CHAPTER

4

OPTION MARKETS AND CONTRACTS

LEARNING OUTCOMES

After completing this chapter, you will be able to do the following:

- Identify the basic elements and characteristics of option contracts.
- Define European option, American option, moneyness, payoff, intrinsic value, and time value.
- Differentiate between exchange-traded options and over-the-counter options.
- Identify the different varieties of options in terms of the types of instruments underlying them.
- Compare and contrast interest rate options to forward rate agreements (FRAs).
- Explain how option payoffs are determined, and show how interest rate option payoffs differ from the payoffs of other types of options.
- Define interest rate caps and floors.
- Identify the minimum and maximum values of European options and American options.
- Illustrate how the lower bounds of European calls and puts are determined by constructing portfolio combinations that prevent arbitrage, and calculate an option's lower bound.
- Determine the lowest prices of European and American calls and puts based on the rules for lower bounds.
- Illustrate how a portfolio (combination) of options establishes the relationship between options that differ only by exercise price.
- Explain how option prices are affected by differences in the time to expiration.
- Illustrate how put–call parity for European options is established by comparing the payoffs on a fiduciary call and a protective put, explain how to use this result to create synthetic instruments, and explain why an investor would want to do so.
- Illustrate how violations of put–call parity for European options can be exploited and how those violations are eliminated.
- Explain the relationship between American options and European options in terms of the lower bounds on option prices and the possibility of early exercise.
- Explain how cash flows on the underlying asset affect put–call parity and the lower bounds on option prices.
- Identify the directional effect of an interest rate change on an option's price.

- Explain how an option price is determined in a one-period binomial model.
- Illustrate how an arbitrage opportunity can be exploited in a one-period binomial model.
- Explain how an option price is determined in a two-period binomial model.
- Calculate prices of options on bonds and interest rate options in one- and two-period binomial models.
- Explain how the binomial model value converges as time periods are added.
- List and briefly explain the assumptions underlying the Black–Scholes–Merton model.
- Calculate the value of a European option using the Black–Scholes–Merton model.
- Explain how an option price, as represented by the Black–Scholes–Merton model, is affected by each of the input values (the Greeks).
- Explain and illustrate the concept of an option's delta and how it is used in dynamic hedging.
- Explain the gamma effect on an option's price and delta.
- Discuss how cash flows on the underlying asset affect an option's price.
- Explain and illustrate the two methods for estimating the volatility of the underlying.
- Illustrate how put–call parity for options on forwards (or futures) is established.
- Explain how American options on forwards and futures are alike, and explain how they differ from European options.
- Calculate the value of a European option on forwards (or futures) using the Black model.
- Calculate the value of a European interest rate option using the Black model.
- Discuss the role of options markets in financial systems and society.

1 INTRODUCTION

In Chapter 1, we provided a general introduction to derivative markets. In Chapter 2 we examined forward contracts, and in Chapter 3 we looked at futures contracts. We noted how similar forward and futures contracts are: Both are commitments to buy an underlying asset at a fixed price at a later date. Forward contracts, however, are privately created, over-the-counter customized instruments that carry credit risk. Futures contracts are publicly traded, exchange-listed standardized instruments that effectively have no credit risk. Now we turn to options. Like forwards and futures, they are derivative instruments that provide the opportunity to buy or sell an underlying asset with a specific expiration date. But in contrast, buying an option gives the *right*, not the obligation, to buy or sell an underlying asset. And whereas forward and futures contracts involve no exchange of cash up front, options require a cash payment from the option buyer to the option seller.

Yet options contain several features common to forward and futures contracts. For one, options can be created by any two parties with any set of terms they desire. In this sense, options can be privately created, over-the-counter, customized instruments that are subject to credit risk. In addition, however, there is a large market for publicly traded, exchange-listed, standardized options, for which credit risk is essentially eliminated by the clearinghouse.

Just as we examined the pricing of forwards and futures in the last two chapters, we shall examine option pricing in this chapter. We shall also see that options can be created out of forward contracts, and that forward contracts can be created out of options. With some simplifying assumptions, options can be created out of futures contracts and futures contracts can be created out of options.

Finally, we note that options also exist that have a futures or forward contract as the underlying. These instruments blend some of the features of both options and forwards/futures.

As background, we discuss the definitions and characteristics of options.

2 BASIC DEFINITIONS AND ILLUSTRATIONS OF OPTIONS CONTRACTS

In Chapter 1, we defined an option as a financial derivative contract that provides a party the right to buy or sell an underlying at a fixed price by a certain time in the future. The party holding the right is the option buyer; the party granting the right is the option seller. There are two types of options, a **call** and a **put**. A call is an option granting the right to buy the underlying; a put is an option granting the right to sell the underlying. With the exception of some advanced types of options, a given option contract is either a call, granting the right to buy, or a put, granting the right to sell, but not both.[1] We emphasize that this right to buy or sell is held by the option buyer, also called the long or option holder, and granted by the option seller, also called the short or option writer.

To obtain this right, the option buyer pays the seller a sum of money, commonly referred to as the **option price**. On occasion, this option price is called the **option premium** or just the **premium**. This money is paid when the option contract is initiated.

2.1 Basic Characteristics of Options

The fixed price at which the option holder can buy or sell the underlying is called the **exercise price**, **strike price**, **striking price**, or **strike**. The use of this right to buy or sell the underlying is referred to as **exercise** or **exercising the option**. Like all derivative contracts, an option has an **expiration date**, giving rise to the notion of an option's **time to expiration**. When the expiration date arrives, an option that is not exercised simply expires.

What happens at exercise depends on the whether the option is a call or a put. If the buyer is exercising a call, she pays the exercise price and receives either the underlying or an equivalent cash settlement. On the opposite side of the transaction is the seller, who receives the exercise price from the buyer and delivers the underlying, or alternatively, pays an equivalent cash settlement. If the buyer is exercising a put, she delivers the stock and receives the exercise price or an equivalent cash settlement. The seller, therefore, receives the underlying and must pay the exercise price or the equivalent cash settlement.

As noted in the above paragraph, cash settlement is possible. In that case, the option holder exercising a call receives the difference between the market value of the underlying and the exercise price from the seller in cash. If the option holder exercises a put, she receives the difference between the exercise price and the market value of the underlying in cash.

[1] Of course, a party could buy both a call and a put, thereby holding the right to buy *and* sell the underlying.

There are two primary exercise styles associated with options. One type of option has **European-style exercise**, which means that the option can be exercised only on its expiration day. In some cases, expiration could occur during that day; in others, exercise can occur only when the option has expired. In either case, such an option is called a **European option**. The other style of exercise is **American-style exercise**. Such an option can be exercised on any day through the expiration day and is generally called an **American option**.[2]

Option contracts specify a designated number of units of the underlying. For exchange-listed, standardized options, the exchange establishes each term, with the exception of the price. The price is negotiated by the two parties. For an over-the-counter option, the two parties decide each of the terms through negotiation.

In an over-the-counter option—one created off of an exchange by any two parties who agree to trade—the buyer is subject to the possibility of the writer defaulting. When the buyer exercises, the writer must either deliver the stock or cash if a call, or pay for the stock or pay cash if a put. If the writer cannot do so for financial reasons, the option holder faces a credit loss. Because the option holder paid the price up front and is not required to do anything else, the seller does not face any credit risk. Thus, although credit risk is bilateral in forward contracts—the long assumes the risk of the short defaulting, and the short assumes the risk of the long defaulting—the credit risk in an option is unilateral. Only the buyer faces credit risk because only the seller can default. As we discuss later, in exchange-listed options, the clearinghouse guarantees payment to the buyer.

2.2 Some Examples of Options

Consider some call and put options on Sun Microsystems (SUNW). The date is 13 June and Sun is selling for $16.25. Exhibit 4-1 gives information on the closing prices of four options, ones expiring in July and October and ones with exercise prices of 15.00 and 17.50. The July options expire on 20 July and the October options expire on 18 October. In the parlance of the profession, these are referred to as the July 15 calls, July 17.50 calls, October 15 calls, and October 17.50 calls, with similar terminology for the puts. These particular options are American style.

EXHIBIT 4-1 Closing Prices of Selected Options on SUNW, 13 June

Exercise Price	July Calls	October Calls	July Puts	October Puts
15.00	2.35	3.30	0.90	1.85
17.50	1.00	2.15	2.15	3.20

Note: Stock price is $16.25; July options expire on 20 July; October options expire on 18 October.

Consider the July 15 call. This option permits the holder to buy SUNW at a price of $15 a share any time through 20 July. To obtain this option, one would pay a price of $2.35. Therefore, a writer received $2.35 on 13 June and must be ready to sell SUNW to the buyer for $15 during the period through 20 July. Currently, SUNW trades above $15 a share, but as we shall see in more detail later, the option holder has no reason to exercise the option

[2] It is worthwhile to be aware that these terms have nothing to do with Europe or America. Both types of options are found in Europe and America. The names are part of the folklore of options markets, and there is no definitive history to explain how they came into use.

right now.[3] To justify purchase of the call, the buyer must be anticipating that SUNW will increase in price before the option expires. The seller of the call must be anticipating that SUNW will not rise sufficiently in price before the option expires.

Note that the option buyer could purchase a call expiring in July but permitting the purchase of SUNW at a price of $17.50. This price is more than the $15.00 exercise price, but as a result, the option, which sells for $1.00, is considerably cheaper. The cheaper price comes from the fact that the July 17.50 call is less likely to be exercised, because the stock has a higher hurdle to clear. A buyer is not willing to pay as much and a seller is more willing to take less for an option that is less likely to be exercised.

Alternatively, the option buyer could choose to purchase an October call instead of a July call. For any exercise price, however, the October calls would be more expensive than the July calls because they allow a longer period for the stock to make the move that the buyer wants. October options are more likely to be exercised than July options; therefore, a buyer would be willing to pay more and the seller would demand more for the October calls.

Suppose the buyer expects the stock price to go down. In that case, he might buy a put. Consider the October 17.50 put, which would cost the buyer $3.20. This option would allow the holder to sell SUNW at a price of $17.50 any time up through 18 October.[4] He has no reason to exercise the option right now, because it would mean he would be buying the option for $3.20 and selling a stock worth $16.25 for $17.50. In effect, the option holder would part with $19.45 (the cost of the option of $3.20 plus the value of the stock of $16.25) and obtain only $17.50.[5] The buyer of a put obviously must be anticipating that the stock will fall before the expiration day.

If he wanted a cheaper option than the October 17.50 put, he could buy the October 15 put, which would cost only $1.85 but would allow him to sell the stock for only $15.00 a share. The October 15 put is less likely to be exercised than the October 17.50, because the stock price must fall below a lower hurdle. Thus, the buyer is not willing to pay as much and the seller is willing to take less.

For either exercise price, purchase of a July put instead of an October put would be much cheaper but would allow less time for the stock to make the downward move necessary for the transaction to be worthwhile. The July put is cheaper than the October put; the buyer is not willing to pay as much and the seller is willing to take less because the option is less likely to be exercised.

In observing these option prices, we have obtained our first taste of some principles involved in pricing options.

Call options have a lower premium the higher the exercise price.

Put options have a lower premium the lower the exercise price.

Both call and put options are cheaper the shorter the time to expiration.[6]

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[3] The buyer paid $2.35 for the option. If he exercised it right now, he would pay $15.00 for the stock, which is worth only $16.25. Thus, he would have effectively paid $17.35 (the cost of the option of $2.35 plus the exercise price of $15) for a stock worth $16.25. Even if he had purchased the option previously at a much lower price, the current option price of $2.35 is the opportunity cost of exercising the option—that is, he can always sell the option for $2.35. Therefore, if he exercised the option, he would be throwing away the $2.35 he could receive if he sold it.

[4] Even if the option holder did not own the stock, he could use the option to sell the stock short.

[5] Again, even if the option were purchased in the past at a much lower price, the $3.20 current value of the option is an opportunity cost. Exercise of the option is equivalent to throwing away the opportunity cost.

[6] There is an exception to the rule that put options are cheaper the shorter the time to expiration. This statement is always true for American options but not always for European options. We explore this point later.

These results should be intuitive, but later in this chapter we show unequivocally why they must be true.

2.3 The Concept of Moneyness of an Option

An important concept in the study of options is the notion of an option's **moneyness**, which refers to the relationship between the price of the underlying and the exercise price. We use the terms **in-the-money**, **out-of-the-money**, and **at-the-money**. We explain the concept in Exhibit 4-2 with examples from the SUNW options. Note that in-the-money options are those in which exercising the option would produce a cash inflow that exceeds the cash outflow. Thus, calls are in-the-money when the value of the underlying exceeds the exercise price. Puts are in-the-money when the exercise price exceeds the value of the underlying. In our example, there are no at-the-money SUNW options, which would require that the stock value equal the exercise price; however, an at-the-money option can effectively be viewed as an out-of-the-money option, because its exercise would not bring in more money than is paid out.

EXHIBIT 4-2 Moneyness of an Option

In-the-Money		Out-of-the-Money	
Option	Justification	Option	Justification
July 15 call	16.25 > 15.00	July 17.50 call	16.25 < 17.50
October 15 call	16.25 > 15.00	October 17.50 call	16.25 < 17.50
July 17.50 put	17.50 > 16.25	July 15 put	15.00 < 16.25
October 17.50 put	17.50 > 16.25	October 15 put	15.00 < 16.25

Notes: Sun Microsystems options on 13 June; stock price is 16.25. See Exhibit 4-1 for more details. There are no options with an exercise price of 16.25, so no options are at-the-money.

As explained above, *one would not necessarily exercise an in-the-money option, but one would never exercise an out-of-the-money option.*

We now move on to explore how options markets are organized.

3 THE STRUCTURE OF GLOBAL OPTIONS MARKETS

Although no one knows exactly how options first got started, contracts similar to options have been around for thousands of years. In fact, insurance is a form of an option. The insurance buyer pays the insurance writer a premium and receives a type of guarantee that covers losses. This transaction is similar to a put option, which provides coverage of a portion of losses on the underlying and is often used by holders of the underlying. The first true options markets were over-the-counter options markets in the United States in the 19th century.

3.1 Over-the-Counter Options Markets

In the United States, customized over-the-counter options markets were in existence in the early part of the 20th century and lasted well into the 1970s. An organization called the Put and Call Brokers and Dealers Association consisted of a group of firms that served as brokers and dealers. As brokers, they attempted to match buyers of options with sellers, thereby earning a commission. As dealers, they offered to take either side of the option transaction, usually laying off (hedging) the risk in another transaction. Most of these transactions were retail, meaning that the general public were their customers.

As we discuss in Section 3.2 below, the creation of the Chicago Board Options Exchange was a revolutionary event, but it effectively killed the Put and Call Brokers and Dealers Association. Subsequently, the increasing use of swaps facilitated a rebirth of the customized over-the-counter options market. Currency options, a natural extension to currency swaps, were in much demand. Later, interest rate options emerged as a natural outgrowth of interest rate swaps. Soon bond, equity, and index options were trading in a vibrant over-the-counter market. In contrast to the previous over-the-counter options market, however, the current one emerged as a largely wholesale market. Transactions are usually made with institutions and corporations and are rarely conducted directly with individuals. This market is much like the forward market described in Chapter 2, with dealers offering to take either the long or short position in options and hedging that risk with transactions in other options or derivatives. There are no guarantees that the seller will perform; hence, the buyer faces credit risk. As such, option buyers must scrutinize sellers' credit risk and may require some risk reduction measures, such as collateral.

As previously noted, customized options have *all* of their terms—such as price, exercise price, time to expiration, identification of the underlying, settlement or delivery terms, size of the contract, and so on—determined by the two parties.

Like forward markets, over-the-counter options markets are essentially unregulated. In most countries, participating firms, such as banks and securities firms, are regulated by the appropriate authorities but there is usually no particular regulatory body for the over-the-counter options markets. In some countries, however, there are regulatory bodies for these markets.

Exhibit 4-3 provides information on the leading dealers in over-the-counter currency and interest rate options as determined by *Risk* magazine in its annual surveys of banks and investment banks and also end users.

EXHIBIT 4-3 *Risk* Magazine Surveys of Banks, Investment Banks, and Corporate End Users to Determine the Top Three Dealers in Over-the-Counter Currency and Interest Rate Options

	Respondents	
Currencies	Banks and Investment Banks	Corporate End Users
Currency Options		
\$/€	UBS Warburg	Citigroup
	Citigroup/Deutsche Bank	Royal Bank of Scotland
		Deutsche Bank
\$/¥	UBS Warburg	Citigroup
	Credit Suisse First Boston	JP Morgan Chase
	JP Morgan Chase/Royal Bank of Scotland	UBS Warburg
\$/£	Royal Bank of Scotland	Royal Bank of Scotland
	UBS Warburg	Citigroup
	Citigroup	Hong Kong Shanghai Banking Corp.
\$/SF	UBS Warburg	UBS Warburg
	Credit Suisse First Boston	Credit Suisse First Boston
	Citigroup	Citigroup

Interest Rate Options		
$	JP Morgan Chase	JP Morgan Chase
	Deutsche Bank	Citigroup
	Bank of America	Deutsche Bank/Lehman Brothers
€	JP Morgan Chase	JP Morgan Chase
	Credit Suisse First Boston/ Morgan Stanley	Citigroup UBS Warburg
¥	JP Morgan Chase/Deutsche Bank	UBS Warburg
	Bank of America	Barclays Capital
		Citigroup
£	Barclays Capital	Royal Bank of Scotland
	Societe Generale Groupe	Citigroup
	Bank of America/Royal Bank of Scotland	Hong Kong Shanghai Banking Corp.
SF	UBS Warburg	UBS Warburg
	JP Morgan Chase	JP Morgan
	Credit Suisse First Boston	Goldman Sachs

Notes: $ = U.S. dollar, € = euro, ¥ = Japanese yen, £ = U.K. pound sterling, SF = Swiss franc

Source: Risk, September 2002, pp. 30–67 for Banks and Investment Banking dealer respondents, and June 2002, pp. 24–34 for Corporate End User respondents.

Results for Corporate End Users for Interest Rate Options are from *Risk*, July 2001, pp. 38–46. *Risk* omitted this category from its 2002 survey.

3.2 Exchange-Listed Options Markets

As briefly noted above, the Chicago Board Options Exchange was formed in 1973. Created as an extension of the Chicago Board of Trade, it became the first organization to offer a market for standardized options. In the United States, standardized options also trade on the Amex–Nasdaq, the Philadelphia Stock Exchange, and the Pacific Stock Exchange.[7] On a worldwide basis, standardized options are widely traded on such exchanges as LIFFE (the London International Financial Futures and Options Exchange) in London, Eurex in Frankfurt, and most other foreign exchanges. Exhibit 4-4 shows the 20 largest options exchanges in the world. Note, perhaps surprisingly, that the leading options exchange is in Korea.

EXHIBIT 4-4 World's 20 Largest Options Exchanges

Exchange and Location	Volume in 2001
Korea Stock Exchange (Korea)	854,791,792
Chicago Board Options Exchange (United States)	306,667,851
MONEP (France)	285,667,686
Eurex (Germany and Switzerland)	239,016,516
American Stock Exchange (United States)	205,103,884
Pacific Stock Exchange (United States)	102,701,752
Philadelphia Stock Exchange (United States)	101,373,433
Chicago Mercantile Exchange (United States)	95,740,352

[7] You may wonder why the New York Stock Exchange is not mentioned. Standardized options did trade on the NYSE at one time but were not successful, and the right to trade these options was sold to another exchange.

Amsterdam Exchange (Netherlands)	66,400,654
LIFFE (United Kingdom)	54,225,652
Chicago Board of Trade (United States)	50,345,068
OM Stockholm (Sweden)	39,327,619
South African Futures Exchange (South Africa)	24,307,477
MEFF Renta Variable (Spain)	23,628,446
New York Mercantile Exchange (United States)	17,985,109
Korea Futures Exchange (Korea)	11,468,991
Italian Derivatives Exchange (Italy)	11,045,804
Osaka Securities Exchange (Japan)	6,991,908
Bourse de Montreal (Canada)	5,372,930
Hong Kong Futures Exchange (China)	4,718,880

Note: Volume given is in number of contracts.
Source: Data supplied by *Futures Industry* magazine.

As described in Chapter 2 on futures, the exchange fixes all terms of standardized instruments except the price. Thus, the exchange establishes the expiration dates and exercise prices as well as the minimum price quotation unit. The exchange also determines whether the option is European or American, whether the exercise is cash settlement or delivery of the underlying, and the contract size. In the United States, an option contract on an individual stock covers 100 shares of stock. Terminology such as "one option" is often used to refer to one option contract, which is really a set of options on 100 shares of stock. Index option sizes are stated in terms of a multiplier, indicating that the contract covers a hypothetical number of shares, as though the index were an individual stock. Similar specifications apply for options on other types of underlyings.

The exchange generally allows trading in exercise prices that surround the current stock price. As the stock price moves, options with exercise prices around the new stock price are usually added. The majority of trading occurs in options that are close to being at-the-money. Options that are far in-the-money or far out-of-the-money, called **deep-in-the-money** and **deep-out-of-the-money** options, are usually not very actively traded and are often not even listed for trading.

Most exchange-listed options have fairly short-term expirations, usually the current month, the next month, and perhaps one or two other months. Most of the trading takes place for the two shortest expirations. Some exchanges list options with expirations of several years, which have come to be called LEAPS, for **long-term equity anticipatory securities**. These options are fairly actively purchased, but most investors tend to buy and hold them and do not trade them as often as they do the shorter-term options.

The exchanges also determine on which companies they will list options for trading. Although specific requirements do exist, generally the exchange will list the options of any company for which it feels the options would be actively traded. The company has no voice in the matter. Options of a company can be listed on more than one exchange in a given country.

In Chapter 3, we described the manner in which futures are traded. The procedure is very similar for exchange-listed options. Some exchanges have pit trading, whereby parties meet in the pit and arrange a transaction. Some exchanges use electronic trading, in which transactions are conducted through computers. In either case, the transactions are guaranteed by the clearinghouse. In the United States, the clearinghouse is an independent company called the Options Clearing Corporation or OCC. The OCC guarantees to the buyer that the clearinghouse will step in and fulfill the obligation if the seller reneges at exercise.

When the buyer purchases the option, the premium, which one might think would go to the seller, instead goes to the clearinghouse, which maintains it in a margin account. In addition, the seller must post some margin money, which is based on a formula that reflects whether the seller has a position that hedges the risk and whether the option is in- or out-of-the-money. If the price moves against the seller, the clearinghouse will force the seller to put up additional margin money. Although defaults are rare, the clearinghouse has always been successful in paying when the seller defaults. Thus, exchange-listed options are effectively free of credit risk.

Because of the standardization of option terms and participants' general acceptance of these terms, exchange-listed options can be bought and sold at any time prior to expiration. Thus, a party who buys or sells an option can re-enter the market before the option expires and offset the position with a sale or a purchase of the identical option. From the clearinghouse's perspective, the positions cancel.

As in futures markets, traders on the options exchange are generally either market makers or brokers. Some slight technical distinctions exist between different types of market makers in different options markets, but the differences are minor and do not concern us here. Like futures traders, option market makers attempt to profit by scalping (holding positions very short term) to earn the bid–ask spread and sometimes holding positions longer, perhaps closing them overnight or leaving them open for days or more.

When an option expires, the holder decides whether or not to exercise it. When the option is expiring, there are no further gains to waiting, so in-the-money options are always exercised, assuming they are in-the-money by more than the transaction cost of buying or selling the underlying or arranging a cash settlement when exercising. Using our example of the SUNW options, if at expiration the stock is at 16, the calls with an exercise price of 15 would be exercised. Most exchange-listed stock options call for actual delivery of the stock. Thus, the seller delivers the stock and the buyer pays the seller, through the clearinghouse, $15 per share. If the exchange specifies that the contract is cash settled, the seller simply pays the buyer $1. For puts requiring delivery, the buyer tenders the stock and receives the exercise price from the seller. If the option is out-of-the-money, it simply expires unexercised and is removed from the books. If the put is cash settled, the writer pays the buyer the equivalent cash amount.

Some nonstandardized exchange-traded options exist in the United States. In an attempt to compete with the over-the-counter options market, some exchanges permit some options to be individually customized and traded on the exchange, thereby benefiting from the advantages of the clearinghouse's credit guarantee. These options are primarily available only in large sizes and tend to be traded only by large institutional investors.

Like futures markets, exchange-listed options markets are typically regulated at the federal level. In the United States, federal regulation of options markets is the responsibility of the Securities and Exchange Commission; similar regulatory structures exist in other countries.

4 TYPES OF OPTIONS

Almost anything with a random outcome can have an option on it. Note that by using the word *anything*, we are implying that the underlying does not even need to be an asset. In this section, we shall discover the different types of options, identified by the nature of the underlying. Our focus in this book is on financial options, but it is important, nonetheless, to gain some awareness of other types of options.

4.1 Financial Options

Financial options are options in which the underlying is a financial asset, an interest rate, or a currency.

4.1.1 Stock Options

Options on individual stocks, also called **equity options**, are among the most popular. Exchange-listed options are available on most widely traded stocks and an option on any stock can potentially be created on the over-the-counter market. We have already given examples of stock options in an earlier section; we now move on to index options.

4.1.2 Index Options

Stock market indices are well known, not only in the investment community but also among many individuals who are not even directly investing in the market. Because a stock index is just an artificial portfolio of stocks, it is reasonable to expect that one could create an option on a stock index. Indeed, we have already covered forward and futures contracts on stock indices; options are no more difficult in structure.

For example, consider options on the S&P 500 Index, which trade on the Chicago Board Options Exchange and have a designated index contract multiplier of 250. On 13 June of a given year, the S&P 500 closed at 1241.60. A call option with an exercise price of $1,250 expiring on 20 July was selling for $28. The option is European style and settles in cash. The underlying, the S&P 500, is treated as though it were a share of stock worth $1,241.60, which can be bought, using the call option, for $1,250 on 20 July. At expiration, if the option is in-the-money, the buyer exercises it and the writer pays the buyer the $250 contract multiplier times the difference between the index value at expiration and $1,250.

