)$

HYPERBOLIC FUNCTIONS

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$= 2\cosh^2 x - 1$$
$$= 1 + 2\sinh^2 x$$

TRIGONOMETRY

Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = +\cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Compound angle addition and subtraction formulae:

$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\sin(A - B) = \sin A \cos B - \cos A \sin B$
$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles:

$\sin 2A = 2 \sin A \cos A$
$\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$
$\sin^2 A = ½(1 - \cos 2A)$
$\cos^2 A = ½(1 + \cos 2A)$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences.

$\sin A \cos B = ½(\sin(A + B) + \sin(A - B))$
$\cos A \sin B = ½(\sin(A + B) - \sin(A - B))$
$\cos A \cos B = ½(\cos(A + B) + \cos(A - B))$
$\sin A \sin B = -½(\cos(A + B) - \cos(A - B))$

Sums or differences of sines and cosines into products:

$$\sin x + \sin y = 2\sin\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\sin x - \sin y = 2\cos\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

$$\cos x + \cos y = 2\cos\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\cos x - \cos y = -2\sin\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

DIFFERENTIATION

1. $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
2. $\frac{d}{dx} k = 0$
3. $\frac{d}{dx} ax^n = anx^{n-1}$
4. $\frac{d}{dx} f.g = f.g' + g.f'$
5. $\frac{d}{dx} \frac{f}{g} = \frac{g.f' - f\,g'}{g^2}$
6. $\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1} f'(x)$
7. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$
8. Parametric equations

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$$

9. Maximum/minimum
   For turning points $f'(x) = 0$
   Let $x = a$ be a solution for the above
   If $f'(a) > 0$, then a minimum
   If $f'(a) < 0$, then a maximum
   For points of inflection $f''(x) = 0$
   Let $x = b$ be a solution for the above
   Test for inflection: $f(b - h)$ and $f(b + h)$
   Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists

10. $\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
11. $\frac{d}{dx} \cos^{-1} f(x) = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
12. $\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1+[f(x)]^2}$
13. $\frac{d}{dx} \cot^{-1} f(x) = \frac{-f'(x)}{1+[f(x)]^2}$
14. $\frac{d}{dx} \sec^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$
15. $\frac{d}{dx} \text{cosec}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{[f(x)]^2 - 1}}$
16. $\frac{d}{dx} \sinh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 + 1}}$
17. $\frac{d}{dx} \cosh^{-1} f(x) = \frac{f'(x)}{\sqrt{[f(x)]^2 - 1}}$
18. $\frac{d}{dx} \tanh^{-1} f(x) = \frac{f'(x)}{1-[f(x)]^2}$
19. $\frac{d}{dx} \coth^{-1} f(x) = \frac{f'(x)}{[f(x)]^2 - 1}$
20. $\frac{d}{dx} \text{sech}^{-1} f(x) = \frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$
21. $\frac{d}{dx} \text{cosech}^{-1} f(x) = \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 + 1}}$
22. Increments. $\delta z = \frac{\partial z}{\partial x}.\delta x + \frac{\partial z}{\partial y}.\delta y + \frac{\partial z}{\partial w}.\delta w$
23. Rate of change:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt}$$

INTEGRATION

1. By parts: $\int u\,dv = uv - \int v\,du$
2. $\int_a^b f(x)dx = F(b) - F(a)$
3. Mean value $= \frac{1}{b-a} \int_a^b y\,dx$
4. $(\text{R.M.S.})^2 = \frac{1}{b-a} \int_a^b y^2\,dx$