In the United States, there are also options on the Dow Jones Industrial Average, the Nasdaq, and various other indices. There are nearly always options on the best-known stock indices in most countries.

Just as there are options on stocks, there are also options on bonds.

4.1.3 Bond Options

Options on bonds, usually called **bond options**, are primarily traded in the over-the-counter markets. Options exchanges have attempted to generate interest in options on bonds, but have not been very successful. Corporate bonds are not very actively traded; most are purchased and held to expiration. Government bonds, however, are very actively traded; nevertheless, options on them have not gained widespread acceptance on options exchanges. Options exchanges generate much of their trading volume from individual investors, who have far more interest in and understanding of stocks than bonds.

Thus, bond options are found almost exclusively in the over-the-counter market and are almost always options on government bonds. Consider, for example, a U.S. Treasury bond maturing in 27 years. The bond has a coupon of 5.50 percent, a yield of 5.75 percent, and is selling for $0.9659 per $1 par. An over-the-counter options dealer might sell a put or call option on the bond with an exercise price of $0.98 per $1.00 par. The option could be European or American. Its expiration day must be significantly before the maturity date of the bond. Otherwise, as the bond approaches maturity, its price will move toward par, thereby removing much of the uncertainty in its price. The option could be specified to settle with actual delivery of the bond or with a cash settlement. The parties would also specify that the contract covered a given notional principal, expressed in terms of a face value of the underlying bond.

Continuing our example, let us assume that the contract covers $5 million face value of bonds and is cash settled. Suppose the buyer exercises a call option when the bond price is at \$0.995. Then the option is in-the-money by \$0.995 − \$0.98 = \$0.015 per \$1 par. The seller pays the buyer 0.015(\$5,000,000) = \$75,000. If instead the contract called for delivery, the seller would deliver \$5 million face value of bonds, which would be worth \$5,000,000(\$0.995) = \$4,975,000. The buyer would pay \$5,000,000(\$0.98) = \$4,900,000. Because the option is created in the over-the-counter market, the option buyer would assume the risk of the seller defaulting.

Even though bond options are not very widely traded, another type of related option is widely used, especially by corporations. This family of options is called **interest rate options**. These are quite different from the options we have previously discussed, because the underlying is not a particular financial instrument.

4.1.4 Interest Rate Options

In Chapter 2, we devoted considerable effort to understanding the Eurodollar spot market and forward contracts on the Eurodollar rate or LIBOR, called FRAs. In this chapter, we cover options on LIBOR. Although these are not the only interest rate options, their characteristics are sufficiently general to capture most of what we need to know about options on other interest rates. First recall that a Eurodollar is a dollar deposited outside of the United States. The primary Eurodollar rate is LIBOR, and it is considered the best measure of an interest rate paid in dollars on a nongovernmental borrower. These Eurodollars represent dollar-denominated time deposits issued by banks in London borrowing from other banks in London.

Before looking at the characteristics of interest rate options, let us set the perspective by recalling that FRAs are forward contracts that pay off based on the difference between the underlying rate and the fixed rate embedded in the contract when it is constructed. For example, consider a 3 × 9 FRA. This contract expires in three months. The underlying rate is six-month LIBOR. Hence, when the contract is constructed, the underlying Eurodollar instrument matures in nine months. *When the contract expires, the payoff is made immediately*, but the rate on which it is based, 180-day LIBOR, is set in the spot market, where it is assumed that interest will be paid 180 days later. Hence, the payoff on an FRA is discounted by the spot rate on 180-day LIBOR to give a present value for the payoff as of the expiration date.

Just as an FRA is a forward contract in which the underlying is an interest rate, an **interest rate option** is an option in which the underlying is an interest rate. Instead of an exercise price, it has an **exercise rate** (or **strike rate**), which is expressed on an order of magnitude of an interest rate. At expiration, the option payoff is based on the difference between the underlying rate in the market and the exercise rate. Whereas an FRA is a *commitment* to make one interest payment and receive another at a future date, an interest rate option is the *right* to make one interest payment and receive another. And just as there are call and put options, there is also an **interest rate call** and an **interest rate put**.

An interest rate call is an option in which the holder has the right to make a known interest payment and receive an unknown interest payment. The underlying is the unknown interest rate. If the unknown underlying rate turns out to be higher than the exercise rate at expiration, the option is in-the-money and is exercised; otherwise, the option simply expires. *An interest rate put is an option in which the holder has the right to make an unknown interest payment and receive a known interest payment.* If the unknown underlying rate turns out to be lower than the exercise rate at expiration, the option is in-the-money and is exercised; otherwise, the option simply expires. All interest rate option contracts have a specified size, which, as in FRAs, is called the notional principal. An interest rate option can be European or American style, but most tend to be European style. Interest rate options are settled in cash.

As with FRAs, these options are offered for purchase and sale by dealers, which are financial institutions, usually the same ones who offer FRAs. These dealers quote rates for options of various exercise prices and expirations. When a dealer takes an option position, it usually then offsets the risk with other transactions, often Eurodollar futures.

To use the same example we used in introducing FRAs, consider options expiring in 90 days on 180-day LIBOR. The option buyer specifies whatever exercise rate he desires. Let us say he chooses an exercise rate of 5.5 percent and a notional principal of $10 million.

Now let us move to the expiration day. Suppose that 180-day LIBOR is 6 percent. Then the call option is in-the-money. The payoff to the holder of the option is

$$(\$10,000,000)(0.06 - 0.055)\left(\frac{180}{360}\right) = \$25,000$$

This money is not paid at expiration, however; it is paid 180 days later. There is no reason why the payoff could not be made at expiration, as is done with an FRA. The delay of payment associated with interest rate options actually makes more sense, because these instruments are commonly used to hedge floating-rate loans in which the rate is set on a given day but the interest is paid later. We shall see examples of the convenience of this type of structure in Chapter 7.

Note that the difference between the underlying rate and the exercise rate is multiplied by 180/360 to reflect the fact that the rate quoted is a 180-day rate but is stated as an annual rate. Also, the interest calculation is multiplied by the notional principal.

In general, the payoff of an interest rate call is

$$(\text{Notional Principal})\text{Max}(0,\text{Underlying rate at expiration} - \text{Exercise rate})\left(\frac{\text{Days in underlying rate}}{360}\right) \quad \textbf{(4-1)}$$

The expression Max(0,Underlying rate at expiration − Exercise rate) is similar to a form that we shall commonly see throughout this chapter for all options. The payoff of a call option at expiration is based on the maximum of zero or the underlying minus the exercise rate. If the option expires out-of-the-money, then "Underlying rate at expiration − Exercise rate" is negative; consequently, zero is greater. Thus, the option expires with no value. If the option expires in-the-money, "Underlying rate at expiration − Exercise rate" is positive. Thus, the option expires worth this difference (multiplied by the notional principal and the Days/360 adjustment). The expression "Days in underlying rate," which we used in Chapter 2, refers to the fact that the rate is specified as the rate on an instrument of a specific number of days to maturity, such as a 90-day or 180-day rate, thereby requiring that we multiply by 90/360 or 180/360 or some similar adjustment.

For an interest rate put option, the general formula is

$$(\text{Notional Principal})\text{Max}(0,\text{Exercise rate} - \text{Underlying rate at expiration})\left(\frac{\text{Days in underlying rate}}{360}\right) \quad \textbf{(4-2)}$$

For an exercise rate of 5.5 percent and an underlying rate at expiration of 6 percent, an interest rate put expires out-of-the-money. Only if the underlying rate is less than the exercise rate does the put option expire in-the-money.

As noted above, borrowers often use interest rate call options to hedge the risk of rising rates on floating-rate loans. Lenders often use interest rate put options to hedge the risk of falling rates on floating-rate loans. The form we have seen here, in which the option expires with a single payoff, is not the more commonly used variety of interest rate option. Floating-rate loans usually involve multiple interest payments. Each of those payments is set on a given date. To hedge the risk of interest rates increasing, the borrower would need options expiring on each rate reset date. Thus, the borrower would require a combination of interest rate call options. Likewise, a lender needing to hedge the risk of falling rates on a multiple-payment floating-rate loan would need a combination of interest rate put options.

A combination of interest rate calls is referred to as an **interest rate cap** or sometimes just a **cap**. A combination of interest rate puts is called an **interest rate floor** or sometimes just a **floor**.[8] Specifically, *an interest rate cap is a series of call options on an interest rate, with each option expiring at the date on which the floating loan rate will be reset, and with each option having the same exercise rate.*[9] Each option is independent of the others; thus, exercise of one option does not affect the right to exercise any of the others. Each component call option is called a **caplet**. *An interest rate floor is a series of put options on an interest rate, with each option expiring at the date on which the floating loan rate will be reset, and with each option having the same exercise rate.* Each component put option is called a **floorlet**. The price of an interest rate cap or floor is the sum of the prices of the options that make up the cap or floor.

A special combination of caps and floors is called an **interest rate collar**. *An interest rate collar is a combination of a long cap and a short floor or a short cap and a long floor.* Consider a borrower in a floating rate loan who wants to hedge the risk of rising interest rates but is concerned about the requirement that this hedge must have a cash outlay up front: the option premium. A collar, which adds a short floor to a long cap, is a way of reducing and even eliminating the up-front cost of the cap. The sale of the floor brings in cash that reduces the cost of the cap. It is possible to set the exercise rates such that the price received for the sale of the floor precisely offsets the price paid for the cap, thereby completely eliminating the up-front cost. This transaction is sometimes called a **zero-cost collar**. The term is a bit misleading, however, and brings to mind the importance of noting the true cost of a collar. Although the cap allows the borrower to be paid from the call options when rates are high, the sale of the floor requires the borrower to pay the counterparty when rates are low. Thus, the cost of protection against rising rates is the loss of the advantage of falling rates. Caps, floors, and collars are popular instruments in the interest rate markets. We shall explore strategies using them in Chapter 7.

Although interest rate options are primarily written on such rates as LIBOR, Euribor, and Euroyen, the underlying can be any interest rate.

4.1.5 Currency Options

As we noted in Chapter 2, the currency forward market is quite large. The same is true for the currency options market. A **currency option** allows the holder to buy (if a call) or sell (if a put) an underlying currency at a fixed exercise rate, expressed as an exchange rate. Many companies, knowing that they will need to convert a currency X at a future date into a currency Y, will buy a call option on currency Y specified in terms of currency X. For example, say that a U.S. company will be needing €50 million for an expansion project in three months. Thus, it will be buying euros and is exposed to the risk of the euro rising against the dollar. Even though it has that concern, it would also like to benefit if the euro weakens against the dollar. Thus, it might buy a call option on the euro. Let us say it specifies an exercise rate of $0.90. So it pays cash up front for the right to buy €50 million at a rate of $0.90 per euro. If the option expires with the euro above $0.90, it can buy euros

[8] It is possible to construct caps and floors with options on any other type of underlying, but they are very often used when the underlying is an interest rate.

[9] Technically, each option need not have the same exercise rate, but they generally do.

at $0.90 and avoid any additional cost over $0.90. If the option expires with the euro below $0.90, it does not exercise the option and buys euros at the market rate.

Note closely these two cases:

Euro expires above $0.90
Company buys €50 million at $0.90

Euro expires at or below $0.90
Company buys €50 million at the market rate

These outcomes can also be viewed in the following manner:

Dollar expires below €1.1111, that is, €1 > $0.90
Company sells $45 million (€50 million × $0.90) at €1.1111, equivalent to buying €50 million

Dollar expires above €1.1111, that is, €1 < $0.90
Company sells sufficient dollars to buy €50 million at the market rate

This transaction looks more like a put in which the underlying is the dollar and the exercise rate is expressed as €1.1111. Thus, the call on the euro can be viewed as a put on the dollar. Specifically, a call to buy €50 million at an exercise price of $0.90 is also a put to sell €50 million × $0.90 = $45 million at an exercise price of 1/$0.90, or €1.1111.

Most foreign currency options activity occurs on the customized over-the-counter markets. Some exchange-listed currency options trade on a few exchanges, but activity is fairly low.

4.2 Options on Futures

In Chapter 2 we covered futures markets. One of the important innovations of futures markets is options on futures. These contracts originated in the United States as a result of a regulatory structure that separated exchange-listed options and futures markets. The former are regulated by the Securities and Exchange Commission, and the latter are regulated by the Commodity Futures Trading Commission (CFTC). SEC regulations forbid the trading of options side by side with their underlying instruments. Options on stocks trade on one exchange, and the underlying trades on another or on Nasdaq.

The futures exchanges got the idea that they could offer options in which the underlying is a futures contract; no such prohibitions for side-by-side trading existed under CFTC rules. As a result, the futures exchanges were able to add an attractive instrument to their product lines. The side-by-side trading of the option and its underlying futures made for excellent arbitrage linkages between these instruments. Moreover, some of the options on futures are designed to expire on the same day the underlying futures expires. Thus, the options on the futures are effectively options on the spot asset that underlies the futures.

A call option on a futures gives the holder the right to enter into a long futures contract at a fixed futures price. A put option on a futures gives the holder the right to enter into a short futures contract at a fixed futures price. The fixed futures price is, of course, the exercise price. Consider an option on the Eurodollar futures contract trading at the Chicago Mercantile Exchange. On 13 June of a particular year, an option expiring on 13 July was based on the July Eurodollar futures contract. That futures contract expires on 16 July, a few days after the option expires.[10] The call option with exercise price of 95.75 had a price of $4.60. The underlying futures price was 96.21. Recall that this price is the IMM index value, which means that the price is based on a discount rate of 100 − 96.21 = 3.79. The contract size is $1 million.

[10] Some options on futures expire a month or so before the futures expires. Others expire very close to, if not at, the futures expiration.

The buyer of this call option on a futures would pay 0.046($1,000,000) = $46,000 and would obtain the right to buy the July futures contract at a price of 95.75. Thus, at that time, the option was in the money by 96.21 − 95.75 = 0.46 per $100 face value. Suppose that when the option expires, the futures price is 96.00. Then the holder of the call would exercise it and obtain a long futures position at a price of 95.75. The price of the underlying futures is 96.00, so the margin account is immediately marked to market with a credit of 0.25 or $625.[11] The party on the short side of the contract is immediately set up with a short futures contract at the price of 95.75. That party will be charged the $625 gain that the long made. If the option is a put, exercise of it establishes a short position. The exchange assigns the put writer a long futures position.

4.3 Commodity Options

Options in which the asset underlying the futures is a commodity, such as oil, gold, wheat, or soybeans, are also widely traded. There are exchange-traded as well as over-the-counter versions. Over-the-counter options on oil are widely used.

Our focus in this book is on financial instruments so we will not spend any time on commodity options, but readers should be aware of the existence and use of these instruments by companies whose business involves the buying and selling of these commodities.

4.4 Other Types of Options

As derivative markets develop, options (and even some other types of derivatives) have begun to emerge on such underlyings as electricity, various sources of energy, and even weather. These instruments are almost exclusively customized over-the-counter instruments. Perhaps the most notable feature of these instruments is how the underlyings are often instruments that cannot actually be held. For example, electricity is not considered a storable asset because it is produced and almost immediately consumed, but it is nonetheless an asset and certainly has a volatile price. Consequently, it is ideally suited for options and other derivatives trading.

Consider weather. It is hardly an asset at all but simply a random factor that exerts an enormous influence on economic activity. The need to hedge against and speculate on the weather has created a market in which measures of weather activity, such as economic losses from storms or average temperature or rainfall, are structured into a derivative instrument. Option versions of these derivatives are growing in importance and use. For example, consider a company that generates considerable revenue from outdoor summer activities, provided that it does not rain. Obviously a certain amount of rain will occur, but the more rain, the greater the losses for the company. It could buy a call option on the amount of rainfall with the exercise price stated as a quantity of rainfall. If actual rainfall exceeds the exercise price, the company exercises the option and receives an amount of money related to the excess of the rainfall amount over the exercise price.

Another type of option, which is not at all new but is increasingly recognized in practice, is the real option. A real option is an option associated with the flexibility inherent in capital investment projects. For example, companies may invest in new projects that have the option to defer the full investment, expand or contract the project at a later date, or even terminate the project. In fact, most capital investment projects have numerous elements of flex-

[11] If the contract is in-the-money by 96 − 95.75 = 0.25 per $100 par, it is in-the-money by 0.25/100 = 0.0025, or 0.25 percent of the face value. Because the face value is $1 million, the contract is in the money by (0.0025)(90/360)($1,000,000) = $625. (Note the adjustment by 90/360.) Another way to look at this calculation is that the futures price at 95.75 is 1 − (0.0425)(90/360) = $0.989375 per $1 par, or $989,375. At 96, the futures price is 1 − 0.04(90/360) = $0.99 per $1 par or $990,000. The difference is $625. So, exercising this option is like entering into a futures contract at a price of $989,375 and having the price immediately go to $990,000, a gain of $625. The call holder must deposit money to meet the Eurodollar futures margin, but the exercise of the option gives him $625. In other words, assuming he meets the minimum initial margin requirement, he is immediately credited with $625 more.

ibility that can be viewed as options. Of course, these options do not trade in markets the same way as financial and commodity options, and they must be evaluated much more carefully. They are, nonetheless, options and thus have the potential for generating enormous value.

Again, our emphasis is on financial options, but readers should be aware of the growing role of these other types of options in our economy. Investors who buy shares in companies that have real options are, in effect, buying real options. In addition, commodity and other types of options are sometimes found in investment portfolios in the form of "alternative investments" and can provide significant diversification benefits.

To this point, we have examined characteristics of options markets and contracts. Now we move forward to the all-important topic of how options are priced.

5 PRINCIPLES OF OPTION PRICING

In Chapters 2 and 3, we discussed the pricing and valuation of forward and futures contracts. Recall that the value of a contract is what someone must pay to buy into it or what someone would receive to sell out of it. A forward or futures contract has zero value at the start of the contract, but the value turns positive or negative as prices or rates change. A contract that has positive value to one party and negative value to the counterparty can turn around and have negative value to the former and positive value to the latter as prices or rates change. The forward or futures price is the price that the parties agree will be paid on the future date to buy and sell the underlying.

With options, these concepts are different. An option has a positive value at the start. The buyer must pay money and the seller receives money to initiate the contract. Prior to expiration, the option always has positive value to the buyer and negative value to the seller. In a forward or futures contract, the two parties agree on the fixed price the buyer will pay the seller. This fixed price is set such that the buyer and seller do not exchange any money. The corresponding fixed price at which a call holder can buy the underlying or a put holder can sell the underlying is the exercise price. It, too, is negotiated between buyer and seller but still results in the buyer paying the seller money up front in the form of an option premium or price.[12]

Thus, what we called the forward or futures price corresponds more to the exercise price of an option. The option price *is* the option value: With a few exceptions that will be clearly noted, in this chapter we do not distinguish between the option price and value.

In this section of the chapter, we examine the principles of option pricing. These principles are characteristics of option prices that are governed by the rationality of investors. These principles alone do not allow us to calculate the option price. We do that in Section 6.

Before we begin, it is important to remind the reader that we assume all participants in the market behave in a rational manner such that they do not throw away money and that they take advantage of arbitrage opportunities. As such, we assume that markets are sufficiently competitive that no arbitrage opportunities exist.

Let us start by developing the notation, which is very similar to what we have used previously. Note that time 0 is today and time T is the expiration.

S_0, S_T = price of the underlying asset at time 0 (today) and time T (expiration)

X = exercise price

r = risk-free rate

[12] For a call, there is no finite exercise price that drives the option price to zero. For a put, the unrealistic example of a zero exercise price would make the put price be zero.

T = time to expiration, equal to number of days to expiration divided by 365

c_0, c_T = price of European call today and at expiration

C_0, C_T = price of American call today and at expiration

p_0, p_T = price of European put today and at expiration

P_0, P_T = price of American put today and at expiration

On occasion, we will introduce some variations of the above as well as some new notation. For example, we start off with no cash flows on the underlying, but we shall discuss the effects of cash flows on the underlying in Section 5.7.

5.1 Payoff Values

The easiest time to determine an option's value is at expiration. At that point, there is no future. Only the present matters. An option's value at expiration is called its **payoff**. We introduced this material briefly in our basic descriptions of types of options; now we cover it in more depth.

At expiration, a call option is worth either zero or the difference between the underlying price and the exercise price, whichever is greater:

$$c_T = \text{Max}(0, S_T - X) \qquad \textbf{(4-3)}$$
$$C_T = \text{Max}(0, S_T - X)$$

Note that at expiration, a European option and an American option have the same payoff because they are equivalent instruments at that point.

The expression $\text{Max}(0,S_T - X)$ means to take the greater of zero or $S_T - X$. Suppose the underlying price exceeds the exercise price, $S_T > X$. In this case, the option is expiring in-the-money and the option is worth $S_T - X$. Suppose that at the instant of expiration, it is possible to buy the option for less than $S_T - X$. Then one could buy the option, immediately exercise it, and immediately sell the underlying. Doing so would cost c_T (or C_T) for the option and X to buy the underlying but would bring in S_T for the sale of the underlying. If c_T (or C_T) $< S_T - X$, this transaction would net an immediate risk-free profit. The collective actions of all investors doing this would force the option price up to $S_T - X$. The price could not go higher than $S_T - X$, because all that the option holder would end up with an instant later when the option expires is $S_T - X$. If $S_T < X$, meaning that the call is expiring out-of-the-money, the formula says the option should be worth zero. It cannot sell for less than zero because that would mean that the option seller would have to pay the option buyer. A buyer would not pay more than zero, because the option will expire an instant later with no value.

At expiration, a put option is worth either zero or the difference between the exercise price and the underlying price, whichever is greater:

$$p_T = \text{Max}(0, X - S_T) \qquad \textbf{(4-4)}$$
$$P_T = \text{Max}(0, X - S_T)$$

Suppose $S_T < X$, meaning that the put is expiring in-the-money. At the instant of expiration, suppose the put is selling for less than $X - S_T$. Then an investor buys the put for p_T (or P_T) and the underlying for S_T and exercises the put, receiving X. If p_T (or P_T) $< X - S_T$, this transaction will net an immediate risk-free profit. The combined actions of participants doing this will force the put price up to $X - S_T$. It cannot go any higher, because the put buyer will end up an instant later with only $X - S_T$ and would not pay more than this. If $S_T > X$, meaning that the put is expiring out-of the-money, it is worth zero. It cannot be worth less than zero because the option seller would have to pay the option buyer.

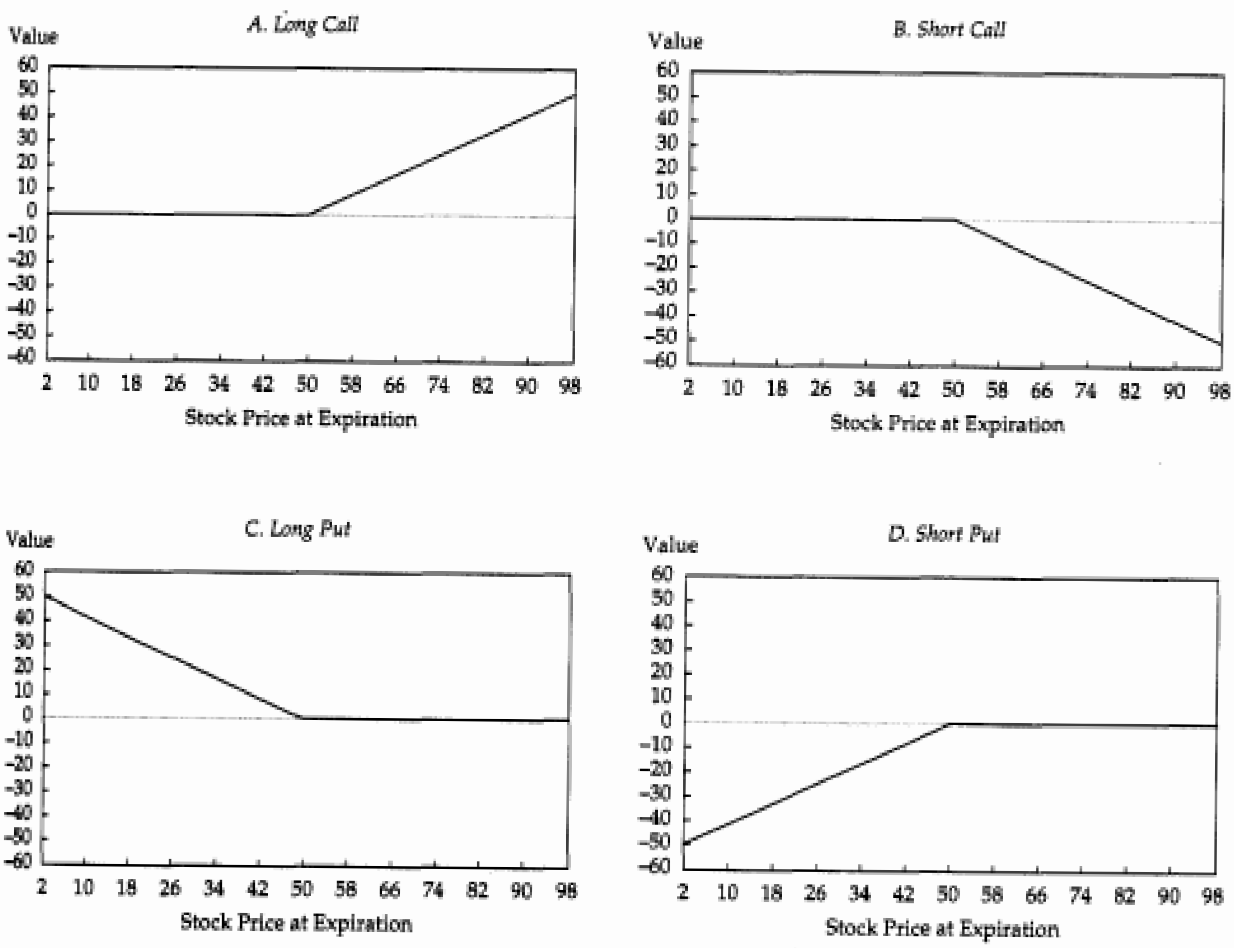
It cannot be worth more than zero because the buyer would not pay for a position that, an instant later, will be worth nothing.

These important results are summarized along with an example in Exhibit 4-5. The payoff diagrams for the short positions are also shown and are obtained as the negative of the long positions. For the special case of $S_T = X$, meaning that both call and put are expiring at-the-money, we can effectively treat the option as out-of-the-money because it is worth zero at expiration.

EXHIBIT 4-5 Option Values at Expiration (Payoffs)

		Example (X = 50)	
Option	Value	$S_T = 52$	$S_T = 48$
European call	$c_T = \text{Max}(0, S_T - X)$	$c_T = \text{Max}(0, 52 - 50) = 2$	$c_T = \text{Max}(0, 48 - 50) = 0$
American call	$C_T = \text{Max}(0, S_T - X)$	$C_T = \text{Max}(0, 52 - 50) = 2$	$C_T = \text{Max}(0, 48 - 50) = 0$
European put	$p_T = \text{Max}(0, X - S_T)$	$p_T = \text{Max}(0, 50 - 52) = 0$	$p_T = \text{Max}(0, 50 - 48) = 2$
American put	$P_T = \text{Max}(0, X - S_T)$	$P_T = \text{Max}(0, 50 - 52) = 0$	$P_T = \text{Max}(0, 50 - 48) = 2$

Notes: Results for the European and American calls correspond to Graph A. Results for Graph B are the negative of Graph A. Results for the European and American puts correspond to Graph C, and results for Graph D are the negative of Graph C.



The value $\text{Max}(0,S_T - X)$ for calls or $\text{Max}(0,X - S_T)$ for puts is also called the option's **intrinsic value** or **exercise value**. We shall use the former terminology. Intrinsic value is what the option is worth to exercise it based on current conditions. In this section, we have talked only about the option at expiration. Prior to expiration, an option will normally sell for more than its intrinsic value.[13] The difference between the market price of the option and its intrinsic value is called its **time value** or **speculative value**. We shall use the former terminology. The time value reflects the potential for the option's intrinsic value at expiration to be greater than its current intrinsic value. At expiration, of course, the time value is zero.

PRACTICE PROBLEM 1

For Parts A through E, determine the payoffs of calls and puts under the conditions given.

A. The underlying is a stock index and is at 5,601.19 when the options expire. The multiplier is 500. The exercise price is
 i. 5,500
 ii. 6,000

B. The underlying is a bond and is at $1.035 per $1 par when the options expire. The contract is on $100,000 face value of bonds. The exercise price is
 i. $1.00
 ii. $1.05

C. The underlying is a 90-day interest rate and is at 9 percent when the options expire. The notional principal is $50 million. The exercise rate is
 i. 8 percent
 ii. 10.5 percent

D. The underlying is the Swiss franc and is at $0.775 when the options expire. The options are on SF500,000. The exercise price is
 i. $0.75
 ii. $0.81

E. The underlying is a futures contract and is at 110.5 when the options expire. The options are on a futures contract covering $1 million of the underlying. These prices are percentages of par. The exercise price is
 i. 110
 ii. 115

For Parts F and G, determine the payoffs of the strategies indicated and describe the payoff graph.

F. The underlying is a stock priced at $40. A call option with an exercise price of $40 is selling for $7. You buy the stock and sell the call. At expiration, the stock price is
 i. $52
 ii. $38

[13] We shall later see an exception to this statement for European puts, but for now take it as the truth.

G. The underlying is a stock priced at \$60. A put option with an exercise price of \$60 is priced at \$5. You buy the stock and buy the put. At expiration, the stock price is
  i. \$68
  ii. \$50

SOLUTIONS

A. **i.** Calls: Max(0,5601.19 − 5500) × 500 = 50,595
Puts: Max(0,5500 − 5601.19) × 500 = 0
ii. Calls: Max(0,5601.19 − 6000) × 500 = 0
Puts: Max(0,6000 − 5601.19) × 500 = 199,405

B. **i.** Calls: Max(0,1.035 − 1.00) × \$100,000 = \$3,500
Puts: Max(0,1.00 − 1.035) × \$100,000 = \$0
ii. Calls: Max(0,1.035 − 1.05) × \$100,000 = \$0
Puts: Max(0,1.05 − 1.035) × \$100,000 = \$1,500

C. **i.** Calls: Max(0,0.09 − 0.08) × (90/360) × \$50,000,000 = \$125,000
Puts: Max(0,0.08 − 0.09) × (90/360) × \$50,000,000 = \$0
ii. Calls: Max(0,0.09 − 0.105) × (90/360) × \$50,000,000 = \$0
Puts: Max(0,0.105 − 0.09) × (90/360) × \$50,000,000 = \$187,500

D. **i.** Calls: Max(0,0.775 − 0.75) × SF500,000 = SF12,500
Puts: Max(0,0.75 − 0.775) × SF500,000 = SF0
ii. Calls: Max(0,0.775 − 0.81) × SF500,000 = SF0
Puts: Max(0,0.81 − 0.775) × SF500,000 = SF17,500

E. **i.** Calls: Max(0,110.5 − 110) × (1/100) × \$1,000,000 = \$5,000
Puts: Max(0,110 − 110.5) × (1/100) × \$1,000,000 = \$0
ii. Calls: Max(0,110.5 − 115) × (1/100) × \$1,000,000 = \$0
Puts: Max(0,115 − 110.5) × (1/100) × \$1,000,000 = \$45,000

F. **i.** 52 − Max(0,52 − 40) = 40
ii. 38 − Max(0,38 − 40) = 38

For any value of the stock price at expiration of 40 or above, the payoff is constant at 40. For stock price values below 40 at expiration, the payoff declines with the stock price. The graph would look similar to the short put in Panel D of Exhibit 4-5. This strategy is known as a covered call and is discussed in Chapter 7.

G. **i.** 68 + Max(0,60 − 68) = 68
ii. 50 + Max(0,60 − 50) = 60

For any value of the stock price at expiration of 60 or below, the payoff is constant at 60. For stock price values above 60 at expiration, the payoff increases with the stock price at expiration. The graph will look similar to the long call in Panel A of Exhibit 4-5. This strategy is known as a protective put and is covered later in this chapter and in Chapter 7.

There is no question that everyone agrees on the option's intrinsic value; after all, it is based on the current stock price and exercise price. It is the time value that we have more difficulty estimating. So remembering that *Option price = Intrinsic value + Time value*, let us move forward and attempt to determine the value of an option today, prior to expiration.

5.2 Boundary Conditions

We start by examining some simple results that establish minimum and maximum values for options prior to expiration.

5.2.1 Minimum and Maximum Values

The first and perhaps most obvious result is one we have already alluded to: *The minimum value of any option is zero.* We state this formally as

$$c_0 \geq 0, C_0 \geq 0 \tag{4-5}$$
$$p_0 \geq 0, P_0 \geq 0$$

No option can sell for less than zero, for in that case the writer would have to pay the buyer.

Now consider the maximum value of an option. It differs somewhat depending on whether the option is a call or a put and whether it is European or American. *The maximum value of a call is the current value of the underlying*:

$$c_0 \leq S_0, C_0 \leq S_0 \tag{4-6}$$

A call is a means of buying the underlying. It would not make sense to pay more for the right to buy the underlying than the value of the underlying itself.

For a put, it makes a difference whether the put is European or American. One way to see the maximum value for puts is to consider the best possible outcome for the put holder. The best outcome is that the underlying goes to a value of zero. Then the put holder could sell a worthless asset for X. For an American put, the holder could sell it immediately and capture a value of X. For a European put, the holder would have to wait until expiration; consequently, we must discount X from the expiration day to the present. Thus, *the maximum value of a European put is the present value of the exercise price. The maximum value of an American put is the exercise price,*

$$p_0 \leq X/(1 + r)^T, P_0 \leq X \tag{4-7}$$

where r is the risk-free interest rate and T is the time to expiration. These results for the maximums and minimums for calls and puts are summarized in Exhibit 4-6, which also includes a numerical example.

EXHIBIT 4-6 Minimum and Maximum Values of Options

Option	Minimum Value	Maximum Value	Example (S_0 = 52, X = 50, r = 5%, T = 1/2 year)
European call	$c_0 \geq 0$	$c_0 \leq S_0$	$0 \leq c_0 \leq 52$
American call	$C_0 \geq 0$	$C_0 \leq S_0$	$0 \leq C_0 \leq 52$
European put	$p_0 \geq 0$	$p_0 \leq X/(1 + r)^T$	$0 \leq p_0 \leq 48.80$ [$48.80 = 50/(1.05)^{0.5}$]
American put	$P_0 \geq 0$	$P_0 \leq X$	$0 \leq P_0 \leq 50$

5.2.2 Lower Bounds

The results we established in Section 5.2.1 do not put much in the way of restrictions on the option price. They tell us that the price is somewhere between zero and the maximum, which is either the underlying price, the exercise price, or the present value of the exercise price—a fairly wide range of possibilities. Fortunately, we can tighten the range up a little on the low side: We can establish a **lower bound** on the option price.

For American options, which are exercisable immediately, we can state that the lower bound of an American option price is its current intrinsic value:[14]

$$C_0 \geq \text{Max}(0, S_0 - X) \quad \textbf{(4-8)}$$

$$P_0 \geq \text{Max}(0, X - S_0)$$

The reason these results hold today is the same reason we have already shown for why they must hold at expiration. If the option is in-the-money and is selling for less than its intrinsic value, it can be bought and exercised to net an immediate risk-free profit.[15] The collective actions of market participants doing this will force the American option price up to at least the intrinsic value.

Unfortunately, we cannot make such a statement about European options—but we can show that the lower bound is either zero or the current underlying price minus the present value of the exercise price, whichever is greater. They cannot be exercised early; thus, there is no way for market participants to exercise an option selling for too little with respect to its intrinsic value. Fortunately, however, there is a way to establish a lower bound for European options. We can combine options with risk-free bonds and the underlying in such a way that a lower bound for the option price emerges.

First, we need the ability to buy and sell a risk-free bond with a face value equal to the exercise price and current value equal to the present value of the exercise price. This procedure is simple but perhaps not obvious. If the exercise price is X (say, 100), we buy a bond with a face value of X (100) maturing on the option expiration day. The current value of that bond is the present value of X, which is $X/(1 + r)^T$. So we buy the bond today for $X/(1 + r)^T$ and hold it until it matures on the option expiration day, at which time it will pay off X. We assume that we can buy or sell (issue) this type of bond. Note that this transaction involves borrowing or lending an amount of money equal to the present value of the exercise price with repayment of the full exercise price.

Exhibit 4-7 illustrates the construction of a special combination of instruments. We buy the European call and the risk-free bond and sell short the underlying asset. Recall that short selling involves borrowing the asset and selling it. At expiration, we shall buy back

EXHIBIT 4-7 A Lower Bound Combination for European Calls

		Value at Expiration	
Transaction	Current Value	$S_T \leq X$	$S_T > X$
Buy call	c_0	0	$S_T - X$
Sell short underlying	$-S_0$	$-S_T$	$-S_T$
Buy bond	$X/(1 + r)^T$	X	X
Total	$c_0 - S_0 + X/(1 + r)^T$	$X - S_T \geq 0$	0

[14] Normally we have italicized sentences containing important results. This one, however, is a little different: We are stating it temporarily. We shall soon show that we can override one of these results with a lower bound that is higher and, therefore, is a better lower bound.

[15] Consider, for example, an in-the-money call selling for less than $S_0 - X$. One can buy the call for C_0, exercise it, paying X, and sell the underlying netting a gain of $S_0 - X - C_0$. This value is positive and represents an immediate risk-free gain. If the option is an in-the-money put selling for less than $X - S_0$, one can buy the put for P_0, buy the underlying for S_0, and exercise the put to receive X, thereby netting an immediate risk-free gain of $X - S_0 - P_0$.

the asset. In order to illustrate the logic behind the lower bound for a European call in the simplest way, we assume that we can sell short without any restrictions.

In Exhibit 4-7 the two right-hand columns contain the value of each instrument when the option expires. The rightmost column is the case of the call expiring in-the-money, in which case it is worth $S_T - X$. In the other column, the out-of-the-money case, the call is worth zero. The underlying is worth $-S_T$ (the negative of its current value) in either case, reflecting the fact that we buy it back to cover the short position. The bond is worth X in both cases. The sum of all the positions is positive when the option expires out-of-the-money and zero when the option expires in-the-money. Therefore, in no case does this combination of instruments have a negative value. That means that we never have to pay out any money at expiration. We are guaranteed at least no loss at expiration and possibly something positive.

If there is a possibility of a positive outcome from the combination and if we know we shall never have to pay anything out from holding a combination of instruments, the cost of that combination must be positive—it must cost us something to enter into the position. We cannot take in money to enter into the position. In that case, we would be receiving money up front and never having to pay anything out. The cost of entering the position is shown in the second column, labeled the "Current Value." Because that value must be positive, we therefore require that $c_0 - S_0 + X/(1 + r)^T \geq 0$. Rearranging this equation, we obtain $c_0 \geq S_0 - X/(1 + r)^T$. Now we have a statement about the minimum value of the option, which can serve as a lower bound. This result is solid, because if the call is selling for less than $S_0 - X/(1 + r)^T$, an investor can buy the call, sell short the underlying, and buy the bond. Doing so would bring in money up front and, as we see in Exhibit 4-7, an investor would not have to pay out any money at expiration and might even get a little more money. Because other investors would do the same, the call price would be forced up until it is at least $S_0 - X/(1 + r)^T$.

But we can improve on this result. Suppose $S_0 - X/(1 + r)^T$ is negative. Then we are stating that the call price is greater than a negative number. But we already know that the call price cannot be negative. So we can now say that

$$c_0 \geq \text{Max}[0, S_0 - X/(1 + r)^T]$$

In other words, *the lower bound on a European call price is either zero or the underlying price minus the present value of the exercise price, whichever is greater*. Notice how this lower bound differs from the minimum value for the American call, $\text{Max}(0, S_0 - X)$. For the European call, we must wait to pay the exercise price and obtain the underlying. Therefore, the expression contains the current underlying value—the present value of its future value—as well as the present value of the exercise price. For the American call, we do not have to wait until expiration; therefore, the expression reflects the potential to immediately receive the underlying price minus the exercise price. We shall have more to say, however, about the relationship between these two values.

To illustrate the lower bound, let X = 50, r = 0.05, and T = 0.5. If the current underlying price is 45, then the lower bound for the European call is

$$\text{Max}[0, 45 - 50/(1.05)^{0.5}] = \text{Max}(0, 45 - 48.80) = \text{Max}(0, -3.80) = 0$$

All this calculation tells us is that the call must be worth no less than zero, which we already knew. If the current underlying price is 54, however, the lower bound for the European call is

$$\text{Max}(0, 54 - 48.80) = \text{Max}(0, 5.20) = 5.20$$

which tells us that the call must be worth no less than 5.20. With European puts, we can also see that the lower bound differs from the lower bound on American puts in this same use of the present value of the exercise price.

EXHIBIT 4-8 A Lower Bound Combination for European Puts

Transaction	Current Value	Value at Expiration	
		$S_T < X$	$S_T \geq X$
Buy put	p_0	$X - S_T$	0
Buy underlying	S_0	S_T	S_T
Issue bond	$-X/(1 + r)^T$	$-X$	$-X$
Total	$p_0 + S_0 - X/(1 + r)^T$	0	$S_T - X \geq 0$

Exhibit 4-8 constructs a similar type of portfolio for European puts. Here, however, we buy the put and the underlying and borrow by issuing the zero-coupon bond. The payoff of each instrument is indicated in the two rightmost columns. Note that the total payoff is never less than zero. Consequently, the initial value of the combination must not be less than zero. Therefore, $p_0 + S_0 - X/(1 + r)^T \geq 0$. Isolating the put price gives us $p_0 \geq X/(1 + r)^T - S_0$. But suppose that $X/(1 + r)^T - S_0$ is negative. Then, the put price must be greater than a negative number. We know that the put price must be no less than zero. So we can now formally say that

$$p_0 \geq \text{Max}[0, X/(1 + r)^T - S_0]$$

In other words, *the lower bound of a European put is the greater of either zero or the present value of the exercise price minus the underlying price.* For the American put, recall that the expression was $\text{Max}(0, X - S_0)$. So for the European put, we adjust this value to the present value of the exercise price. The present value of the asset price is already adjusted to S_0.

Using the same example we did for calls, let $X = 50$, $r = 0.05$, and $T = 0.5$. If the current underlying price is 45, then the lower bound for the European put is

$$\text{Max}(0, 50/(1.05)^{0.5} - 45) = \text{Max}(0, 48.80 - 45) = \text{Max}(0, 3.80) = 3.80$$

If the current underlying price is 54, however, the lower bound is

$$\text{Max}(0, 48.80 - 54) = \text{Max}(0, -5.20) = 0$$

At this point let us reconsider what we have found. The lower bound for a European call is $\text{Max}[0, S_0 - X/(1 + r)^T]$. We also observed that an American call must be worth at least $\text{Max}(0, S_0 - X)$. But except at expiration, the European lower bound is greater than the minimum value of the American call.[16] We could not, however, expect an American call to be worth less than a European call. Thus the lower bound of the European call holds for American calls as well. Hence, we can conclude that

$$c_0 \geq \text{Max}[0, S_0 - X/(1 + r)^T] \qquad \textbf{(4-9)}$$

$$C_0 \geq \text{Max}[0, S_0 - X/(1 + r)^T]$$

[16] We discuss this point more formally and in the context of whether it is ever worthwhile to exercise an American call early in Section 5.6.

For European puts, the lower bound is $\text{Max}[0,X/(1 + r)^T - S_0]$. For American puts, the minimum price is $\text{Max}(0,X - S_0)$. The European lower bound is lower than the minimum price of the American put, so the American put lower bound is not changed to the European lower bound, the way we did for calls. Hence,

$$p_0 \geq \text{Max}[0,X/(1 + r)^T - S_0] \quad \textbf{(4-10)}$$

$$P_0 \geq \text{Max}(0,X - S_0)$$

These results tell us the lowest possible price for European and American options.

Recall that we previously referred to an option price as having an intrinsic value and a time value. For American options, the intrinsic value is the value if exercised, $\text{Max}(0,S_0 - X)$ for calls and $\text{Max}(0,X - S_0)$ for puts. The remainder of the option price is the time value. For European options, the notion of a time value is somewhat murky, because it first requires recognition of an intrinsic value. Because a European option cannot be exercised until expiration, in a sense, all of the value of a European option is time value. The notion of an intrinsic value and its complement, a time value, is therefore inappropriate for European options, though the concepts are commonly applied to European options. Fortunately, understanding European options does not require that we separate intrinsic value from time value. We shall include them together as they make up the option price.

PRACTICE PROBLEM 2

Consider call and put options expiring in 42 days, in which the underlying is at 72 and the risk-free rate is 4.5 percent. The underlying makes no cash payments during the life of the options.

A. Find the lower bounds for European calls and puts with exercise prices of 70 and 75.

B. Find the lower bounds for American calls and puts with exercise prices of 70 and 75.

SOLUTIONS

A. 70 call: $\text{Max}[0,72 - 70/(1.045)^{0.1151}] = \text{Max}(0,2.35) = 2.35$
75 call: $\text{Max}[0,72 - 75/(1.045)^{0.1151}] = \text{Max}(0,-2.62) = 0$
70 put: $\text{Max}[0,70/(1.045)^{0.1151} - 72] = \text{Max}(0,-2.35) = 0$
75 put: $\text{Max}[0,75/(1.045)^{0.1151} - 72] = \text{Max}(0,2.62) = 2.62$

B. 70 call: $\text{Max}[0,72 - 70/(1.045)^{0.1151}] = \text{Max}(0,2.35) = 2.35$
75 call: $\text{Max}[0,72 - 75/(1.045)^{0.1151}] = \text{Max}(0,-2.62) = 0$
70 put: $\text{Max}(0,70 - 72) = 0$
75 put: $\text{Max}(0,75 - 72) = 3$

5.3 The Effect of a Difference in Exercise Price

Now consider two options on the same underlying with the same expiration day but different exercise prices. Generally, the higher the exercise price, the lower the value of a call and the higher the price of a put. To see this, let the two exercise prices be X_1 and X_2, with X_1 being the smaller. Let $c_0(X_1)$ be the price of a European call with exercise price X_1 and $c_0(X_2)$ be the price of a European call with exercise price X_2. We refer to these as the X_1

call and the X_2 call. In Exhibit 4-9, we construct a combination in which we buy the X_1 call and sell the X_2 call.[17]

EXHIBIT 4-9 Portfolio Combination for European Calls Illustrating the Effect of Differences in Exercise Prices

		Value at Expiration		
Transaction	Current Value	$S_T \le X_1$	$X_1 < S_T < X_2$	$S_T \ge X_2$
Buy call ($X = X_1$)	$c_0(X_1)$	0	$S_T - X_1$	$S_T - X_1$
Sell call ($X = X_2$)	$-c_0(X_2)$	0	0	$-(S_T - X_2)$
Total	$c_0(X_1) - c_0(X_2)$	0	$S_T - X_1 > 0$	$X_2 - X_1 > 0$

Note first that the three outcomes are all non-negative. This fact establishes that the current value of the combination, $c_0(X_1) - c_0(X_2)$ has to be non-negative. We have to pay out at least as much for the X_1 call as we take in for the X_2 call; otherwise, we would get money up front, have the possibility of a positive value at expiration, and never have to pay any money out. Thus, because $c_0(X_1) - c_0(X_2) \ge 0$, we restate this result as

$$c_0(X_1) \ge c_0(X_2)$$

This expression is equivalent to the statement that *a call option with a higher exercise price cannot have a higher value than one with a lower exercise price*. The option with the higher exercise price has a higher hurdle to get over; therefore, the buyer is not willing to pay as much for it. Even though we demonstrated this result with European calls, it is also true for American calls. Thus,[18]

$$C_0(X_1) \ge C_0(X_2)$$

In Exhibit 4-10 we construct a similar portfolio for puts, except that we buy the X_2 put (the one with the higher exercise price) and sell the X_1 put (the one with the lower exercise price).

EXHIBIT 4-10 Portfolio Combination for European Puts Illustrating the Effect of Differences in Exercise Prices

		Value at Expiration		
Transaction	Current Value	$S_T \le X_1$	$X_1 < S_T < X_2$	$S_T \ge X_2$
Buy put ($X = X_2$)	$p_0(X_2)$	$X_2 - S_T$	$X_2 - S_T$	0
Sell put ($X = X_1$)	$-p_0(X_1)$	$-(X_1 - S_T)$	0	0
Total	$p_0(X_2) - p_0(X_1)$	$X_2 - X_1 > 0$	$X_2 - S_T > 0$	0

[17] In Chapter 7, when we cover option strategies, this transaction will be known as a bull spread.

[18] It is possible to use the results from this table to establish a limit on the difference between the prices of these two options, but we shall not do so here.

Observe that the value of this combination is never negative at expiration; therefore, it must be non-negative today. Hence, $p_0(X_2) - p_0(X_1) \geq 0$. We restate this result as

$$p_0(X_2) \geq p_0(X_1)$$

Thus, *the value of a European put with a higher exercise price must be at least as great as the value of a European put with a lower exercise price*. These results also hold for American puts. Therefore,

$$P_0(X_2) \geq P_0(X_1)$$

Even though it is technically possible for calls and puts with different exercise prices to have the same price, *generally we can say that the higher the exercise price, the lower the price of a call and the higher the price of a put*. For example, refer back to Exhibit 4-1 and observe how the most expensive calls and least expensive puts have the lower exercise prices.

5.4 The Effect of a Difference in Time to Expiration

Option prices are also affected by the time to expiration of the option. Intuitively, one might expect that the longer the time to expiration, the more valuable the option. A longer-term option has more time for the underlying to make a favorable move. In addition, if the option is in-the-money by the end of a given period of time, it has a better chance of moving even further in-the-money over a longer period of time. If the additional time gives it a better chance of moving out-of-the-money or further out-of-the-money, the limitation of losses to the amount of the option premium means that the disadvantage of the longer time is no greater. In most cases, a longer time to expiration is beneficial for an option. We will see that longer-term American and European calls and longer-term American puts are worth no less than their shorter-term counterparts.

First let us consider each of the four types of options: European calls, American calls, European puts, and American puts. We shall introduce options otherwise identical except that one has a longer time to expiration than the other. The one expiring earlier has an expiration of T_1 and the one expiring later has an expiration of T_2. The prices of the options are $c_0(T_1)$ and $c_0(T_2)$ for the European calls, $C_0(T_1)$ and $C_0(T_2)$ for the American calls, $p_0(T_1)$ and $p_0(T_2)$ for the European puts, and $P_0(T_1)$ and $P_0(T_2)$ for the American puts.

When the shorter-term call expires, the European call is worth $\text{Max}(0,S_{T_1} - X)$, but we have already shown that the longer-term European call is worth *at least* $\text{Max}(0,S_{T_1} - X/(1 + r)^{(T_2-T_1)})$, which is at least as great as this amount.[19] Thus, the longer-term European call is worth at least the value of the shorter-term European call. These results are not altered if the call is American. When the shorter-term American call expires, it is worth $\text{Max}(0,S_{T_1} - X)$. The longer-term American call must be worth at least the value of the European call, so it is worth *at least* $\text{Max}[0,S - X/(1 + r)^{T_2-T_1}]$. Thus, the longer-term call, European or American, is worth no less than the shorter-term call when the shorter-term call expires. Because this statement is always true, the longer-term call, European or American, is worth no less than the shorter-term call at any time prior to expiration. Thus,

$$c_0(T_2) \geq c_0(T_1) \qquad \textbf{(4-11)}$$

$$C_0(T_2) \geq C_0(T_1)$$

Notice that these statements do not mean that the longer-term call is always worth more; it means that the longer-term call can be worth no less. With the exception of the rare case

[19] Technically, we showed this calculation using a time to expiration of T, but here the time to expiration is $T_2 - T_1$.

in which both calls are so far out-of-the-money or in-the-money that the additional time is of no value, the longer-term call will be worth more.