TABLE OF INTEGRALS

1. $\int x^n dx = \frac{x^{(n+1)}}{n+1} + c, \quad n \neq -1$

2. $\int (u+v)dx = \int udx + \int vdx$

3. $\int audx = a\int udx$, a a constant

4. $\int [f(x)]^n . f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$

5. $\int \frac{f'(x)}{f(x)} dx = \ell n|f(x)| + c$

6. $\int f'(x).e^{f(x)} dx = e^{f(x)} + c$

7. $\int f'(x).a^{f(x)} dx = \frac{a^{f(x)}}{\ell na} + c$

8. $\int f'(x).\sin f(x)dx = -\cos f(x) + c$

9. $\int f'(x).\cos f(x)dx = \sin f(x) + c$

10. $\int f'(x).\tan f(x)dx = \ell n \sec f(x) + c$

11. $\int f'(x).\cot f(x)dx = \ell n \sin f(x) + c$

12 $\int f'(x).\sec f(x)dx = \ell n[\sec f(x) + \tan f(x)] + c$

13. $\int f'(x).\text{cosec} f(x)dx = \ell n[\text{cosec} f(x) - \cot f(x)] + c$

14. $\int f'(x).\sec^2 f(x)dx = \tan f(x) + c$

15. $\int f'(x).\text{cosec}^2 f(x)dx = -\cot f(x) + c$

16. $\int f'(x).\sec f(x) \tan f(x)dx = \sec f(x) + c$

17. $\int f'(x).\text{cosec} f(x).\cot f(x)dx = -\text{cosec} f(x) + c$

18. $\int f'(x).\sinh f(x)dx = \cosh f(x) + c$

19. $\int f'(x).\cosh f(x)dx = \sinh f(x) + c$

20. $\int f'(x).\tanh f(x)dx = \ell n \cosh f(x) + c$

21. $\int f'(x).\coth f(x)dx = \ell n \sinh f(x) + c$

22. $\int f'(x).\text{sech}^2 f(x)dx = \tanh f(x) + c$

[TURN OVER]

23 $\int f'(x)\operatorname{cosech}^2 f(x)dx = -\coth f(x) + c$

24. $\int f'(x)\operatorname{sech} f(x).\tanh f(x)dx = -\operatorname{sech} f(x) + c$

25 $\int f'(x)\operatorname{cosech} f(x).\coth f(x)dx = -\operatorname{cosech} f(x) + c$

26. $\int \frac{f'(x)}{[f(x)]^2 - a^2}dx = -\frac{1}{a}\operatorname{arc}\coth\left(\frac{f(x)}{a}\right) + c$

27. $\int \frac{f'(x)}{a^2 - [f(x)]^2}dx = \frac{1}{a}\operatorname{arc}\tanh\left(\frac{f(x)}{a}\right) + c$

28 $\int \frac{f'(x)}{[f(x)]^2 + a^2}dx = \frac{1}{a}\operatorname{arc}\tan\left(\frac{f(x)}{a}\right) + c$

29. $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \operatorname{arc}\sin\left(\frac{f(x)}{a}\right) + c$

30. $\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}dx = \operatorname{arc}\sinh\left(\frac{f(x)}{a}\right) + c$

31. $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}}dx = \operatorname{arc}\cosh\left(\frac{f(x)}{a}\right) + c$

32. $\int f'(x)\sqrt{a^2 - [f(x)]^2}dx = \frac{a^2}{2}\operatorname{arc}\sin\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{a^2 - [f(x)]^2} + c$

33 $\int f'(x)\sqrt{[f(x)]^2 + a^2}dx = \frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 + a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 + a^2} + c$

34. $\int f'(x)\sqrt{[f(x)]^2 - a^2}dx = -\frac{a^2}{2}\ell n\left|f(x) + \sqrt{[f(x)]^2 - a^2}\right| + \frac{f(x)}{2}\sqrt{[f(x)]^2 - a^2} + c$

TABLE OF LAPLACE TRANSFORMS

Study Guide 2 page 20 and page 51

$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}, \quad n=1,2,3,\ ..$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}, \quad n=1,2,3,$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t\sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t\cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt}\sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt}\cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt}\sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt}\cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c)F(t-c)$	$e^{-cs}f(s)$
$\delta(t-a)$	e^{-as}

Study Guide 2 page 55

$\mathcal{L}\{f(t)\}=F(s)$

$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$

$\mathcal{L}\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$

$\mathcal{L}\{f'''(t)\}=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$