For European puts, we have a slight problem. For calls, the longer term gives additional time for a favorable move in the underlying to occur. For puts, this is also true, but there is one disadvantage to waiting the additional time. When a put is exercised, the holder receives money. The lost interest on the money is a disadvantage of the additional time. For calls, there is no lost interest. In fact, a call holder earns additional interest on the money by paying out the exercise price later. Therefore, it is not always true that additional time is beneficial to the holder of a European put. It is true, however, that the additional time is beneficial to the holder of an American put. An American put can always be exercised; there is no penalty for waiting. Thus, we have

$p_0(T_2)$ can be either greater or less than $p_0(T_1)$

$$P_0(T_2) \geq P_0(T_1) \qquad \textbf{(4-12)}$$

So for European puts, either the longer-term or the shorter-term option can be worth more. The longer-term European put will tend to be worth more when volatility is greater and interest rates are lower.

Referring back to Exhibit 4-1, observe that the longer-term put and call options are more expensive than the shorter-term ones. As noted, we might observe an exception to this rule for European puts, but these are all American options.

5.5 Put–Call Parity

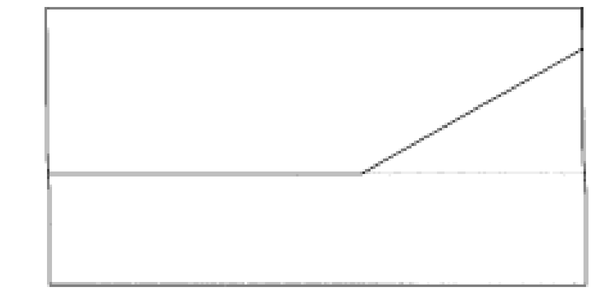
So far we have been working with puts and calls separately. To see how their prices must be consistent with each other and to explore common option strategies, let us combine puts and calls with each other or with a risk-free bond. We shall put together some combinations that produce equivalent results.

5.5.1 Fiduciary Calls and Protective Puts

First we consider an option strategy referred to as a **fiduciary call**. It consists of a European call and a risk-free bond, just like the ones we have been using, that matures on the option expiration day and has a face value equal to the exercise price of the call. The upper part of the table in Exhibit 4-11 shows the payoffs at expiration of the fiduciary call. We see that if the price of the underlying is below X at expiration, the call expires worthless and the bond is worth X. If the price of the underlying is above X at expiration, the call expires and is worth S_T (the underlying price) − X. So at expiration, the fiduciary call will end up worth X or S_T, whichever is greater.

EXHIBIT 4-11 Portfolio Combinations for Equivalent Packages of Puts and Calls

		Value at Expiration	
Transaction	Current Value	$S_T \leq X$	$S_T > X$
Fiduciary Call			
Buy call	c_0	0	$S_T - X$
Buy bond	$X/(1 + r)^T$	X	X
Total	$c_0 + X/(1 + r)^T$	X	S_T
Protective Put			
Buy put	p_0	$X - S_T$	0
Buy underlying asset	S_0	S_T	S_T
Total	$p_0 + S_0$	X	S_T



This type of combination is called a fiduciary call because it allows protection against downside losses and is thus faithful to the notion of preserving capital.

Now we construct a strategy known as a **protective put**, which consists of a European put and the underlying asset. If the price of the underlying is below X at expiration, the put expires and is worth $X - S_T$ and the underlying is worth S_T. If the price of the underlying is above X at expiration, the put expires with no value and the underlying is worth S_T. So at expiration, the protective put is worth X or S_T, whichever is greater. The lower part of the table in Exhibit 4-11 shows the payoffs at expiration of the protective put.

Thus, the fiduciary call and protective put end up with the same value. They are, therefore, identical combinations. To avoid arbitrage, their values today must be the same. The value of the fiduciary call is the cost of the call, c_0, and the cost of the bond, $X/(1 + r)^T$. The value of the protective put is the cost of the put, p_0, and the cost of the underlying, S_0. Thus,

$$c_0 + X/(1 + r)^T = p_0 + S_0 \quad \textbf{(4-13)}$$

This equation is called **put–call parity** and is one of the most important results in options. It does not say that puts and calls are equivalent, but it does show an equivalence (parity) of a call/bond portfolio and a put/underlying portfolio.

Put–callparity can be written in a number of other ways. By rearranging the four terms to isolate one term, we can obtain some interesting and important results. For example,

$$c_0 = p_0 + S_0 - X/(1 + r)^T$$

means that a call is equivalent to a long position in the put, a long position in the asset, and a short position in the risk-free bond. The short bond position simply means to borrow by issuing the bond, rather than lend by buying the bond as we did in the fiduciary call portfolio. We can tell from the sign whether we should go long or short. Positive signs mean to go long; negative signs mean to go short.

5.5.2 Synthetics

Because the right-hand side of the above equation is equivalent to a call, we often refer to it as a **synthetic call**. To see that the synthetic call is equivalent to the actual call, look at Exhibit 4-12:

EXHIBIT 4-12 Call and Synthetic Call

		Value at Expiration	
Transaction	Current Value	$S_T \le X$	$S_T > X$
Call			
Buy call	c_0	0	$S_T - X$
Synthetic Call			
Buy put	p_0	$X - S_T$	0
Buy underlying asset	S_0	S_T	S_T
Issue bond	$-X/(1 + r)^T$	$-X$	$-X$
Total	$p_0 + S_0 - X/(1 + r)^T$	0	$S_T - X$

The call produces the value of the underlying minus the exercise price or zero, whichever is greater. The synthetic call does the same thing, but in a different way. When the call expires in-the-money, the synthetic call produces the underlying value minus the payoff on the bond, which is X. When the call expires out-of-the-money, the put covers the loss on the underlying and the exercise price on the put matches the amount of money needed to pay off the bond.

Similarly, we can isolate the put as follows:

$$p_0 = c_0 - S_0 + X/(1 + r)^T$$

which says that a put is equivalent to a long call, a short position in the underlying, and a long position in the bond. Because the left-hand side is a put, it follows that the right-hand side is a **synthetic put**. The equivalence of the put and synthetic put is shown in Exhibit 4-13.

EXHIBIT 4-13 Put and Synthetic Put

		Value at Expiration	
Transaction	Current Value	$S_T \le X$	$S_T > X$
Put			
Buy put	p_0	$X - S_T$	0
Synthetic Put			
Buy call	c_0	0	$S_T - X$
Short underlying asset	$-S_0$	$-S_T$	$-S_T$
Buy bond	$X/(1 + r)^T$	X	X
Total	$c_0 - S_0 + X/(1 + r)^T$	$X - S_T$	0

As you can well imagine, there are numerous other combinations that can be constructed. Exhibit 4-14 shows a number of the more important combinations. There are two primary reasons that it is important to understand synthetic positions in option pricing. Synthetic positions enable us to price options, because they produce the same results as options and have known prices. Synthetic positions also tell how to exploit mispricing of options relative to their underlying assets. Note that we can not only synthesize a call or a put, but

EXHIBIT 4-14 Alternative Equivalent Combinations of Calls, Puts, the Underlying, and Risk-Free Bonds

Strategy	Consisting of	Worth	Equates to	Strategy	Consisting of	Worth
Fiduciary call	Long call + Long bond	$c_0 + X/(1 + r)^T$	=	Protective put	Long put + Long underlying	$p_0 + S_0$
Long call	Long call	c_0	=	Synthetic call	Long put + Long underlying + Short bond	$p_0 + S_0 - X/(1 + r)^T$
Long put	Long put	p_0	=	Synthetic put	Long call + Short underlying + Long bond	$c_0 - S_0 + X/(1 + r)^T$
Long underlying	Long underlying	S_0	=	Synthetic underlying	Long call + Long bond + Short put	$c_0 + X/(1 + r)^T - p_0$
Long bond	Long bond	$X/(1 + r)^T$	=	Synthetic bond	Long put + Long underlying + Short call	$p_0 + S_0 - c_0$

we can also synthesize the underlying or the bond. As complex as it might seem to do this, it is really quite easy. First, we learn that *a fiduciary call is a call plus a risk-free bond maturing on the option expiration day with a face value equal to the exercise price of the option*. Then we learn that *a protective put is the underlying plus a put*. Then we learn the basic put–call parity equation: *A fiduciary call is equivalent to a protective put*:

$$c_0 + X/(1 + r)^T = p_0 + S_0$$

Learn the put–call parity equation this way, because it is the easiest form to remember and has no minus signs.

Next, we decide which instrument we want to synthesize. We use simple algebra to isolate that instrument, with a plus sign, on one side of the equation, moving all other instruments to the other side. We then see what instruments are on the other side, taking plus signs as long positions and minus signs as short positions. Finally, to check our results, we should construct a table like Exhibits 4-11 or 4-12, with the expiration payoffs of the instrument we wish to synthesize compared with the expiration payoffs of the equivalent combination of instruments. We then check to determine that the expiration payoffs are the same.

5.5.3 An Arbitrage Opportunity

In this section we examine the arbitrage strategies that will push prices to put–call parity. Suppose that in the market, prices do not conform to put–call parity. This is a situation in which price does not equal value. Recalling our basic equation, $c_0 + X/(1 + r)^T = p_0 + S_0$, we should insert values into the equation and see if the equality holds. If it does not, then obviously one side is greater than the other. We can view one side as overpriced and the other as underpriced, which suggests an arbitrage opportunity. To exploit this mispricing, we buy the underpriced combination and sell the overpriced combination.

Consider the following example involving call options with an exercise price of $100 expiring in half a year ($T = 0.5$). The risk-free rate is 10 percent. The call is priced at $7.50, and the put is priced at $4.25. The underlying price is $99.

The left-hand side of the basic put–call parity equation is $c_0 + X/(1 + r)^T = 7.50 + 100/(1.10)^{0.5} = 7.50 + 95.35 = 102.85$. The right-hand side is $p_0 + S_0 = 4.25 + 99 = 103.25$. So the right-hand side is greater than the left-hand side. This means that the protective put is overpriced. Equivalently, we could view this as the fiduciary call being underpriced. Either way will lead us to the correct strategy to exploit the mispricing.

We sell the overpriced combination, the protective put. This means that we sell the put and sell short the underlying. Doing so will generate a cash inflow of \$103.25. We buy the fiduciary call, paying out \$102.85. This series of transactions nets a cash inflow of \$103.25 − \$102.85 = \$0.40. Now, let us see what happens at expiration.

The options expire with the underlying above 100:

The bond matures, paying \$100.

Use the \$100 to exercise the call, receiving the underlying.

Deliver the underlying to cover the short sale.

The put expires with no value.

Net effect: No money in or out.

The options expire with the underlying below 100:

The bond matures, paying \$100.

The put expires in-the-money; use the \$100 to buy the underlying.

Use the underlying to cover the short sale.

The call expires with no value.

Net effect: No money in our out.

So we receive \$0.40 up front and do not have to pay anything out. The position is perfectly hedged and represents an arbitrage profit. The combined effects of other investors performing this transaction will result in the value of the protective put going down and/or the value of the covered call going up until the two strategies are equivalent in value. Of course, it is possible that transaction costs might consume any profit, so small discrepancies will not be exploited.

It is important to note that regardless of which put–call parity equation we use, we will arrive at the same strategy. For example, in the above problem, the synthetic put (a long call, a short position in the underlying, and a long bond) is worth \$7.50 − \$99 + \$95.35 = \$3.85. The actual put is worth \$4.25. Thus, we would conclude that we should sell the actual put and buy the synthetic put. To buy the synthetic put, we would buy the call, short the underlying, and buy the bond—precisely the strategy we used to exploit this price discrepancy.

In all of these examples based on put–call parity, we used only European options. Put–call parity using American options is considerably more complicated. The resulting parity equation is a complex combination of inequalities. Thus, we cannot say that a given combination exactly equals another; we can say only that one combination is more valuable than another. Exploitation of any such mispricing is somewhat more complicated, and we shall not explore it here.

PRACTICE PROBLEM 3

European put and call options with an exercise price of 45 expire in 115 days. The underlying is priced at 48 and makes no cash payments during the life of the options. The risk-free rate is 4.5 percent. The put is selling for 3.75, and the call is selling for 8.00.

A. Identify the mispricing by comparing the price of the actual call with the price of the synthetic call.

B. Based on your answer in Part A, demonstrate how an arbitrage transaction is executed.

SOLUTIONS

A. Using put–call parity, the following formula applies:

$$c_0 = p_0 + S_0 - X/(1 + r)^T$$

The time to expiration is $T = 115/365 = 0.3151$. Substituting values into the right-hand side:

$$c_0 = 3.75 + 48 - 45/(1.045)^{0.3151} = 7.37$$

Hence, the synthetic call is worth 7.37, but the actual call is selling for 8.00 and is, therefore, overpriced.

B. Sell the call for 8.00 and buy the synthetic call for 7.37. To buy the synthetic call, buy the put for 3.75, buy the underlying for 48.00, and issue a zero-coupon bond paying 45.00 at expiration. The bond will bring in $45.00/(1.045)^{0.3151} = 44.38$ today. This transaction will bring in $8.00 - 7.37 = 0.63$.

At expiration, the following payoffs will occur:

	$S_T < 45$	$S_T \geq 45$
Short call	0	$-(S_T - 45)$
Long put	$45 - S_T$	0
Underlying	S_T	S_T
Bond	-45	-45
Total	0	0

Thus there will be no cash in or out at expiration. The transaction will net a risk-free gain of $8.00 - 7.37 = 0.63$ up front.

5.6 American Options, Lower Bounds, and Early Exercise

As we have noted, American options can be exercised early and in this section we specify cases in which early exercise can have value. Because early exercise is never mandatory, the right to exercise early may be worth something but could never hurt the option holder. Consequently, the prices of American options must be no less than the prices of European options:

$$C_0 \geq c_0 \quad \textbf{(4-14)}$$

$$P_0 \geq p_0$$

Recall that we already used this result in establishing the minimum price from the lower bounds and intrinsic value results in Section 5.2.2. Now, however, our concern is understanding the conditions under which early exercise of an American option might occur.

Suppose today, time 0, we are considering exercising early an in-the-money American call. If we exercise, we pay X and receive an asset worth S_0. But we already determined that a European call is worth at least $S_0 - X/(1 + r)^T$—that is, the underlying price minus the present value of the exercise price, which is more than $S_0 - X$. Because we just argued that the American call must be worth no less than the European call, it therefore must also be worth at least $S_0 - X/(1 + r)^T$. This means that the value we could obtain by selling it to someone else is more than the value we could obtain by exercising it. Thus, there is no reason to exercise the call early.

Some people fail to see the logic behind not exercising early. Exercising a call early simply gives the money to the call writer and throws away the right to decide at expiration if you want the underlying. It is like renewing a magazine subscription before the current subscription expires. Not only do you lose the interest on the money, you also lose the right to decide later if you want to renew. Without offering an early exercise incentive, the American call would have a price equal to the European call price. Thus, we must look at another case to see the value of the early exercise option.

If the underlying makes a cash payment, there may be reason to exercise early. If the underlying is a stock and pays a dividend, there may be sufficient reason to exercise just before the stock goes ex-dividend. By exercising, the option holder throws away the time value but captures the dividend. We shall skip the technical details of how this decision is made and conclude by stating that

- *When the underlying makes no cash payments, $C_0 = c_0$.*
- *When the underlying makes cash payments during the life of the option, early exercise can be worthwhile and C_0 can thus be higher than c_0.*

We emphasize the word *can*. It is possible that the dividend is not high enough to justify early exercise.

For puts, there is nearly always a possibility of early exercise. Consider the most obvious case, an investor holding an American put on a bankrupt company. The stock is worth zero. It cannot go any lower. Thus, the put holder would exercise immediately. As long as there is a possibility of bankruptcy, the American put will be worth more than the European put. But in fact, bankruptcy is not required for early exercise. The stock price must be very low, although we cannot say exactly how low without resorting to an analysis using option pricing models. Suffice it to say that *the American put is nearly always worth more than the European put*: $P_0 > p_0$.

5.7 The Effect of Cash Flows on the Underlying Asset

Both the lower bounds on puts and calls and the put–call parity relationship must be modified to account for cash flows on the underlying asset. In Chapters 2 and 3, we discussed situations in which the underlying has cash flows. Stocks pay dividends, bonds pay interest, foreign currencies pay interest, and commodities have carrying costs. As we have done in the previous chapters, we shall assume that these cash flows are either known or can be expressed as a percentage of the asset price. Moreover, as we did previously, we can remove the present value of those cash flows from the price of the underlying and use this adjusted underlying price in the results we have obtained above.

In the previous chapters, we specified these cash flows in the form of the accumulated value at T of all cash flows incurred on the underlying over the life of the derivative contract. When the underlying is a stock, we specified these cash flows more precisely in the form of dividends, using the notation FV(D,0,T) as the future value, or alternatively PV(D,0,T) as the present value, of these dividends. When the underlying was a bond, we used the notation FV(CI,0,T) or PV(CI,0,T), where CI stands for "coupon interest." When the cash flows can be specified in terms of a yield or rate, we used the notation δ where $S_0/(1 + \delta)^T$ is the underlying price reduced by the present value of the cash flows.[20] Using

[20] We actually used several specifications of the dividend yield in Chapter 3, but we shall use just one here.

continuous compounding, the rate can be specified as δ^c so that $S_0e^{-\delta^c T}$ is the underlying price reduced by the present value of the dividends. For our purposes in this chapter on options, let us just write this specification as PV(CF,0,T), which represents the present value of the cash flows on the underlying over the life of the options. Therefore, we can restate the lower bounds for European options as

$$c_0 \geq \text{Max}\{0,[S_0 - \text{PV(CF,0,T)}] - X/(1 + r)^T\}$$

$$p_0 \geq \text{Max}\{0,X/(1 + r)^T - [S_0 - \text{PV(CF,0,T)}]\}$$

and put–call parity as

$$c_0 + X/(1 + r)^T = p_0 + [S_0 - \text{PV(CF,0,T)}]$$

which reflects the fact that, as we said, we simply reduce the underlying price by the present value of its cash flows over the life of the option.

5.8 The Effect of Interest Rates and Volatility

It is important to know that interest rates and volatility exert an influence on option prices. *When interest rates are higher, call option prices are higher and put option prices are lower.* This effect is not obvious and strains the intuition somewhat. When investors buy call options instead of the underlying, they are effectively buying an indirect leveraged position in the underlying. When interest rates are higher, buying the call instead of a direct leveraged position in the underlying is more attractive. Moreover, by using call options, investors save more money by not paying for the underlying until a later date. For put options, however, higher interest rates are disadvantageous. When interest rates are higher, investors lose more interest while waiting to sell the underlying when using puts. Thus, the opportunity cost of waiting is higher when interest rates are higher. Although these points may not seem completely clear, fortunately they are not critical. Except when the underlying is a bond or interest rate, interest rates do not have a very strong effect on option prices.

Volatility, however, has an extremely strong effect on option prices. *Higher volatility increases call and put option prices because it increases possible upside values and increases possible downside values of the underlying.* The upside effect helps calls and does not hurt puts. The downside effect does not hurt calls and helps puts. The reasons calls are not hurt on the downside and puts are not hurt on the upside is that when options are out-of-the-money, it does not matter if they end up more out-of-the-money. But when options are in-the-money, it does matter if they end up more in-the-money.

Volatility is a critical variable in pricing options. It is the only variable that affects option prices that is not directly observable either in the option contract or in the market. It must be estimated. We shall have more to say about volatility later in this chapter.

5.9 Option Price Sensitivities

Later in this chapter, we will study option price sensitivities in more detail. These sensitivity measures have Greek names:

- *Delta* is the sensitivity of the option price to a change in the price of the underlying.
- *Gamma* is a measure of how well the delta sensitivity measure will approximate the option price's response to a change in the price of the underlying.
- *Rho* is the sensitivity of the option price to the risk-free rate.
- *Theta* is the rate at which the time value decays as the option approaches expiration.
- *Vega* is the sensitivity of the option price to volatility.

6 DISCRETE-TIME OPTION PRICING: THE BINOMIAL MODEL

Until now, we have looked only at some basic principles of option pricing. Other than put–call parity, all we examined were rules and conditions, often suggesting limitations, on option prices. With put–call parity, we found that we could price a put or a call based on the prices of the combinations of instruments that make up the synthetic version of the instrument. If we wanted to determine a call price, we had to have a put; if we wanted to determine a put price, we had to have a call. What we need to be able to do is price a put or a call without the other instrument. In this section, we introduce a simple means of pricing an option. It may appear that we oversimplify the situation, but we shall remove the simplifying assumptions gradually, and eventually reach a more realistic scenario.

The approach we take here is called the **binomial model**. The word "binomial" refers to the fact that there are only two outcomes. In other words, we let the underlying price move to only one of two possible new prices. As noted, this framework oversimplifies things, but the model can eventually be extended to encompass all possible prices. In addition, we refer to the structure of this model as **discrete time**, which means that time moves in distinct increments. This is much like looking at a calendar and observing only the months, weeks, or days. Even at its smallest interval, we know that time moves forward at a rate faster than one day at a time. It moves in hours, minutes, seconds, and even fractions of seconds, and fractions of fractions of seconds. When we talk about time moving in the tiniest increments, we are talking about **continuous time**. We will see that the discrete time model can be extended to become a continuous time model. Although we present the continuous time model (Black–Scholes–Merton) in Section 7, we must point out that the binomial model has the advantage of allowing us to price American options. In addition, the binomial model is a simple model requiring a minimum of mathematics. Thus it is worthy of study in its own right.

6.1 The One-Period Binomial Model

We start off by having only one binomial period. This means that the underlying price starts off at a given level, then moves forward to a new price, at which time the option expires. Here we need to change our notation slightly from what we have been using previously. We let S be the current underlying price. One period later, it can move up to S^+ or down to S^-. Note that we are removing the time subscript, because it will not be necessary here. We let X be the exercise price of the option and r be the one period risk-free rate. The option is European style.

6.1.1 The Model

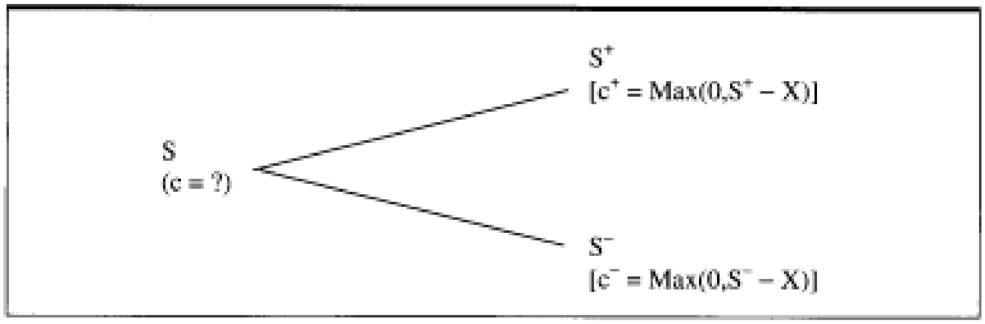
We start with a call option. If the underlying goes up to S^+, the call option will be worth c^+. If the underlying goes down to S^-, the option will be worth c^-. We know that if the option is expiring, its value will be the intrinsic value. Thus,

$$c^+ = \text{Max}(0, S^+ - X)$$

$$c^- = \text{Max}(0, S^- - X)$$

Exhibit 4-15 illustrates this scenario with a diagram commonly known as a **binomial tree**. Note how we indicate that the current option price, c, is unknown.

EXHIBIT 4-15 One-Period Binomial Model



Now let us specify how the underlying moves. We identify a factor, u, as the up move on the underlying and d as the down move:

$$u = \frac{S^+}{S}$$

$$d = \frac{S^-}{S}$$

so that u and d represent 1 plus the rate of return if the underlying goes up and down, respectively. Thus, $S^+ = Su$ and $S^- = Sd$. To avoid an obvious arbitrage opportunity, we require that[21]

$$d < 1 + r < u$$

We are now ready to determine how to price the option. We assume that we have all information except for the current option price. In addition, we do not know in what direction the price of the underlying will move. We start by constructing an arbitrage portfolio consisting of one short call option. Let us now purchase an unspecified number of units of the underlying. Let that number be n. Although at the moment we do not know the value of n, we can figure it out quickly. We call this portfolio a hedge portfolio. In fact, n is sometimes called the **hedge ratio**. Its current value is H, where

$$H = nS - c$$

This specification reflects the fact that we own n units of the underlying worth S and we are short one call.[22] One period later, this portfolio value will go to either H^+ or H^-:

$$H^+ = nS^+ - c^+$$

$$H^- = nS^- - c^-$$

[21] This statement says that if the price of the underlying goes up, it must do so at a rate better than the risk-free rate. If it goes down, it must do so at a rate lower than the risk-free rate. If the underlying always does better than the risk-free rate, it would be possible to buy the underlying, financing it by borrowing at the risk-free rate, and be assured of earning a greater return from the underlying than the cost of borrowing. This would make it possible to generate an unlimited amount of money. If the underlying always does worse than the risk-free rate, one can buy the risk-free asset and finance it by shorting the underlying. This would also make it possible to earn an unlimited amount of money. Thus, the risky underlying asset cannot dominate or be dominated by the risk-free asset.

[22] Think of this specification as a plus sign indicating assets and a minus sign indicating liabilities.

Because we can choose the value of n, let us do so by setting H^+ equal to H^-. This specification means that regardless of which way the underlying moves, the portfolio value will be the same. Thus, the portfolio will be hedged. We do this by setting

$H^+ = H^-$, which means that

$$nS^+ - c^+ = nS^- - c^-$$

We then solve for n to obtain

$$n = \frac{c^+ - c^-}{S^+ - S^-} \quad \textbf{(4-15)}$$

Because the values on the right-hand side are known, we can easily set n according to this formula. If we do so, the portfolio will be hedged. A hedged portfolio should grow in value at the risk-free rate.

$H^+ = H(1 + r)$, or

$$H^- = H(1 + r)$$

We know that $H^+ = nS^+ - c^+$, $H^- = nS^- - c^-$, and $H = nS - c$. We know the values of n, S^+, S^-, c^+, and c^-, as well as r. We can substitute and solve either of the above for c to obtain

$$c = \frac{\pi c^+ + (1 - \pi)c^-}{1 + r} \quad \textbf{(4-16)}$$

where

$$\pi = \frac{1 + r - d}{u - d} \quad \textbf{(4-17)}$$

We see that the call price today, c, is a weighted average of the next two possible call prices, c^+ and c^-. The weights are π and $1 - \pi$. This weighted average is then discounted one period at the risk-free rate.

It might appear that π and $1 - \pi$ are probabilities of the up and down movements, but they are not. In fact, the probabilities of the up and down movements are not required. It is important to note, however, that π and $1 - \pi$ are the probabilities that would exist if investors were risk neutral. Risk-neutral investors value assets by computing the expected future value and discounting that value at the risk-free rate. Because we are discounting at the risk-free rate, it should be apparent that π and $1 - \pi$ would indeed be the probabilities if the investor were risk neutral. In fact, we shall refer to them as **risk-neutral probabilities** and the process of valuing an option is often called **risk-neutral valuation**.[23]

[23] It may be helpful to contrast risk neutrality with risk aversion, which characterizes nearly all individuals. People who are risk neutral value an asset, such as an option or stock, by discounting the expected value at the risk-free rate. People who are risk averse discount the expected value at a higher rate, one that consists of the risk-free rate plus a risk premium. In the valuation of options, we are not making the assumption that people are risk neutral, but the fact that options can be valued by finding the expected value, using these special probabilities, and discounting at the risk-free rate creates the *appearance* that investors are assumed to be risk neutral. We emphasize the word "appearance," because no such assumption is being made. The terms "risk neutral probabilities" and "risk neutral valuation" are widely used in options valuation, although they give a misleading impression of the assumptions underlying the process.

6.1.2 One-Period Binomial Example

Suppose the underlying is a non-dividend-paying stock currently valued at \$50. It can either go up by 25 percent or go down by 20 percent. Thus, u = 1.25 and d = 0.80.

$$S^+ = Su = 50(1.25) = 62.50$$

$$S^- = Sd = 50(0.80) = 40$$

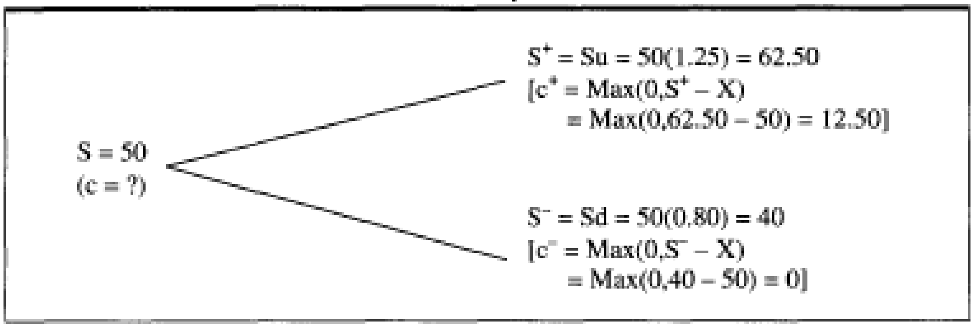
Assume that the call option has an exercise price of 50 and the risk-free rate is 7 percent. Thus, the option values one period later will be

$$c^+ = \text{Max}(0, S^+ - X) = \text{Max}(0, 62.50 - 50) = 12.50$$

$$c^- = \text{Max}(0, S^- - X) = \text{Max}(0, 40 - 50) = 0$$

Exhibit 4-16 depicts the situation.

EXHIBIT 4-16 One-Period Binomial Example



First we calculate π:

$$\pi = \frac{1 + r - d}{u - d} = \frac{1.07 - 0.80}{1.25 - 0.80} = 0.6$$

and, hence, $1 - \pi = 0.4$. Now, we can directly calculate the option price:

$$c = \frac{0.6(12.50) + 0.4(0)}{1.07} = 7.01$$

Thus, the option should sell for \$7.01.

6.1.3 One-Period Binomial Arbitrage Opportunity

Suppose the option is selling for \$8. If the option should be selling for \$7.01 and it is selling for \$8, it is overpriced—a clear case of price not equaling value. Investors would exploit this opportunity by selling the option and buying the underlying. The number of units of the underlying purchased for each option sold would be the value n:

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{12.50 - 0}{62.50 - 40} = 0.556$$

Thus, for every option sold, we would buy 0.556 units of the underlying. Suppose we sell 1,000 calls and buy 556 units of the underlying. Doing so would require an initial outlay of H = 556(\$50) − 1,000(\$8) = \$19,800. One period later, the portfolio value will be either

$$H^+ = nS^+ - c^+ = 556(\$62.50) - 1{,}000(\$12.50) = \$22{,}250, \text{ or}$$

$$H^- = nS^- - c^- = 556(\$40) - 1{,}000(\$0) = \$22{,}240$$

These two values are not exactly the same, but the difference is due only to rounding the hedge ratio, n. We shall use the \$22,250 value. If we invest \$19,800 and end up with \$22,250, the return is

$$\frac{\$22{,}250}{\$19{,}800} - 1 = 0.1237$$

that is, a risk-free return of more than 12 percent in contrast to the actual risk-free rate of 7 percent. Thus we could borrow \$19,800 at 7 percent to finance the initial net cash outflow, capturing a risk-free profit of $(0.1237 - 0.07) \times \$19{,}800 = \$1{,}063$ (to the nearest dollar) without any net investment of money. Other investors will recognize this opportunity and begin selling the option, which will drive down its price. When the option sells for \$7.01, the initial outlay would be $H = 556(\$50) - 1{,}000(\$7.01) = \$20{,}790$. The payoffs at expiration would still be \$22,250. This transaction would generate a return of

$$\frac{\$22{,}250}{\$20{,}790} - 1 \approx 0.07$$

Thus, *when the option is trading at the price given by the model, a hedge portfolio would earn the risk-free rate*, which is appropriate because the portfolio would be risk free.

If the option sells for less than \$7.01, investors would buy the option and sell short the underlying, which would generate cash up front. At expiration, the investor would have to pay back an amount less than 7 percent. All investors would perform this transaction, generating a demand for the option that would push its price back up to \$7.01.

PRACTICE PROBLEM 4

Consider a one-period binomial model in which the underlying is at 65 and can go up 30 percent or down 22 percent. The risk-free rate is 8 percent.

A. Determine the price of a European call option with exercise prices of 70.

B. Assume that the call is selling for 9 in the market. Demonstrate how to execute an arbitrage transaction and calculate the rate of return. Use 10,000 call options.

SOLUTIONS

A. First find the underlying prices in the binomial tree. We have $u = 1.30$ and $d = 1 - 0.22 = 0.78$.

$$S^+ = Su = 65(1.30) = 84.50$$

$$S^- = Sd = 65(0.78) = 50.70$$

Then find the option values at expiration:

$$c^+ = \text{Max}(0, 84.50 - 70) = 14.50$$

$$c^- = \text{Max}(0, 50.70 - 70) = 0$$

The risk-neutral probability is

$$\pi = \frac{1.08 - 0.78}{1.30 - 0.78} = 0.5769$$

and $1 - \pi = 0.4231$. The call's price today is

$$c = \frac{0.5769(14.50) + 0.4231(0)}{1.08} = 7.75$$

B. We need the value of n for calls:

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{14.50 - 0}{84.50 - 50.70} = 0.4290$$

The call is overpriced, so we should sell 10,000 call options and buy 4,290 units of the underlying.

Sell 10,000 calls at 9	+90,000
Buy 4,290 units of the underlying at 65	−278,850
Net cash flow	−188,850

So we invest 188,850. The value of this combination at expiration will be

If $S_T = 84.50$,

$$4{,}290(84.50) - 10{,}000(14.50) = 217{,}505$$

If $S_T = 50.70$,

$$4{,}290(50.70) - 10{,}000(0) = 217{,}503$$

These values differ by only a rounding error.
The rate of return is

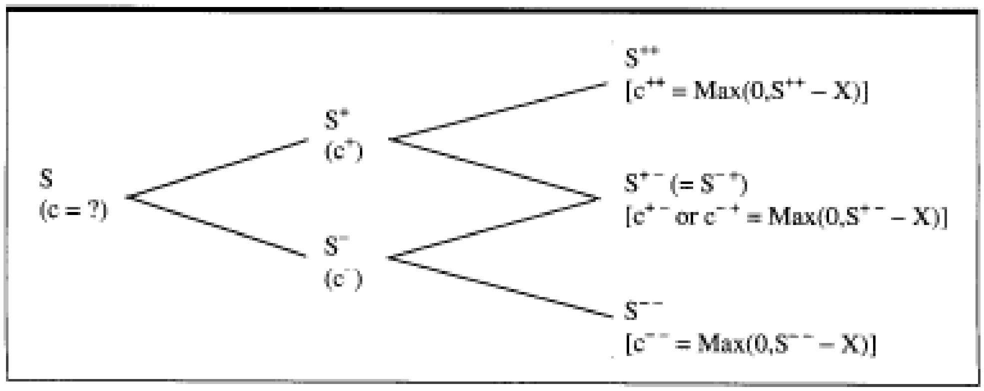
$$\frac{217{,}505}{188{,}850} - 1 = 0.1517$$

Thus, we receive a risk-free return almost twice the risk-free rate. We could borrow the initial outlay of $188,850 at the risk-free rate and capture a risk-free profit without any net investment of money.

6.2 The Two-Period Binomial Model

In the example above, the movements in the underlying were depicted over one period, and there were only two outcomes. We can extend the model and obtain more-realistic results with more than two outcomes. Exhibit 4-17 shows how to do so with a two-period binomial tree.

EXHIBIT 4-17 Two-Period Binomial Model



In the first period, we let the underlying price move from S to S^+ or S^- in the manner we did in the one-period model. That is, if u is the up factor and d is the down factor,

$$S^+ = Su$$

$$S^- = Sd$$

Then, with the underlying at S^+ after one period, it can either move up to S^{++} or down to S^{+-}. Thus,

$$S^{++} = S^+u$$

$$S^{+-} = S^+d$$

If the underlying is at S^- after one period, it can either move up to S^{-+} or down to S^{--}.

$$S^{-+} = S^-u$$

$$S^{--} = S^-d$$

We now have three unique final outcomes instead of two. Actually, we have four final outcomes, but S^{+-} is the same as S^{-+}. We can relate the three final outcomes to the starting price in the following manner:

$$S^{++} = S^+u = Suu = Su^2$$

$$S^{+-} \text{ (or } S^{-+}) = S^+d \text{ (or } S^-u) = Sud \text{ (or } Sdu)$$

$$S^{--} = S^-d = Sdd = Sd^2$$

Now we move forward to the end of the first period. Suppose we are at the point where the underlying price is S^+. Note that now we are back into the one-period model we previously derived. There is one period to go and two outcomes. The call price is c^+ and can go up to c^{++} or down to c^{+-}. Using what we know from the one-period model, the call price must be

$$c^+ = \frac{\pi c^{++} + (1 - \pi)c^{+-}}{1 + r} \qquad \textbf{(4-18)}$$

where again we see that the call price is a weighted average of the next two possible call prices, then discounted back one period. If the underlying price is at S^-, the call price would be

$$c^- = \frac{\pi c^{-+} + (1 - \pi)c^{--}}{1 + r} \qquad \textbf{(4-19)}$$

where in both cases the formula for π is still Equation 4-17:

$$\pi = \frac{1 + r - d}{u - d}$$

Now we step back to the starting point and find that the option price is still given as Equation 4-16:

$$c = \frac{\pi c^+ + (1 - \pi)c^-}{1 + r}$$

again, using the general form that the call price is a weighted average of the next two possible call prices, discounted back to the present. Other than requiring knowledge of the formula for π, the call price formula is simple and intuitive. It is an average, weighted by the risk-neutral probabilities, of the next two outcomes, then discounted to the present.[24]

Recall that the hedge ratio, n, was given as the difference in the next two call prices divided by the difference in the next two underlying prices. This will be true in all cases throughout the binomial tree. Hence, we have different hedge ratios at each time point:

$$n = \frac{c^+ - c^-}{S^+ - S^-}$$

$$n^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}} \qquad \textbf{(4-20)}$$

$$n^- = \frac{c^{-+} - c^{--}}{S^{-+} - S^{--}}$$

6.2.1 Two-Period Binomial Example

We can continue with the example presented in Section 6.1.2 in which the underlying goes up 25 percent or down 20 percent. Let us, however, alter the example a little. Suppose the underlying goes up 11.8 percent and down 10.56 percent, and we extend the number of periods to two. So, the up factor is 1.118 and the down factor is 1 − 0.1056 = 0.8944. If the underlying goes up for two consecutive periods, it rises by a factor of 1.118(1.118) = 1.25 (25 percent). If it goes down in both periods, it falls by a factor of (0.8944)(0.8944) = 0.80 (20 percent). This specification makes the highest and lowest prices unchanged. Let the risk-free rate be 3.44 percent per period. The π becomes (1.0344 − 0.8944)/(1.118 − 0.8944) = 0.6261. The underlying prices at expiration will be

$$S^{++} = Su^2 = 50(1.118)(1.118) = 62.50$$

$$S^{+-} = Sud = 50(1.118)(0.8944) = 50$$

$$S^{--} = Sd^2 = 50(0.8944)(0.8944) = 40$$

When the options expire, they will be worth

$$c^{++} = \text{Max}(0,S^{++} - 50) = \text{Max}(0,62.50 - 50) = 12.50$$

$$c^{+-} = \text{Max}(0,S^{+-} - 50) = \text{Max}(0,50 - 50) = 0$$

$$c^{--} = \text{Max}(0,S^{--} - 50) = \text{Max}(0,40 - 50) = 0$$

The option values after one period are, therefore,

$$c^+ = \frac{\pi c^{++} + (1 - \pi)c^{+-}}{1 + r} = \frac{0.6261(12.50) + 0.3739(0)}{1.0344} = 7.57$$

$$c^- = \frac{\pi c^{-+} + (1 - \pi)c^{--}}{1 + r} = \frac{0.6261(0) + 0.3739(0)}{1.0344} = 0.0$$

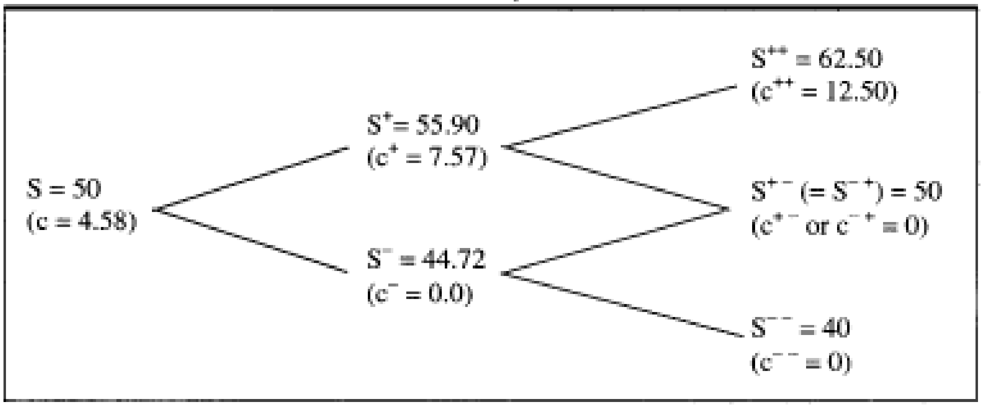
[24] It is also possible to express the price today as a weighted average of the three final option prices discounted two periods, thereby skipping the intermediate step of finding c^+ and c^-; but little is gained by doing so and this approach is somewhat more technical.

So the option price today is

$$c = \frac{\pi c^{+} + (1 - \pi)c^{-}}{1 + r} = \frac{0.6261(7.57) + 0.3739(0)}{1.0344} = 4.58$$

These results are summarized in Exhibit 4-18.

EXHIBIT 4-18 Two-Period Binomial Example



We shall not illustrate an arbitrage opportunity, because doing so requires a very long and detailed example that goes beyond our needs. Suffice it to say that if the option is mispriced, one can construct a hedged portfolio that will capture a return in excess of the risk-free rate.

PRACTICE PROBLEM 5

Consider a two-period binomial model in which the underlying is at 30 and can go up 14 percent or down 11 percent each period. The risk-free rate is 3 percent per period.

A. Find the value of a European call option expiring in two periods with an exercise price of 30.

B. Find the number of units of the underlying that would be required at each point in the binomial tree to construct a risk-free hedge using 10,000 calls.

SOLUTIONS

A. First find the underlying prices in the binomial tree: We have $u = 1.14$ and $d = 1 - 0.11 = 0.89$.

$$S^{+} = Su = 30(1.14) = 34.20$$

$$S^{-} = Sd = 30(0.89) = 26.70$$

$$S^{++} = Su^2 = 30(1.14)^2 = 38.99$$

$$S^{+-} = Sud = 30(1.14)(0.89) = 30.44$$

$$S^{--} = Sd^2 = 30(0.89)^2 = 23.76$$

Then find the option prices at expiration:

$$c^{++} = \text{Max}(0{,}38.99 - 30) = 8.99$$

$$c^{+-} = \text{Max}(0{,}30.44 - 30) = 0.44$$

$$c^{--} = \text{Max}(0{,}23.76 - 30) = 0$$

We will need the value of π:

$$\pi = \frac{1.03 - 0.89}{1.14 - 0.89} = 0.56$$

and $1 - \pi = 0.44$. Then step back and find the option prices at time 1:

$$c^{+} = \frac{0.56(8.99) + 0.44(0.44)}{1.03} = 5.08$$

$$c^{-} = \frac{0.56(0.44) + 0.44(0)}{1.03} = 0.24$$

The price today is

$$c = \frac{0.56(5.08) + 0.44(0.24)}{1.03} = 2.86$$

B. The number of units of the underlying at each point in the tree is found by first computing the values of n.

$$n = \frac{5.08 - 0.24}{34.20 - 26.70} = 0.6453$$

$$n^{+} = \frac{8.99 - 0.44}{38.99 - 30.44} = 1.00$$

$$n^{-} = \frac{0.44 - 0}{30.44 - 23.76} = 0.0659$$

The number of units of the underlying required for 10,000 calls would thus be 6,453 today, 10,000 at time 1 if the underlying is at 34.20, and 659 at time 1 if the underlying is at 26.70.

6.3 Binomial Put Option Pricing

In Section 6.2, the option was a call. It is a simple matter to make the option a put. We could step back through the entire example, changing all c's to p's and using the formulas for the payoff values of a put instead of a call. We should note, however, that if the same formula used for a call is used to calculate the hedge ratio, the minus sign should be ignored as it would suggest being long the stock (put) and short the put (stock) when the hedge portfolio should actually be long both instruments or short both instruments. The put moves opposite to the stock in the first place; hence, long or short positions in both instruments are appropriate.

PRACTICE PROBLEM 6

Repeating the data from Practice Problem 4, consider a one-period binomial model in which the underlying is at 65 and can go up 30 percent or down 22 percent. The risk-free rate is 8 percent. Determine the price of a European put option with exercise price of 70.

SOLUTION

First find the underlying prices in the binomial tree. We have u = 1.30 and d = 1 − 0.22 = 0.78.

$$S^+ = Su = 65(1.30) = 84.50$$

$$S^- = Sd = 65(0.78) = 50.70$$

Then find the option values at expiration:

$$p^+ = \text{Max}(0, 70 - 84.50) = 0$$

$$p^- = \text{Max}(0, 70 - 50.70) = 19.30$$

The risk-neutral probability is

$$\pi = \frac{1.08 - 0.78}{1.30 - 0.78} = 0.5769$$

and $1 - \pi = 0.4231$. The put price today is

$$p = \frac{0.5769(0) + 0.423(19.30)}{1.08} = 7.56$$

6.4 Binomial Interest Rate Option Pricing

In the examples above, the applications were appropriate for options on a stock, currency, or commodity.[25] Now we take a brief look at options on bonds and interest rates. A model for pricing these options must start with a model for the one-period interest rate and the prices of zero-coupon bonds.

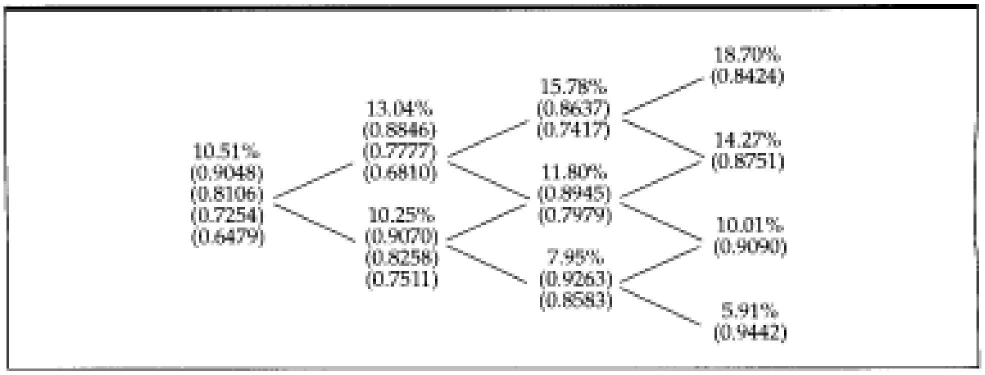
We look at such a model in Exhibit 4-19. Note that this binomial tree is the first one we have seen with more than two time periods. At each point in the tree, we see a group of numbers. The first number is the one-period interest rate. The second set of numbers, which are in parentheses, represents the prices of $1 face value zero-coupon bonds of various maturities. At time 0, 0.9048 is the price of a one-period zero-coupon bond, 0.8106 is the price of a two-period zero-coupon bond, 0.7254 is the price of a three-period zero-coupon bond, and 0.6479 is the price of a four-period zero-coupon bond. The one-period bond price can be determined from the one-period rate—that is, 0.9048 = 1/1.1051, subject to some rounding off. The other prices cannot be determined solely from the one-period rate; we would have to see a tree of the two-, three-, and four-period rates. As we move forward in time, we lose one bond as the one-period bond matures.[26] Thus, at time 1, when the one-period rate is 13.04 percent, the two-period bond from the previous

[25] We have also been assuming that there are no cash flows on the underlying.

[26] Technically we could show the bond we are losing as a bond with a price of $1.00, its face value, at its point of maturity.

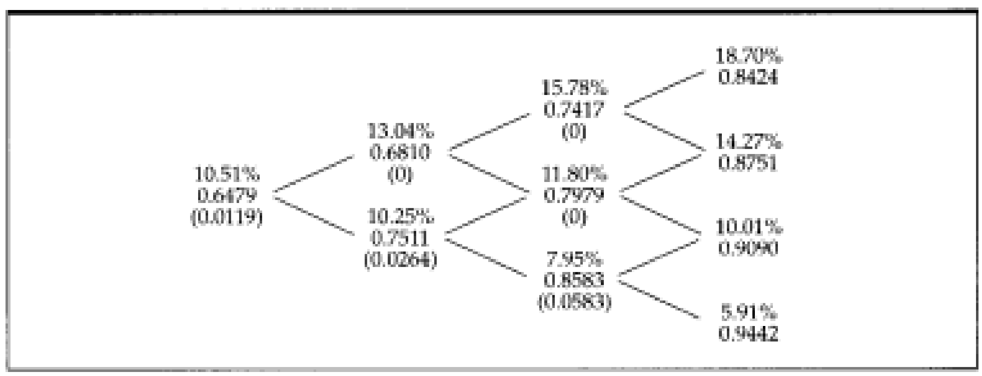
period, whose price was 0.8106, is now a one-period bond whose price is 1/1.1304 = 0.8846. Although we present these prices and rates here without derivation, they were determined using a model that prevents arbitrage opportunities in buying and selling bonds. We do not cover the actual derivation of the model here.

EXHIBIT 4-19 Binomial Interest Rate Tree



Now let us price a European option on a zero-coupon bond. First note that we need the option to expire before the bond matures, and it should have a reasonable exercise price. We shall work with the four-period zero-coupon bond. Exhibit 4-20 contains its price and the price of a two-period call option with an exercise price of $0.80 per $1 of par, as well as the one-period interest rate. The binomial interest rate tree in Exhibit 4-20 is based on the data in Exhibit 4-19. In parentheses in Exhibit 4-20 are the prices of the call option expiring at time 2.

EXHIBIT 4-20 Four-Period Zero-Coupon Bond and Two-Period Call Option with Exercise Price of 0.80



First note that in binomial term structure models, the models are usually fit such that the risk-neutral probability, π, is 0.5. Thus we do not have to calculate π, as in the examples above. We must, however, do one thing quite differently. Whereas we have used a constant interest rate, we must now discount at a different interest rate, the one-period rate, given in Exhibit 4-19, depending on where we are in the tree.

The payoff values at time 2 of the call with exercise price of 0.80 are

$$c^{++} = \text{Max}(0, 0.7417 - 0.80) = 0$$

$$c^{+-} = \text{Max}(0, 0.7979 - 0.80) = 0$$

$$c^{--} = \text{Max}(0, 0.8583 - 0.80) = 0.0583$$

These numbers appear in Exhibit 4-20 at time 2 along with the underlying bond prices and the one-period interest rates. Stepping back to time 1, we find the option prices as follows:

$$c^{+} = \frac{0.5(0) + 0.5(0)}{1.1304} = 0$$

$$c^{-} = \frac{0.5(0) + 0.5(0.0583)}{1.1025} = 0.0264$$

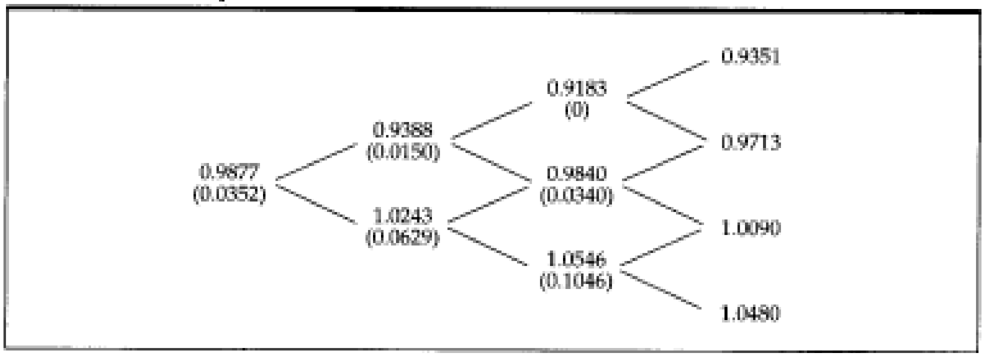
Note how we discount by the appropriate one-period rate, which is 10.25 percent for the bottom outcome at time 1 and 13.04 percent for the top outcome at time 1. Stepping back to time 0, the option price is, therefore,

$$c = \frac{0.5(0) + 0.5(0.0264)}{1.1051} = 0.0119$$

using the one-period rate of 10.51 percent. The call option is thus worth $0.0119 when the underlying zero-coupon bond paying $1 at time 4 is currently worth $0.6479.

Now let us price an option on a coupon bond. First, however, we must construct the tree of coupon bond prices. Exhibit 4-21 illustrates the price of a $1 face value, 11 percent coupon bond maturing at time 4 along with a call option expiring at time 2 with an exercise price of $0.95 per $1 of par.

EXHIBIT 4-21 Four-Period 11 Percent Coupon Bond and Two-Period Call Option with Exercise Price of 0.95



We obtain the prices of the coupon bond from the prices of zero-coupon bonds. For example, at time 0, a four-period 11 percent coupon bond is equivalent to a combination of zero-coupon bonds with face value of 0.11 maturing at times 1, 2, and 3, and a zero-coupon bond with face value of 1.11 maturing at time 4. Thus, its price can be found by multiplying these face values by the prices of one-, two-, three-, and four-period zero-coupon bonds respectively, the prices of which are taken from Exhibit 4-19.

$$0.11(0.9048) + 0.11(0.8106) + 0.11(0.7254) + 1.11(0.6479) = 0.9877$$

At any other point in the tree, we use the same procedure, but of course fewer coupons remain.[27] Of course, pricing a coupon bond by decomposing it into a combination of zero-coupon bonds is basic fixed income material, which you have learned elsewhere in the CFA curriculum.

Now let us find the option prices. At time 2, the prices are

$$c^{++} = \text{Max}(0, 0.9183 - 0.95) = 0$$
$$c^{+-} = \text{Max}(0, 0.9840 - 0.95) = 0.0340$$
$$c^{--} = \text{Max}(0, 1.0546 - 0.95) = 0.1046$$

Stepping back to time 1, the prices are

$$c^{+} = \frac{0.5(0.0) + 0.5(0.340)}{1.304} = 0.0150$$
$$c^{-} = \frac{0.5(0.340) + 0.5(0.1046)}{1.025} = 0.0629$$

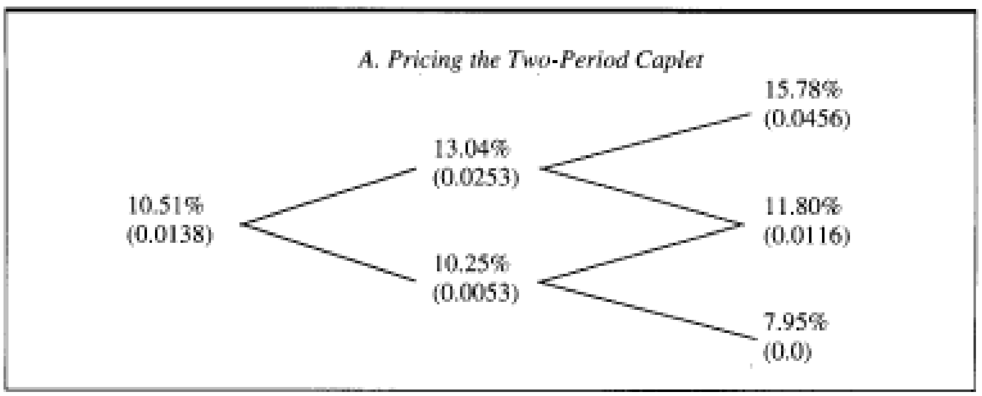
Stepping back to time 0, the option price is

$$c = \frac{0.5(0.0150) + 0.5(0.0629)}{1.1051} = 0.0352$$

Now let us look at options on interest rates. Recall that in Section 4.1.4, we illustrated how these options work. Their payoffs are based on the difference between the interest rate and an exercise rate. When the option expires, the payoff does not occur for one additional period. Thus, we have to discount the intrinsic value at expiration by the one-period interest rate. Recall that an interest rate cap is a set of interest rate call options expiring at various points in the life of a loan. The cap is generally set up to hedge the interest rate risk on a floating rate loan.

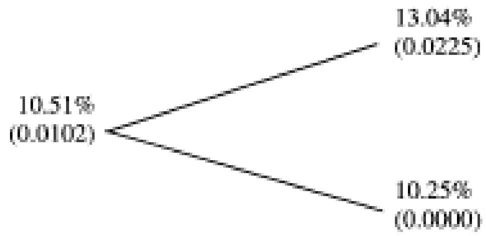
Exhibit 4-22 illustrates the pricing of a two-period cap with an exercise rate of 10.5 percent. This contract consists of two caplets: a one-period call option on the one-period

EXHIBIT 4-22 Two-Period Cap on One-Period Interest Rate with Exercise Rate of 10.5 Percent



[27] For example, consider the middle node at time 2. The coupon bond is now a two-period bond. The one- and two-period zero-coupon bond prices are 0.8945 and 0.7979, respectively (from Exhibit 4-19). Thus, the coupon bond price is 0.11(0.8945) + 1.11(0.7979) = 0.9840 as shown in Exhibit 4-21.

B. Pricing the One-Period Caplet



interest rate with an exercise rate of 10.5 percent, and a two-period call option on the one-period interest rate with an exercise rate of 10.5 percent. We price the cap by pricing these two component options.

In Panel A, we price the two-period caplet. The values at time 2 are

$$c^{++} = \frac{\text{Max}(0,0.1578 - 0.105)}{1.1578} = 0.0456$$

$$c^{+-} = \frac{\text{Max}(0,0.1180 - 0.105)}{1.1180} = 0.0116$$

$$c^{--} = \frac{\text{Max}(0,0.0795 - 0.105)}{1.0795} = 0.0$$

Note especially that we discount the payoff one period at the appropriate one-period rate, because the payoff does not occur until one period later. Stepping back to time 1:

$$c^{+} = \frac{0.5(0.0456) + 0.5(0.0116)}{1.1304} = 0.0253$$

$$c^{-} = \frac{0.5(0.0116) + 0.5(0.0)}{1.1025} = 0.0053$$

At time 0, the option price is

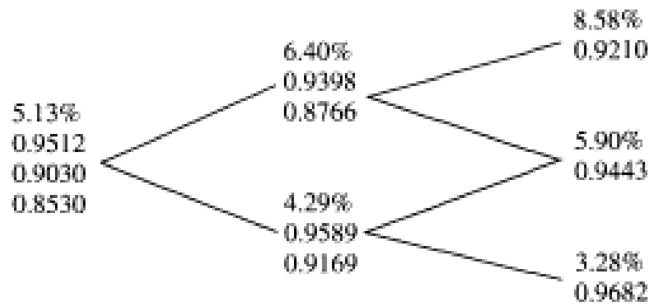
$$c = \frac{0.5(0.0253) + 0.5(0.0053)}{1.1051} = 0.0138$$

Panel B illustrates the same procedure for the one-period caplet. We shall omit the details because they follow precisely the pattern above. The one-period caplet price is 0.0102; thus the cap costs 0.0138 + 0.0102 = 0.0240.

If the option is a floor, the procedure is precisely the same but the payoffs are based on the payoffs of a put instead of a call. Pricing a zero-cost collar, however, is considerably more complex. Remember that a zero-cost collar is a long cap and a short floor with the exercise rates set such that the premium on the cap equals the premium on the floor. We can arbitrarily choose the exercise rate on the cap or the floor, but the exercise rate on the other would have to be found by trial and error so that the premium offsets the premium on the other instrument.

PRACTICE PROBLEM 7

The diagram below is a two-period binomial tree containing the one-period interest rate and the prices of zero-coupon bonds. The first price is a one-period zero-coupon bond, the second is a two-period zero-coupon bond, and the third is a three-period zero-coupon bond. As we move forward, one bond matures and its price is removed. The maturity of each bond is then shorter by one period.



A. Find the price of a European put expiring in two periods with an exercise price of 1.01 on a three-period 6 percent coupon bond with \$1.00 face value.

B. Find the price of a European put option expiring at time 2 with an exercise rate of 6 percent where the underlying is the one-period rate.

SOLUTIONS

A. First we have to find the price of the three-period \$1.00 par, 6 percent coupon bond at expiration of the option (t = 2). We break the coupon bond up into zero-coupon bonds of one, two, and three periods to maturity. The face values of these zero-coupon bonds are 0.06, 0.06, and 1.06, respectively. The bond price at t = 2 is \$1.06 discounted one period at the appropriate discount rate:

Bond prices at time 2:

$++$ outcome: $1.06(0.9210) = 0.9763$

$+-$ outcome: $1.06(0.9443) = 1.0010$

$--$ outcome: $1.06(0.9682) = 1.0263$

Now compute the put option values at expiration:

$++$ outcome: $\text{Max}(0, 1.01 - 0.9763) = 0.0337$

$+-$ outcome: $\text{Max}(0, 1.01 - 1.0010) = 0.0090$

$--$ outcome: $\text{Max}(0, 1.01 - 1.0263) = 0.0000$

Now step back and compute the option values at time 1:

$+$ outcome: $\dfrac{0.5(0.0337) + 0.5(0.0090)}{1.064} = 0.0201$

$-$ outcome: $\dfrac{0.5(0.0090) + 0.5(0.0000)}{1.0429} = 0.0043$

Now step back and compute the option values at time 0:

$$\frac{0.5(0.201) + 0.5(0.0043)}{1.0513} = 0.0116$$

B. First compute the put option values at expiration:

$$p^{++} = \frac{\text{Max}(0,0.06 - 0.0858)}{1.0858} = 0.0000$$

$$p^{+-} = \frac{\text{Max}(0,0.06 - 0.0590)}{1.059} = 0.0009$$

$$p^{--} = \frac{\text{Max}(0,0.06 - 0.0328)}{1.0328} = 0.0263$$

Step back to time 2 and compute the option values:

$$p^{+} = \frac{0.5(0.0000) + 0.5(0.0009)}{1.064} = 0.0004$$

$$p^{-} = \frac{0.5(0.0009) + 0.5(0.0263)}{1.0429} = 0.0130$$

Now step back to time 0 and compute the option price as

$$p = \frac{0.5(0.0004) + 0.5(0.0130)}{1.0513} = 0.0064$$

6.5 American Options

The binomial model is also well suited for handling American-style options. At any point in the binomial tree, we can see whether the calculated value of the option is exceeded by its value if exercised early. If that is the case, we replace the calculated value with the exercise value.[28]

6.6 Extending the Binomial Model

In the examples in this chapter, we divided an option's life into a given number of periods. Suppose we are pricing a one-year option. If we use only one binomial period, it will give us only two prices for the underlying, and we are unlikely to get a very good result. If we use two binomial periods, we will have three prices for the underlying at expiration. This result would probably be better but still not very good. But as we increase the number of periods, the result should become more accurate. In fact, in the limiting case, we are likely to get a very good result. By increasing the number of periods, we are moving from discrete time to continuous time.

Consider the following example of a one-period binomial model for a nine-month option. The asset is priced at 52.75. It can go up by 35.41 percent or down by 26.15 percent, so $u = 1.3541$ and $d = 1 - 0.2615 = 0.7385$. The risk-free rate is 4.88 percent. A call option has an exercise price of 50 and expires in nine months. Using a one-period binomial model would obtain an option price of 10.0259. Exhibit 4-23 shows the results we obtain if we divide the nine-month option life into an increasing number of periods of smaller and smaller length. The manner in which we fit the binomial tree is not arbitrary, however, because we have to alter the values of u, d, and the risk-free rate so that the underlying price move is reasonable for the life of the option. How we alter u and d is related to the volatility, a topic we cover in the next section. In fact, we need not concern ourselves with exactly how to alter any of these values. We need only to observe that our binomial option price appears to be converging to a value of around 8.62.

In the same way a sequence of rapidly taken still photographs converges to what appears to be a continuous sequence of a subject's movements, the binomial model converges to a continuous-time model, the subject of which is in our next section.

[28] See Chapter 4 of *An Introduction to Derivatives and Risk Management*, 6th edition, Don M. Chance (South-Western College Publishing, 2004) for a treatment of this topic.

EXHIBIT 4-23 Binomial Option Prices for Different Numbers of Time Periods

Number of Time Periods	Option Price
1	10.0259
2	8.4782
5	8.8305
10	8.6983
25	8.5862
50	8.6438
100	8.6160
500	8.6162
1000	8.6190

Notes: Call option with underlying price of 52.75, up factor of 1.3541, down factor of 0.7385, risk-free rate of 4.88 percent, and exercise price of 50. The variables u, d, and r are altered accordingly as the number of time periods increases.

7 CONTINUOUS-TIME OPTION PRICING: THE BLACK–SCHOLES–MERTON MODEL

When we move to a continuous-time world, we price options using the famous Black–Scholes–Merton model. Named after its founders Fischer Black, Myron Scholes, and Robert Merton, this model resulted in the award of a Nobel Prize to Scholes and Merton in 1997.[29] (Fischer Black had died in 1995 and thus was not eligible for the prize.) The model can be derived either as the continuous limit of the binomial model, or through taking expectations, or through a variety of highly complex mathematical procedures. We are not concerned with the derivation here and instead simply present the model and its applications. First, however, let us briefly review its underlying assumptions.

7.1 Assumptions of the Model

7.1.1 The Underlying Price Follows a Geometric Lognormal Diffusion Process

This assumption is probably the most difficult to understand, but in simple terms, *the underlying price follows a lognormal probability distribution as it evolves through time.* A lognormal probability distribution is one in which the log return is normally distributed. For example, if a stock moves from 100 to 110, the return is 10 percent but the log return is $\ln(1.10) = 0.0953$ or 9.53 percent. Log returns are often called *continuously compounded returns*. If the log or continuously compounded return follows the familiar normal or bell-shaped distribution, the return is said to be lognormally distributed. The distribution of the return itself is skewed, reaching further out to the right and truncated on the left side, reflecting the limitation that an asset cannot be worth less than zero.

The lognormal distribution is a convenient and widely used assumption. It is almost surely not an exact measure in reality, but it suffices for our purposes.

[29] The model is more commonly called the Black–Scholes model, but we choose to give Merton the credit he is due that led to his co-receipt of the Nobel Prize.

7.1.2 The Risk-Free Rate Is Known and Constant

The Black–Scholes–Merton model does not allow interest rates to be random. Generally, we assume that *the risk-free rate is constant*. This assumption becomes a problem for pricing options on bonds and interest rates, and we will have to make some adjustments then.

7.1.3 The Volatility of the Underlying Asset Is Known and Constant

The volatility of the underlying asset, specified in the form of the standard deviation of the log return, is assumed to be known at all times and does not change over the life of the option. This assumption is the most critical, and we take it up again in a later section. In reality, the volatility is definitely not known and must be estimated or obtained from some other source. In addition, volatility is generally not constant. Obviously, the stock market is more volatile at some times than at others. Nonetheless, the assumption is critical for this model. Considerable research has been conducted with the assumption relaxed, but this topic is an advanced one and does not concern us here.

7.1.4 There Are No Taxes or Transaction Costs

We have made this assumption all along in pricing all types of derivatives. Taxes and transaction costs greatly complicate our models and keep us from seeing the essential financial principles involved in the models. It is possible to relax this assumption, but we shall not do so here.

7.1.5 There Are No Cash Flows on the Underlying

We have discussed this assumption at great length in pricing futures and forwards and earlier in this chapter in studying the fundamentals of option pricing. The basic form of the Black–Scholes–Merton model makes this assumption, but it can easily be relaxed. We will show how to do this in Section 7.4.

7.1.6 The Options Are European

With only a few very advanced variations, the Black–Scholes–Merton model does not price American options. Users of the model must keep this in mind, or they may badly misprice these options. For pricing American options, the best approach is the binomial model with a large number of time periods.

7.2 The Black–Scholes–Merton Formula

Although the mathematics underlying the Black–Scholes–Merton formula are quite complex, the formula itself is not difficult, although it may appear so at first glance. The input variables are some of those we have already used: S_0 is the price of the underlying, X is the exercise price, r^c is the continuously compounded risk-free rate, and T is the time to expiration. The one other variable we need is the standard deviation of the log return on the asset. We denote this as σ and refer to it as the volatility. Then, the Black–Scholes–Merton formulas for the prices of call and put options are

$$c = S_0N(d_1) - Xe^{-r^cT}N(d_2) \qquad \textbf{(4-21)}$$

$$p = Xe^{-r^cT}[1 - N(d_2)] - S_0[1 - N(d_1)]$$

where

$$d_1 = \frac{\ln(S_0/X) + [r^c + (\sigma^2/2)]T}{\sigma\sqrt{T}} \qquad \textbf{(4-22)}$$

$d_2 = d_1 - \sigma\sqrt{T}$

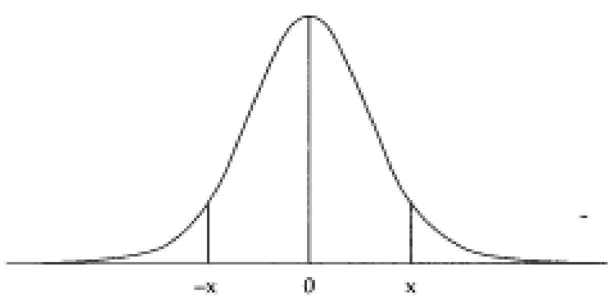
σ = the annualized standard deviation of the continuously compounded return on the stock

r^c = the continuously compounded risk-free rate of return

Of course, we have already seen the terms "ln" and "e" in previous chapters. We do, however, introduce two new and somewhat unusual looking terms, $N(d_1)$ and $N(d_2)$. These terms represent normal probabilities based on the values of d_1 and d_2. We compute the normal probabilities associated with values of d_1 and d_2 using the second equation above and insert these values into the formula as $N(d_1)$ and $N(d_2)$. Exhibit 4-24 presents a brief review of the normal probability distribution and explains how to obtain a probability value. Once we know how to look up a number in a normal probability table, we can then easily calculate d_1 and d_2, look them up in the table to obtain $N(d_1)$ and $N(d_2)$, and then insert the values of $N(d_1)$ and $N(d_2)$ into the above formula.

EXHIBIT 4-24 The Normal Probability Distribution

The normal probability distribution, or bell-shaped curve, gives the probability that a standard normal random variable will be less than or equal to a given value. The graph below shows the normal probability distribution; note that the curve is centered around zero. The values on the horizontal axis run from $-\infty$ to $+\infty$. If we were interested in a value of x of positive infinity, we would have $N(+\infty) = 1$. This expression means that the probability is 1.0 that we would obtain a value less than $+\infty$. If we were interested in a value of x of negative infinity, then $N(-\infty) = 0.0$. This expression means that there is zero probability of a value of x of less than negative infinity. Below, we are interested in the probability of a value less than x, where x is not infinite. We want N(x), which is the area under the curve to the left of x.



We obtain the values of N(x) by looking them up in a table. Below is an excerpt from a table of values of x (the full table is given as Appendix 4A). Suppose x = 1.12. Then we find the

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	**0.7734**	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	**0.8686**	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

row containing the value 1.1 and move over to the column containing 0.02. The sum of the row value and the column value is the value of x. The corresponding probability is seen as the value 0.8686. Thus, N(1.12) = 0.8686. This means that the probability of obtaining a value of less than 1.12 in a normal distribution is 0.8686.

Now, suppose the value of x is negative. Observe in the figure above that the area to the left of −x is the same as the area to the right of +x. Therefore, if x is a negative number, N(x) is found as 1 − N(−x). For example, let x = −0.75. We simply look up N(−x) = N[−(−0.75)] = N(0.75) = 0.7734. Then N(−0.75) = 1 − 0.7734 = 0.2266.

Consider the following example. The underlying price is 52.75 and has a volatility of 0.35. The continuously compounded risk-free rate is 4.88 percent. The option expires in nine months; therefore, T = 9/12 = 0.75. The exercise price is 50. First we calculate the values of d_1 and d_2:

$$d_1 = \frac{\ln(52.75/50) + (0.0488 + (0.35)^2/2)0.75}{0.35\sqrt{0.75}} = 0.4489$$

$$d_2 = 0.4489 - 0.35\sqrt{0.75} = 0.1458$$

To use the normal probability table in Appendix 4A, we must round off d_1 and d_2 to two digits to the right of the decimal. Thus we have $d_1 = 0.45$ and $d_2 = 0.15$. From the table, we obtain

$$N(0.45) = 0.6736$$

$$N(0.15) = 0.5596$$

Then we plug everything into the equation for c:

$$c = 52.75(0.6736) - 50e^{-0.0488(0.75)}(0.5596) = 8.5580$$

The value of a put with the same terms would be

$$p = 50e^{-0.0488(0.75)}(1 - 0.5596) - 52.75(1 - 0.6736) = 4.0110$$

At this point, we should note that the Black–Scholes–Merton model is extremely sensitive to rounding errors. In particular, the process of looking up values in the normal probability table is a major source of error. A number of other ways exist to obtain $N(d_1)$ and $N(d_2)$, such as using Microsoft Excel's function "=normsdist()". Using a more precise method, such as Excel, the value of the call would be 8.619. Note that this is the value to which the binomial option price converged in the example we showed with 1,000 time periods in Exhibit 4-23. Indeed, the Black–Scholes–Merton model is said to be the continuous limit of the binomial model.

PRACTICE PROBLEM 8

Use the Black–Scholes–Merton model to calculate the prices of European call and put options on an asset priced at 68.5. The exercise price is 65, the continuously compounded risk-free rate is 4 percent, the options expire in 110 days, and the volatility is 0.38. There are no cash flows on the underlying.

SOLUTIONS

The time to expiration will be T = 110/365 = 0.3014. Then d_1 and d_2 are

$$d_1 = \frac{\ln(68.5/65) + (0.04 + (0.38)^2/2)(0.3014)}{0.38\sqrt{0.3014}} = 0.4135$$

$$d_2 = 0.4135 - 0.38\sqrt{0.3014} = 0.2049$$

Looking up in the normal probability table, we have

$$N(0.41) = 0.6591$$

$$N(0.20) = 0.5793$$

Plugging into the option price formula,

$$c = 68.5(0.6591) - 65e^{-0.04(0.3014)}(0.5793) = 7.95$$

$$p = 65e^{-0.04(0.3014)}(1 - 0.5793) - 68.5(1 - 0.6591) = 3.67$$

Let us now take a look at the various inputs required in the Black–Scholes–Merton model. We need to know where to obtain the inputs and how the option price varies with these inputs.

7.3 Inputs to the Black–Scholes–Merton Model

The Black–Scholes–Merton model has five inputs: the underlying price, the exercise price, the risk-free rate, the time to expiration, and the volatility.[30] As we have previously seen, call option prices should be higher the higher the underlying price, the longer the time to expiration, the higher the volatility, and the higher the risk-free rate. They should be lower the higher the exercise price. Put option prices should be higher the higher the exercise price and the higher the volatility. They should be lower the higher the underlying price and the higher the risk-free rate. As we saw, European put option prices can be either higher or lower the longer the time to expiration. American put option prices are always higher the longer the time to expiration, but the Black–Scholes–Merton model does not apply to American options.

These relationships are general to any European and American options and do not require the Black–Scholes–Merton model to understand them. Nonetheless, the Black–Scholes–Merton model provides an excellent opportunity to examine these relationships more closely. We can calculate and plot relationships such as those mentioned, which are usually called the option Greeks, because they are often referred to with Greek names. Let us now look at each of the inputs and the various option Greeks.

7.3.1 The Underlying Price: Delta and Gamma

The price of the underlying is generally one of the easiest sources of input information. Suffice it to say that if an investor cannot obtain the price of the underlying, then she should not even be considering the option. The price should generally be obtained as a quote or trade price from a liquid, open market.

[30] Later we shall add one more input, cash flows on the underlying.

The relationship between the option price and the underlying price has a special name: It is called the option **delta**. In fact, the delta can be obtained approximately from the Black–Scholes–Merton formula as the value of $N(d_1)$ for calls and $N(d_1) - 1$ for puts. More formally, the delta is defined as

$$\text{Delta} = \frac{\text{Change in option price}}{\text{Change in underlying price}} \tag{4-23}$$

The above definition for delta is exact; the use of $N(d_1)$ for calls and $N(d_1) - 1$ for puts is approximate. Later in this section, we shall see why $N(d_1)$ and $N(d_2)$ are approximations and when they are good or bad approximations.

Let us consider the example we previously worked, where $S = 52.75$, $X = 50$, $r^c = 0.0488$, $T = 0.75$, and $\sigma = 0.35$. Using a computer to obtain a more precise Black–Scholes–Merton answer, we get a call option price of 8.6186 and a put option price of 4.0717. $N(d_1)$, the call delta, is 0.6733, so the put delta is $0.6733 - 1 = -0.3267$. Given that Delta = (Change in option price/Change in underlying price), we should expect that

Change in option price = Delta × Change in underlying price.

Therefore, for a \$1 change in the price of the underlying, we should expect

Change in call option price = 0.6733(1) = 0.6733

Change in put option price = −0.3267(1) = −0.3267

This calculation would mean that

Approximate new call option price = 8.6186 + 0.6733 = 9.2919

Approximate new put option price = 4.0717 − 0.3267 = 3.7450

To test the accuracy of this approximation, we let the underlying price move up \$1 to \$53.75 and re-insert these values into the Black–Scholes–Merton model. We would then obtain

Actual new call option price = 9.3030

Actual new put option price = 3.7560

The delta approximation is fairly good, but not perfect.

Delta is important as a risk measure. *The delta defines the sensitivity of the option price to a change in the price of the underlying.* Traders, especially dealers in options, use delta to construct hedges to offset the risk they have assumed by buying and selling options. For example, recall from Chapter 2 that FRA dealers offer to take either side of an FRA transaction. They then usually hedge the risk they have assumed by entering into other transactions. These same types of dealers offer to buy and sell options, hedging that risk with other transactions. For example, suppose we are a dealer offering to sell the call option we have been working with above. A customer buys 1,000 options for 8.619. We now are short 1,000 call options, which exposes us to considerable risk if the underlying goes up. So we must buy a certain number of units of the underlying to hedge this risk. We previously showed that the delta is 0.6733, so we would buy 673 units of the underlying at 52.75.[31] Assume for the moment that the delta tells us precisely the movement in the option for a movement in the underlying. Then suppose the underlying moves up $1:

Change in value of 1,000 long units of the underlying: 673(+$1) = $673

Change in value of 1,000 short options: 1,000(+$1)(0.6733) ≈ $673

Because we are long the underlying and short the options, these values offset. At this point, however, the delta has changed. If we recalculate it, we would find it to be 0.6953. This would require that we have 695 units of the underlying, so we would need to buy an additional 22 units. We would borrow the money to do this. In some cases, we would need to sell off units of the underlying, in which case we would invest the money in the risk-free asset.

We shall return to the topic of delta hedging in Chapter 7. For now, however, let us consider how changes in the underlying price will change the delta. In fact, even if the underlying price does not change, the delta would still change as the option moves toward expiration. For a call, the delta will increase toward 1.0 as the underlying price moves up and will decrease toward 0.0 as the underlying price moves down. For a put, the delta will decrease toward −1.0 as the underlying price moves down and increase towards 0.0 as the underlying price moves up.[32] If the underlying price does not move, a call delta will move toward 1.0 if the call is in-the-money or 0.0 if the call is out-of-the-money as the call moves toward the expiration day. A put delta will move toward −1.0 if the put is in-the-money or 0.0 if the put is out-of-the-money as it moves toward expiration.

So the delta is constantly changing, which means that delta hedging is a dynamic process. In fact, delta hedging is often referred to as **dynamic hedging**. In theory, the delta is changing continuously and the hedge should be adjusted continuously, but continuous adjustment is not possible in reality. When the hedge is not adjusted continuously, we are admitting the possibility of much larger moves in the price of the underlying. Let us see what happens in that case.

Using our previous example, we allow an increase in the underlying price of $10 to $62.75. Then the call price should change by 0.6733(10) = 6.733, and the put option price should change by −0.3267(10) = −3.267. Thus, the approximate prices would be

Approximate new call option price = 8.619 + 6.733 = 15.3520

Approximate new put option price = 4.0717 − 3.267 = 0.8047

The actual prices are obtained by recalculating the option values using the Black–Scholes–Merton model with an underlying price of 62.75. Using a computer for greater precision, we find that these prices are

Actual new call option price = 16.3026

Actual new put option price = 1.7557

The approximations based on delta are not very accurate. In general, the larger the move in the underlying, the worse the approximation. This will make delta hedging less effective.

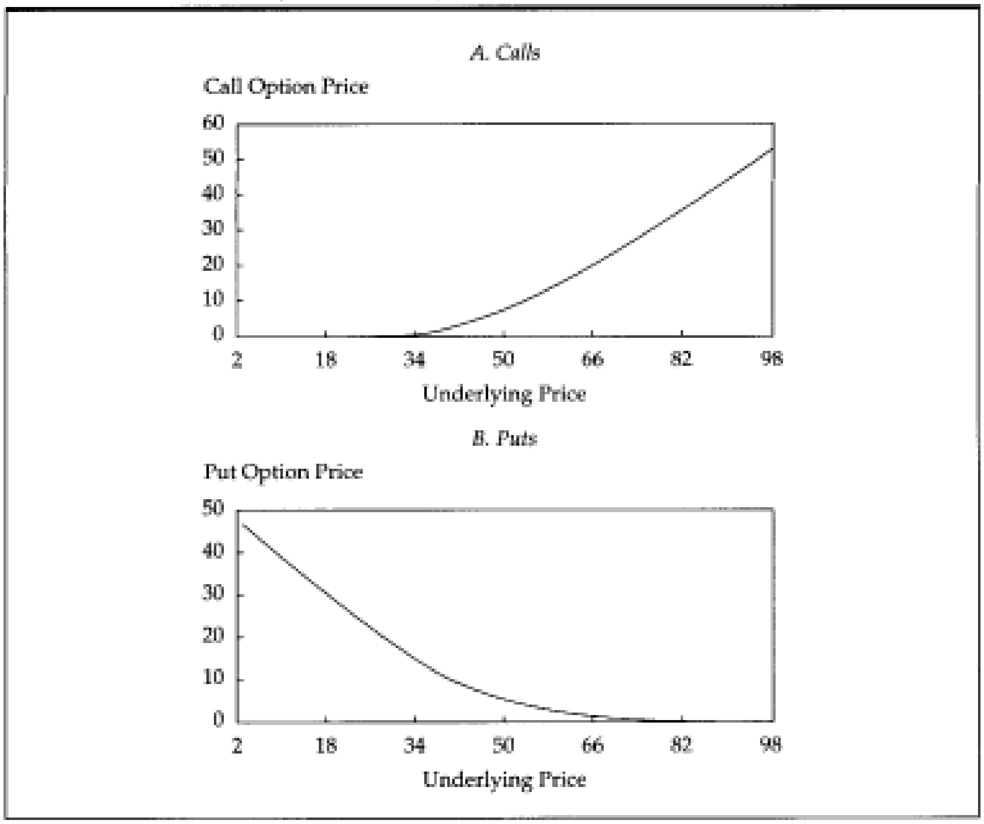
Exhibit 4-25 shows the relationship between the option price and the underlying price. Panel A depicts the relationship for calls and Panel B shows the corresponding relationship for puts. Notice the curvature in the relationship between the option price and the underlying price. Call option values definitely increase the greater the underlying value, and put

[31] This transaction would require 673($52.75) = $35,500, less the 1,000($8.619) = $8,619 received from the sale of the option, for a total investment required of $26,881. We would probably borrow this money.

[32] Remember that the put delta is negative; hence, its movement is down toward −1.0 or up toward 0.0.

option values definitely decrease. But the amount of change is not the same in each direction. $N(d_1)$ measures the slope of this line at a given point. As such, it measures only the slope for a very small change in the underlying. When the underlying changes by more than a very small amount, the curvature of the line comes into play and distorts the relationship between the option price and underlying price that is explained by the delta. The problem here is much like the relationship between a bond price and its yield. This first-order relationship between a bond price and its yield is called the duration; therefore, duration is similar to delta.

EXHIBIT 4-25 The Relationship between Option Price and Underlying Price $X = 50, r^c = 0.0488, T = 0.75, \sigma = 0.35$



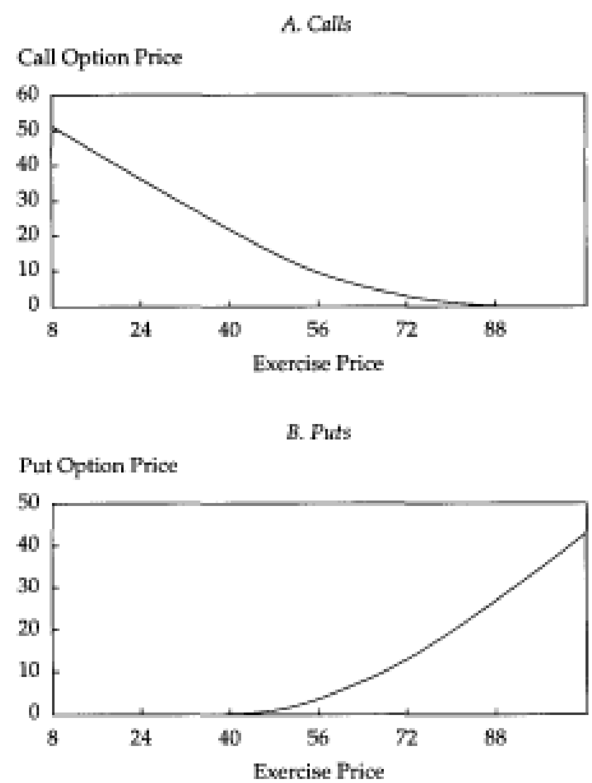
The curvature or second-order effect is known in the fixed income world as the convexity. In the options world, this effect is called **gamma**. Gamma is a numerical measure of how sensitive the delta is to a change in the underlying—in other words, how much the delta changes. When gamma is large, the delta changes rapidly and cannot provide a good approximation of how much the option moves for each unit of movement in the underlying. We shall not concern ourselves with measuring and using gamma, but we should know a few things about the gamma and, therefore, about the behavior of the delta.

Gamma is larger when there is more uncertainty about whether the option will expire in- or out-of-the-money. This means that *gamma will tend to be large when the option is at-the-money and close to expiration.* In turn, this statement means that delta will be a poor approximation for the option's price sensitivity when it is at-the-money and close to the expiration day. Thus, a delta hedge will work poorly. When the gamma is large, we may need to use a gamma-based hedge, which would require that we add a position in another option to the delta-hedge position of the underlying and the option. We shall not take up this advanced topic here.

7.3.2 The Exercise Price

The exercise price is easy to obtain. It is specified in the option contract and does not change. Therefore, it is not worthwhile to speak about what happens when the exercise price changes, but we can talk about how the option price would differ if we choose an option with a different exercise price. As we have previously seen, the call option price will be lower the higher the exercise price and the put option price will be higher. This relationship is confirmed for our sample option in Exhibit 4-26.

EXHIBIT 4-26 The Relationship between Option Price and Exercise Price
$S = 52.75$, $r^c = 0.0488$, $T = 0.75$, $\sigma = 0.35$



7.3.3 The Risk-Free Rate: Rho

The risk-free rate is the continuously compounded rate on the risk-free security whose maturity corresponds to the option's life. We have used the risk-free rate in previous chapters; sometimes we have used the discrete version and sometimes the continuous version. As we have noted, the continuously compounded risk-free rate is the natural log of 1 plus the discrete risk-free rate.

For example, suppose the discrete risk-free rate quoted in annual terms is 5 percent. Then the continuous rate is

$$r^c = \ln(1 + r) = \ln(1.05) = 0.0488$$

Let us recall the difference in these two specifications. Suppose we want to find the present value of \$1 in six months using both the discrete and continuous risk-free rates.

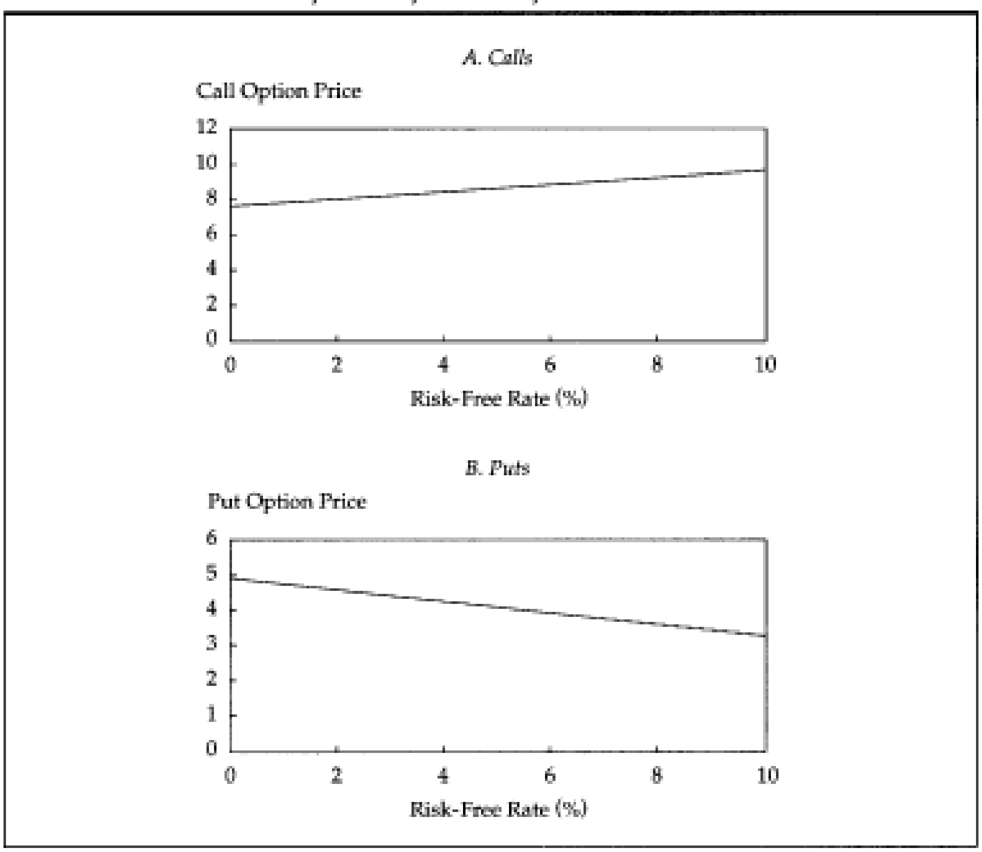
$$\text{Present value using discrete rate} = \frac{1}{(1 + r)^T} = \frac{1}{(1.05)^{0.5}} = 0.9759$$

$$\text{Present value using continuous rate} = e^{-r^cT} = e^{-0.0488(0.5)} = 0.9759$$

Obviously either specification will work. Because of how it uses the risk-free rate in the calculation of d_1, however, the Black–Scholes–Merton model requires the continuous risk-free rate.

The sensitivity of the option price to the risk-free rate is called the **rho**. We shall not concern ourselves with the calculation of rho. Technically, the Black–Scholes–Merton model assumes a constant risk-free rate, so it is meaningless to talk about the risk-free rate changing over the life of the option. We can, however, explore how the option price would differ if the current rate were different. Exhibit 4-27 depicts this effect. Note how little change occurs in the option price over a very broad range of the risk-free rate. Indeed, *the price of a European option on an asset is not very sensitive to the risk-free rate.*[33]

EXHIBIT 4-27 The Relationship between Option Price and Risk-Free Rate $S = 52.75$, $X = 50$, $T = 0.75$, $\sigma = 0.35$



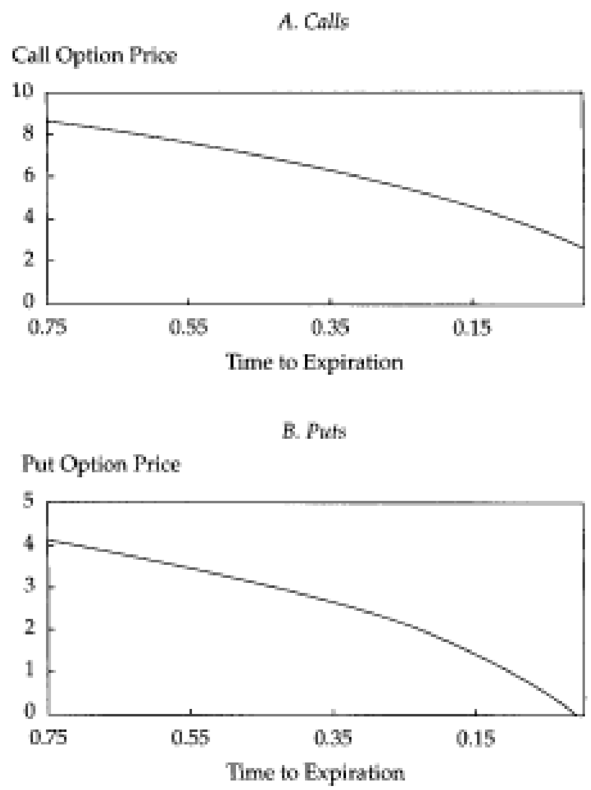
[33] When the underlying is an interest rate, however, there is a strong relationship between the option price and interest rates.

7.3.4 Time to Expiration: Theta

Time to expiration is an easy input to determine. An option has a definite expiration date specified in the contract. We simply count the number of days until expiration and divide by 365, as we have done previously with forward and futures contracts.

Obviously, the time remaining in an option's life moves constantly towards zero. Even if the underlying price is constant, the option price will still change. We noted that American options have both an intrinsic value and a time value. For European options, all of the price can be viewed as time value. In either case, time value is a function of the option's moneyness, its time to expiration, and its volatility. The more uncertainty there is, the greater the time value. As expiration approaches, the option price moves toward the payoff value of the option at expiration, a process known as **time value decay**. The rate at which the time value decays is called the option's **theta**. We shall not concern ourselves with calculating the specific value of theta, but be aware that if the option price decreases as time moves forward, the theta will be negative. Exhibit 4-28 shows the time value decay for our sample option.

EXHIBIT 4-28 The Relationship between Option Price and Time to Expiration $S = 52.75$, $X = 50$, $r^c = 0.0488$, $\sigma = 0.35$. T Starts at 0.75 and Goes toward 0.0



Note that both call and put values decrease as the time to expiration decreases. We previously noted that European put options do not necessarily do this. For some cases, European put options can increase in value as the time to expiration decreases, the case of a positive theta, but that is not so for our put.[34] *Most of the time, option prices are higher the longer the time to expiration. For European puts, however, some exceptions exist.*

7.3.5 Volatility: Vega

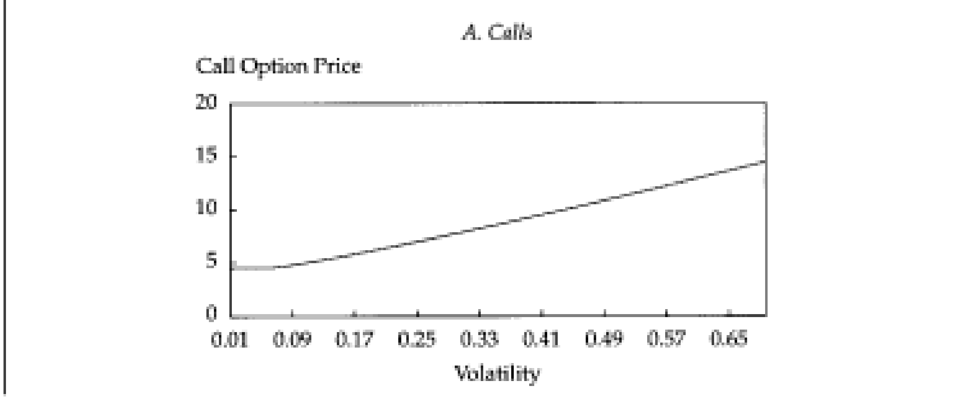
As we have previously noted, volatility is the standard deviation of the continuously compounded return on the stock. We have also noted that the volatility is an extremely important variable in the valuation of an option. It is the only variable that cannot be obtained easily and directly from another source. In addition, as we illustrate here, option prices are extremely sensitive to the volatility. We take up the subject of estimating volatility in Section 7.5.

The relationship between option price and volatility is called the **vega**, which—albeit considered an option Greek—is not actually a Greek word.[35] We shall not concern ourselves with the actual calculation of the vega, but know that the vega is positive for both calls and puts, meaning that if the volatility increases, both call and put prices increase. Also, the vega is larger the closer the option is to being at-the-money.

In the problem we previously worked (S_0 = \$52.75, X = \$50, r^c = 0.0488, T = 0.75), at a volatility of 0.35, the option price was 8.619. Suppose we erroneously use a volatility of 0.40. Then the call price would be 9.446. An error in the volatility of this magnitude would not be difficult to make, especially for a variable that is not directly observable. Yet the result is a very large error in the option price.

Exhibit 4-29 displays the relationship between the option price and the volatility. Note that this relationship is nearly linear and that the option price varies over a very wide range, although this near-linearity is not the case for all options.

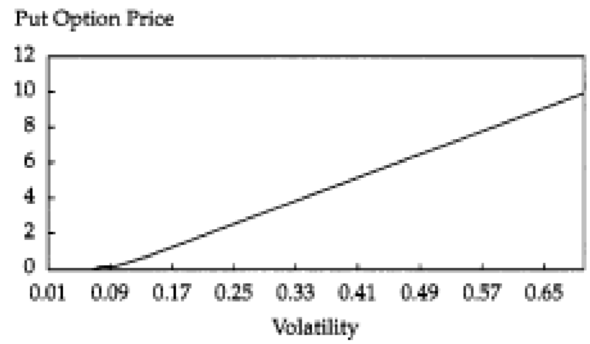
EXHIBIT 4-29 The Relationship between Option Price and Volatility $S = 52.75, X = 50, r^c = 0.0488, T = 0.75$



[34] Positive put thetas tend to occur when the put is deep in-the-money, the volatility is low, the interest rate is high, and the time to expiration is low.

[35] So that all of these effects ("the Greeks") be named after Greek words, the term *kappa* is sometimes used to represent the relationship between an option price and its volatility. As it turns out, however, vega is used far more often than kappa and is probably easier to remember, given the "v" in vega and the "v" in volatility. Vega, however, is a star, not a letter, and its origin is Latin.

B. Puts



7.4 The Effect of Cash Flows on the Underlying

As we saw in Chapters 2 and 3, cash flows on the underlying affect the prices of forward and futures contracts. It should follow that they would affect the prices of options. In studying the option boundary conditions and put–call parity earlier in this chapter, we noted that we subtract the present value of the dividends from the underlying price and use this adjusted price to obtain the boundary conditions or to price the options using put–call parity. We do the same using the Black–Scholes–Merton model. Specifically, we introduced the expression PV(CF,0,T) for the present value of the cash flows on the underlying over the life of the option. So, we simply use S_0 − PV(CF,0,T) in the Black–Scholes–Merton model instead of S_0.

Recall that in previous chapters, we also used continuous compounding to express the cash flows. For stocks, we used a continuously compounded dividend yield; for currencies, we used a continuously compounded interest rate. In the case of stocks, we let δ^c represent the continuously compounded dividend rate. Then we substituted for $S_0 e^{-\delta^c T}$ for S_0 in the Black–Scholes–Merton formula. For a foreign currency, we let S_0 represent the exchange rate, which we discount using r^{fc}, the continuously compounded foreign risk-free rate. Let us work an example involving a foreign currency option.

Let the exchange rate of U.S. dollars for euros be $0.8575. The continuously compounded U.S. risk-free rate, which in this example is r^c, is 5.10 percent. The continuously compounded euro risk-free rate, r^{fc}, is 4.25 percent. A call option expires in 125 days (T = 125/365 = 0.3425) and has an exercise price of $0.90. The volatility of the continuously compounded exchange rate is 0.055.

The first thing we do is obtain the adjusted price of the underlying: $0.8475e^{-0.0425(0.3425)} = 0.8353$. We then use this value as S_0 in the formula for d_1 and d_2:

$$d_1 = \frac{\ln(0.8353/0.90) + [0.051 + (0.055)^2/2](0.3425)}{0.055\sqrt{0.3425}} = -1.7590$$

$$d_2 = -1.7590 - 0.055\sqrt{0.3425} = -1.7912$$

Using the normal probability table, we find that

$$N(d_1) = N(-1.76) = 1 - 0.9608 = 0.0392$$

$$N(d_2) = N(-1.79) = 1 - 0.9633 = 0.0367$$

In discounting the exercise rate to evaluate the second term in the Black–Scholes–Merton expression, we use the domestic (here, U.S.) continuously compounded risk-free rate. The call option price would thus be

$$c = 0.8353(0.0392) - 0.90e^{-0.051(0.3425)}(0.0367) = 0.0003$$

Therefore, this call option on an asset worth \$0.8575 would cost \$0.0003.

PRACTICE PROBLEM 9

Use the Black–Scholes–Merton model adjusted for cash flows on the underlying to calculate the price of a call option in which the underlying is priced at 225, the exercise price is 200, the continuously compounded risk-free rate is 5.25 percent, the time to expiration is three years, and the volatility is 0.15. The effect of cash flows on the underlying is indicated below for two alternative approaches:

A. The present value of the cash flows over the life of the option is 19.72.

B. The continuously compounded dividend yield is 2.7 percent.

SOLUTIONS

A. Adjust the price of the underlying to $S_0 = 225 - 19.72 = 205.28$. Then insert into the Black–Scholes–Merton formula as follows:

$$d_1 = \frac{\ln(205.28/200) + [0.0525 + (0.15)^2/2]3.0}{0.15\sqrt{3.0}} = 0.8364$$

$$d_2 = 0.8364 - 0.15\sqrt{3.0} = 0.5766$$

$$N(0.84) = 0.7995$$

$$N(0.58) = 0.7190$$

$$c = 205.28(0.7995) - 200e^{-0.0525(3.0)}(0.7190) = 41.28$$

B. Adjust the price of the underlying to $S_0 = 225e^{-0.027(3.0)} = 207.49$

$$d_1 = \frac{\ln(207.49/200) + [0.0525 + (0.15)^2/2]3.0}{0.15\sqrt{3.0}} = 0.8776$$

$$d_2 = 0.8776 - 0.15\sqrt{3.0} = 0.6178$$

$$N(0.88) = 0.8106$$

$$N(0.62) = 0.7324$$

$$c = 207.49(0.8106) - 200e^{-0.0525(3.0)}(0.7324) = 43.06$$

7.5 The Critical Role of Volatility

As we have previously stressed, volatility is an extremely important variable in the pricing of options. In fact, with the possible exception of the cash flows on the underlying, it is the only variable that cannot be directly observed and easily obtained. It is, after all, the volatility over the life of the option; therefore, it is not past or current volatility but rather the future volatility. Differences in opinion on option prices nearly always result from differences of opinion about volatility. But how does one obtain a number for the future volatility?

7.5.1 Historical Volatility

The most logical starting place to look for an estimate of future volatility is past volatility. When the underlying is a publicly traded asset, we usually can collect some data over a recent past period and estimate the standard deviation of the continuously compounded return.

Exhibit 4-30 illustrates this process for a sample of 12 monthly prices of a particular stock. We convert these prices to returns, convert the returns to continuously compounded returns, find the variance of the series of continuously compounded returns, and then convert the variance to the standard deviation. In this example, the data are monthly returns, so we must annualize the variance by multiplying it by 12. Then we take the square root to obtain the historical estimate of the annual standard deviation or volatility.

EXHIBIT 4-30 Estimating Historical Volatility

Month	Price	Return	Log Return	(Log Return − Average)^2
0	100			
1	102	0.020000	0.019803	0.000123
2	99	−0.029412	−0.029853	0.001486
3	97	−0.020202	−0.020409	0.000847
4	89	−0.082474	−0.086075	0.008982
5	103	0.157303	0.146093	0.018878
6	104	0.009709	0.009662	0.000001
7	102	−0.019231	−0.019418	0.000790
8	99	−0.029412	−0.029853	0.001486
9	104	0.050505	0.049271	0.001646
10	102	−0.019231	−0.019418	0.000790
11	105	0.029412	0.028988	0.000412
12	111	0.057143	0.055570	0.002197
		Sum	0.104360	0.037639
		Average	0.008697	

The variance is estimated as follows:

$$\sigma^2 = \frac{\sum_{i=1}^{N} (R_i^c - \overline{R}^c)^2}{N - 1}$$

where R_i^c is the continuously compounded return for observation i (shown above in the fourth column and calculated as $\ln(1 + R_i)$, where i goes from 1 to 12) and R_i is the ith return, $\overline{R}^c$ is the average return over the entire sample, and N is the number of observations in the sample (here, N = 12). Then

$$\sigma^2 = \frac{0.037639}{11} = 0.003422$$

Because this sample consists of monthly returns, to obtain the annual variance, we must multiply this number by 12 (or 52 for weekly, or 250—the approximate number of trading days in a year—for daily). Thus

$$\sigma^2 = 12(0.003422) = 0.041064$$

The annual standard deviation or volatility is, therefore,

$$\sigma^2 = \sqrt{0.041064} = 0.2026$$

So the historical volatility estimate is 20.26 percent.

The historical estimate of the volatility is based only on what happened in the past. To get the best estimate, we must use a lot of prices, but that means going back farther in time. The farther back we go, the less current the data become, and the less reliable our estimate of the volatility. We now look at a way of obtaining a more current estimate of the volatility, but one that raises questions as well as answers them.

7.5.2 Implied Volatility

In a market in which options are traded actively, we can reasonably assume that the market price of the option is an accurate reflection of its true value. Thus, by setting the Black–Scholes–Merton price equal to the market price, we can work backwards to infer the volatility. This procedure enables us to determine the volatility that option traders are using to price the option. This volatility is called the **implied volatility**.

Unfortunately, determining implied volatility is not a simple task. We cannot simply solve the Black–Scholes–Merton equation for the volatility. It is a complicated function with the volatility appearing several times, in some cases as σ^2. There are some mathematical techniques that speed up the estimation of the implied volatility. Here, however, we shall look at only the most basic method: trial and error.

Recall the option we have been working with. The underlying price is 52.75, the exercise price is 50, the risk-free rate is 4.88 percent, and the time to expiration is 0.75. In our previous examples, the volatility was 0.35. Using these values in the Black–Scholes–Merton model, we obtained a call option price of 8.619. Suppose we observe the option selling in the market for 9.25. What volatility would produce this price?

We have already calculated a price of 8.619 at a volatility of 0.35. Because the call price varies directly with the volatility, we know that it would take a volatility greater than 0.35 to produce a price higher than 8.619. We do not know how much higher, so we should just take a guess. Let us try a volatility of 0.40. Using the Black–Scholes–Merton formula with a volatility of 0.40, we obtain a price of 9.446. This is too high, so we try a lower volatility. We keep doing this in the following manner:

Volatility	Black–Scholes–Merton Price
0.35	8.619
0.40	9.446
0.39	9.280
0.38	9.114

So now we know that the correct volatility lies between 0.38 and 0.39, closer to 0.39. In solving for the implied volatility, we must decide either how close to the option price we want to be or how many significant digits we want in the implied volatility. If we choose four significant digits in the implied volatility, a value of 0.3882 would produce the option price of 9.2500. Alternatively, if we decide that we want to be within 0.01 of the option price, we would find that the implied volatility is in the range of 38.76 to 38.88 percent.

Thus, if the option is selling for about 9.25, we say that the market is pricing it at a volatility of 0.3882. This number represents the market's best estimate of the true volatility of the option; it can be viewed as a more current source of volatility information than the past volatility. Unfortunately, a circularity exists in the argument. If one uses the Black–Scholes–Merton model to determine if an option is over- or underpriced, the procedure for extracting the implied volatility assumes that the market correctly prices the option. The only way to use the implied volatility in identifying mispriced options is to interpret the implied volatility as either too high or too low, which would require an estimate of true volatility. Nonetheless, the implied volatility is a source of valuable information on the uncertainty in the underlying, and option traders use it routinely.

All of this material on continuous-time option pricing has been focused on options in which the underlying is an asset. As we described earlier in this chapter, there are also options on futures. Let us take a look at the pricing of options on futures, which will pave the way for a continuous-time pricing model for options on interest rates, another case in which the underlying is not an asset.

8 PRICING OPTIONS ON FORWARD AND FUTURES CONTRACTS AND AN APPLICATION TO INTEREST RATE OPTION PRICING

Earlier in this chapter, we discussed how options on futures contracts are active, exchange-traded options in which the underlying is a futures contract. In addition, there are over-the-counter options in which the underlying is a forward contract. In our treatment of these instruments, we assume constant interest rates. As we learned in Chapters 2 and 3, this assumption means that futures and forward contracts will have the same prices. European options on futures and forward contracts will, therefore, have the same prices. American options on forwards will differ in price from American options on futures, and we discuss this later.

First we take a quick look at the basic rules that we previously developed for options on underlying assets. If the underlying asset is a futures contract, the payoff values of the options at expiration are

$$c_T = \text{Max}[0, f_T(T) - X] \qquad \textbf{(4-24)}$$
$$p_T = \text{Max}[0, X - f_T(T)]$$

where $f_T(T)$ is the price of a futures contract at T in which the contract expires at T. Thus, $f_T(T)$ is the futures price at expiration. These formulas are, of course, the same as for options when the underlying is an asset, with the futures price substituted for the asset price. When the option and the futures expire simultaneously, the futures price at expiration, $f_T(T)$, converges to the asset price, S_T, making the above payoffs precisely the same as those of the option on the underlying asset.

The minimum and maximum values for options on forwards or futures are the same as those we obtained for options on assets, substituting the futures price for the asset price. Specifically,

$$0 \le c_0 \le f_0(T)$$
$$0 \le C_0 \le f_0(T) \qquad \textbf{(4-25)}$$
$$0 \le p_0 \le X/(1 + r)^T$$
$$0 \le P_0 \le X$$

We also established lower bounds for European options and intrinsic values for American options and used these results to establish the minimum prices of these options. For options on futures, the lower bounds are

$$c_0 \geq \text{Max}\{0,[f_0(T) - X]/(1 + r)^T\} \quad \textbf{(4-26)}$$
$$p_0 \geq \text{Max}\{0,[X - f_0(T)]/(1 + r)^T\}$$

where $f_0(T)$ is the price at time 0 of a futures contract expiring at T. Therefore, the price of a European call or put on the futures is either zero or the difference between the futures price and exercise price, as formulated above, discounted to the present. For American options on futures, early exercise is possible. Thus, we express their lowest prices as the intrinsic values:

$$C_0 \geq \text{Max}[0,f_0(T) - X] \quad \textbf{(4-27)}$$
$$P_0 \geq \text{Max}[0,X - f_0(T)]$$

Because these values are greater than the lower bounds, we maintain these values as the minimum prices of American calls.[36]

As we have previously pointed out, with the assumption of constant interest rates, futures prices and forward prices are the same. We can treat European options on futures the same way as options on forwards. American options on futures will differ from American options on forwards. Now we explore how put–call parity works for options on forwards.

8.1 Put–Call Parity for Options on Forwards

In an earlier section, we examined put–call parity. Now we take a look at the parity between puts and calls on forward contracts and their underlying forward contracts. First recall the notation: F(0,T) is the price established at time 0 for a forward contract expiring at time T. Let c_0 and p_0 be the prices today of calls and puts on the forward contract. We shall assume that the puts and calls expire when the forward contract expires. The exercise price of the options is X. The payoff of the call is $\text{Max}(0,S_T - X)$, and the payoff of the put is $\text{Max}(0,X - S_T)$.[37] We construct a combination consisting of a long call and a long position in a zero-coupon bond with face value of X − F(0,T). We construct another combination consisting of a long position in a put and a long position in a forward contract. Exhibit 4-31 shows the results.

EXHIBIT 4-31 Portfolio Combinations for Equivalent Packages of Puts, Calls, and Forward Contracts (Put–Call Parity for Forward Contracts)

Transaction	Current Value	Value at Expiration $S_T \leq X$	Value at Expiration $S_T > X$
Call and Bond			
Buy call	c_0	0	$S_T - X$
Buy bond	$[X - F(0,T)]/(1 + r)^T$	$X - F(0,T)$	$X - F(0,T)$
Total	$c_0 + [X - F(0,T)]/(1 + r)^T$	$X - F(0,T)$	$S_T - F(0,T)$

[36] In other words, we cannot use the European lower bound as the lowest price of an American call or put, as we could with calls when the underlying was an asset instead of a futures.

[37] Recall that the option payoffs are given by the underlying price at expiration, because the forward contract expires when the option expires. Therefore, the forward price at expiration is the underlying price at expiration.

Put and Forward			
Buy put	p_0	$X - S_T$	0
Buy forward contract	0	$S_T - F(0,T)$	$S_T - F(0,T)$
Total	p_0	$X - F(0,T)$	$S_T - F(0,T)$

As the exhibit demonstrates, both combinations produce a payoff of either $X - F(0,T)$ or $S_T - F(0,T)$, whichever is greater. The call and bond combination is thus equivalent to the put and forward contract combination. Hence, to prevent an arbitrage opportunity, the initial values of these combinations must be the same. The initial value of the call and bond combination is $c_0 + [X - F(0,T)]/(1 + r)^T$. The forward contract has zero initial value, so the initial value of the put and forward contract combination is only the initial value of the put, p_0. Therefore,

$$c_0 + [X - F(0,T)]/(1 + r)^T = p_0 \qquad \textbf{(4-28)}$$

This equation is **put–call parity for options on forward contracts.**

Note that we seem to have implied that the bond is a long position, but that might not be the case. The bond should have a face value of $X - F(0,T)$. We learned in Chapter 2 that $F(0,T)$ is determined in the market as the underlying price compounded at the risk-free rate.[38] Because there are a variety of options with different exercise prices, any one of which could be chosen, it is clearly possible for X to exceed or be less than $F(0,T)$. If $X > F(0,T)$, we are long the bond, because the payoff of $X - F(0,T)$ is greater than zero, meaning that we get back money from the bond. If $X < F(0,T)$, we issue the bond, because the payoff of $X - F(0,T)$ is less than zero, meaning that we must pay back money. Note the special case when $X = F(0,T)$. The bond is effectively out of the picture. Then $c_0 = p_0$.

Now recall from Chapter 2 that with discrete interest compounding and no storage costs, the forward price is the spot price compounded at the risk-free rate. So,

$$F(0,T) = S_0(1 + r)^T$$

If we substitute this result for $F(0,T)$ in the put–call parity equation for options on forwards, we obtain

$$p_0 + S_0 = c_0 + X/(1 + r)^T$$

which is the put–call parity equation for options on the underlying that we learned earlier in this chapter. Indeed, put–call parity for options on forwards and put–call parity for options on the underlying asset are the same. The only difference is that in the former, the forward contract and the bond replace the underlying. Given the equivalence of options on the forward contract and options on the underlying, we can refer to put–call parity for options on forwards as **put–call–forward parity**. The equation

$$c_0 + [X - F(0,T)]/(1 + r)^T = p_0$$

expresses the relationship between the forward price and the prices of the options on the underlying asset, or alternatively between the forward price and the prices of options on the forward contract. We can also rearrange the equation to isolate the forward price and obtain

$$F(0,T) = (c_0 - p_0)(1 + r)^T + X$$

[38] We are, of course, assuming no cash flows or costs on the underlying asset.

which shows how the forward price is related to the put and call prices and to the exercise price.

Now observe in Exhibit 4-32 how a synthetic forward contract can be created out of options.

EXHIBIT 4-32 Forward Contract and Synthetic Forward Contract

		Value at Expiration	
Transaction	Current Value	$S_T \le X$	$S_T > X$
Forward Contract			
Long forward contract	0	$S_T - F(0,T)$	$S_T - F(0,T)$
Synthetic Forward Contract			
Buy call	c_0	0	$S_T - X$
Sell put	$-p_0$	$-(X - S_T)$	0
Buy (or sell) bond	$[X - F(0,T)]/(1 + r)^T$	$X - F(0,T)$	$X - F(0,T)$
Total	$c_0 - p_0 + [X - F(0,T)]/(1 + r)^T$	$S_T - F(0,T)$	$S_T - F(0,T)$

In the top half of the exhibit is a forward contract. Its payoff at expiration is $S_T - F(0,T)$. In the bottom half of the exhibit is a **synthetic forward contract**, which consists of a long call, a short put, and a long risk-free bond with a face value equal to the exercise price minus the forward price. Note that this bond can actually be short if the exercise price of these options is lower than the forward price. The forward contract and synthetic forward contract have the same payoffs, so their initial values must be equal. The initial value of the forward contract is zero, so the initial value of the synthetic forward contract must be zero. Thus,

$$c_0 - p_0 + [X - F(0,T)]/(1 + r)^T = 0$$

Solving for F(0,T), we obtain the equation for the forward price in terms of the call, put, and bond that was given previously. So a synthetic forward contract is a combination consisting of a long call, a short put, and a zero-coupon bond with face value of $X - F(0,T)$.

Consider the following example: The options and a forward contract expire in 50 days, so $T = 50/365 = 0.1370$. The risk-free rate is 6 percent, and the exercise price is 95. The call price is 5.50, the put price is 10.50, and the forward price is 90.72. Substituting in the above equation, we obtain

$$5.50 - 10.50 + \frac{95 - 90.72}{(1.06)^{0.1370}} = -0.7540$$

which is supposed to be zero. The left-hand side replicates a forward contract. Thus, the synthetic forward is underpriced. We buy it and sell the actual forward contract. So if we buy the call, sell the put, and buy the bond with face value $95 - 90.72 = 4.28$, we bring in 0.7540. At expiration, the payoffs are as follows.

The options and forward expire with the underlying above 95:

The bond matures and pays off $95 - 90.72 = 4.28$.

Exercise the call, paying 95 and obtaining the underlying.

Deliver the underlying and receive 90.72 from the forward contract.

The put expires with no value.

Net effect: No money in or out.

The options and forward expire with the underlying at or below 95:

The bond matures and pays off 95 − 90.72 = 4.28.

Buy the underlying for 95 with the short put.

Deliver the underlying and receive 90.72 from the forward contract.

The call expires with no value.

Net effect: No money in or out.

So we take in 0.7540 up front and never have to pay anything out. The pressure of other investors doing this will cause the call price to increase and the put price to decrease until the above equation equals zero or is at least equal to the transaction costs that would be incurred to exploit any discrepancy from zero.

Similarly, an option can be created from a forward contract. If a long forward contract is equivalent to a long call, short put, and zero-coupon bond with face value of X − F(0,T), then a long call is a long forward, long put, and a zero-coupon bond with face value of F(0,T) − X. A long put is a long call, short forward, and a bond with face value of X − F(0,T). These results are obtained just by rearranging what we learned here about forwards and options.

These results hold strictly for European options; some additional considerations exist for American options, but we do not cover them here.

PRACTICE PROBLEM 10

Determine if a forward contract is correctly priced by using put–call–forward parity. The option exercise price is 90, the risk-free rate is 5 percent, the options and the forward contract expire in two years, the call price is 15.25, the put price is 3.00, and the forward price is 101.43.

SOLUTION

First note that the time to expiration is T = 2.0. There are many ways to express put–call–forward parity. We use the following specification:

$$p_0 = c_0 + [X - F(0,T)]/(1 + r)^T$$

The right-hand side is the synthetic put and consists of a long call, a short forward contract, and a bond with face value of X − F(0,T). Substituting the values into the right-hand side, we obtain

$$p_0 = 15.25 + (90 - 101.43)/(1.05)^{2.0} = 4.88$$

Because the actual put is selling for 3.00, it is underpriced. So we should buy the put and sell the synthetic put. To sell the synthetic put we should sell the call, buy the forward contract, and hold a bond with face value F(0,T) − X. Doing so will generate the following cash flow up front:

Buy put: −3.00

Sell call: +15.25

Buy bond:	$-(101.43 - 90)/(1.05)^{2.0} = -10.37$
Total:	$+1.88$

Thus the transaction brings in 1.88 up front. The payoffs at expiration are

	$S_T < 90$	$S_T \geq 90$
Long put	$90 - S_T$	0
Short call	0	$-(S_T - 90)$
Long bond	$101.43 - 90$	$101.43 - 90$
Long forward	$S_T - 101.43$	$S_T - 101.43$
Total	0	0

Therefore, no money flows in or out at expiration.

8.2 Early Exercise of American Options on Forward and Futures Contracts

As we noted earlier, the holder of an American put option may want to exercise it early. For American call options on underlying assets that make no cash payments, however, there is no justification for exercising the option early. If the underlying asset makes a cash payment, such as a dividend on a stock or interest on a bond, it may be justifiable to exercise the call option early.

For American options on futures, it may be worthwhile to exercise both calls and puts early. Even though early exercise is never justified for American calls on underlying assets that make no cash payments, early exercise can be justified for American call options on futures. Deep-in-the money American call options on futures behave almost identically to the underlying, but the investor has money tied up in the call. If the holder exercises the call and establishes a futures position, he earns interest on the futures margin account. A similar argument holds for deep-in-the-money American put options on futures. The determination of the timing of early exercise is a specialist topic so we do not explore it here.

If the option is on a forward contract instead of a futures contract, however, these arguments are overshadowed by the fact that a forward contract does not pay off until expiration, in contrast to the mark-to-market procedure of futures contracts. Thus, if one exercised either a call or a put on a forward contract early, doing so would only establish a long or short position in a forward contract. This position would not pay any cash until expiration. No justification exists for exercising early if one cannot generate any cash from the exercise. Therefore, an American call on a forward contract is the same as a European call on a forward contract, but American calls on futures are different from European calls on futures and carry higher prices.

8.3 The Black Model

The usual model for pricing European options on futures is called the Black model, named after Fischer Black of Black–Scholes–Merton fame. The formula is

$$c = e^{-r^c T}[f_0(T)N(d_1) - XN(d_2)]$$
$$p = e^{-r^c T}(X[1 - N(d_2)] - f_0(T)[1 - N(d_1)])$$

where

$$d_1 = \frac{\ln(f_0(T)/X) + (\sigma^2/2)T}{\sigma\sqrt{T}}$$

$d_2 = d_1 - \sigma\sqrt{T}$

$f_0(T)$ = the futures price

and the other terms are those we have previously used. The volatility, σ, is the volatility of the continuously compounded change in the futures price.[39]

Although the Black model may appear to give a somewhat different formula, it can be obtained directly from the Black–Scholes–Merton formula. Recall that the futures price in terms of the underlying spot price would be $f_0(T) = S_0e^{r^cT}$. If we substitute the right-hand-side for $f_0(T)$ in the Black formula for d_1, we obtain the Black–Scholes–Merton formula for d_1.[40] Then if we substitute the right-hand side of the above for $f_0(T)$ in the Black formula for c_0 and p_0, we obtain the Black–Scholes–Merton formula for c_0 and p_0. These substitutions should make sense: The prices of options on futures equal the prices of options on the asset when the options and futures expire simultaneously.

The procedure should be straightforward if you have mastered substituting the asset price and other inputs into the Black–Scholes–Merton formula. Also, note that as with the Black–Scholes–Merton formula, the formula applies only to European options. As we noted in the previous section, early exercise of American options on futures is often justified, so we cannot get away with using this formula for American options on futures. We can, however, use the formula for American options on forwards, because they are never exercised early.

PRACTICE PROBLEM 11

The price of a forward contract is 139.19. A European option on the forward contract expires in 215 days. The exercise price is 125. The continuously compounded risk-free rate is 4.25 percent. The volatility is 0.15.

A. Use the Black model to determine the price of the call option.

B. Determine the price of the underlying from the above information and use the Black–Scholes–Merton model to show that the price of an option on the underlying is the same as the price of the option on the forward.

SOLUTIONS

The time to expiration is T = 215/365 = 0.5890.

A. First find d_1 and d_2, then $N(d_1)$ and $N(d_2)$, and then the call price:

$$d_1 = \frac{\ln(139.19/125) + [(0.15)^2/2]0.5890}{0.15\sqrt{0.5890}} = 0.9916$$

$$d_2 = 0.9916 - 0.15\sqrt{0.5890} = 0.8765$$

[39] If we were using the model to price options on forward contracts, we would insert F(0,T), the forward price, instead of the futures price. Doing so would produce some confusion because we have never subscripted the forward price, arguing that it does not change. Therefore, although we could use the formula to price options on forwards at time 0, how could we use the formula to price options on forwards at a later time, say time t, prior to expiration? In that case, we would have to use the price of a newly constructed forward contract that expires at T, F(t,T). Of course, with constant interest rates, these forward prices, F(0,T) and F(t,T), would be identical to the analogous futures price, $f_0(T)$ and $f_t(T)$. So, for ease of exposition we use the futures price.

[40] This action requires us to recognize that $\ln(S_0e^{r^cT}/X) = \ln(S_0/X) + r^cT$.

$N(0.99) = 0.8389$

$N(0.88) = 0.8106$

$$c = e^{-0.0425(0.5890)}[139.19(0.8389) - 125(0.8106)] = 15.06$$

B. We learned in Chapter 2 that if there are no cash flows on the underlying and the interest is compounded continuously, the forward price is given by the formula $F(0, T) = S_0e^{r^cT}$. We can thus find the spot price as

$$S_0 = F(0,T)e^{-r^cT} = 139.19e^{-0.0425(0.5890)} = 135.75$$

Then we simply use the Black–Scholes–Merton formula:

$$d_1 = \frac{\ln(135.75/125) + (0.0425 + (0.15)^2/2)(0.5890)}{0.15\sqrt{0.5890}} = 0.9916$$

$$d_2 = 0.9916 - 0.15\sqrt{0.5890} = 0.8765$$

These are the same values as in Part A, so $N(d_1)$ and $N(d_2)$ will be the same. Plugging into the formula for the call price gives

$$c = 135.75(0.8389) - 125e^{-0.0425(0.5890)}(0.8106) = 15.06$$

This price is the same as in Part A.

8.4 Application of the Black Model to Interest Rate Options

Earlier in this chapter, we described options on interest rates. These derivative instruments parallel the FRAs that we covered in Chapter 2, in that they are derivatives in which the underlying is not a bond but rather an interest rate. Pricing options on interest rates is a challenging task. We showed how this is done using binomial trees. It would be nice if the Black–Scholes–Merton model could be easily used to price interest rate options, but the process is not so straightforward. Pricing options on interest rates requires a sophisticated model that prohibits arbitrage among interest-rate related instruments and their derivatives. The Black–Scholes–Merton model is not sufficiently general to use in this manner. Nonetheless, practitioners often employ the Black model to price interest rate options. Somewhat remarkably, perhaps, it is known to give satisfactory results. Therefore, we provide a quick overview of this practice here.

Suppose we wish to price a one-year interest rate cap, consisting of three caplets. One caplet expires in 90 days, one 180 days, and one in 270 days.[41] The exercise rate is 9 percent. To use the Black model, we use the forward rate as though it were $f_0(T)$.

[41] A one-year cap will have three individual caplets. The first expires in 90 days and pays off in 180 days, the second expires in 180 days and pays off in 270 days, and the third expires in 270 days and pays off in 360 days. The tendency to think that a one-year cap using quarterly periods should have four caplets is incorrect because there is no caplet expiring right now and paying off in 90 days. It would make no sense to create an option that expires immediately. Also, in a one-year loan, the rate is set at the start and reset only three times; hence, only three caplets are required.

Therefore, we also require its volatility and the risk-free rate for the period to the option's expiration.[42] Recalling that there are three caplets and we have to price each one individually, let us first focus on the caplet expiring in 90 days. We first specify that T = 90/365 = 0.2466. Then we need the forward rate today for the period day 90 to day 180. Let this rate be 9.25 percent. We shall assume its volatility is 0.03. We then need the continuously compounded risk-free rate for 90 days, which we assume to be 9.60 percent. So now we have the following input variables:

$$T = 0.2466$$

$$f_0(T) = 0.0925$$

$$\sigma = 0.03$$

$$X = 0.09$$

$$r^c = 0.096$$

Inserting these inputs into the Black model produces

$$d_1 = \frac{\ln(0.0925/0.09) + [(0.03)^2/2](0.2466)}{0.03\sqrt{0.2466}} = 1.8466$$

$$d_2 = 1.8466 - 0.03\sqrt{0.2466} = 1.8317$$

$$N(1.85) = 0.9678$$

$$N(1.83) = 0.9664$$

$$c_0 = e^{-0.096(0.2466)}[0.0925(0.9678) - 0.09(0.9664)] = 0.00248594$$

(Because of the order of magnitude of the inputs, we carry the answer out to eight decimal places.) But this answer is not quite what we need. The formula gives the answer under the assumption that the option payoff occurs at the option expiration. As we know, interest rate options pay off later than their expirations. This option expires in 90 days and pays off 90 days after that. Therefore, we need to discount this result back from day 180 to day 90 using the forward rate of 9.25 percent.[43] We thus have

$$0.00248594e^{-0.0925(0.2466)} = 0.00242988$$

Another adjustment is necessary. Because the underlying price and exercise price are entered as rates, the resulting answer is a rate. Moreover, the underlying rate and exercise rate are expressed as annual rates, so the answer is an annual rate. Interest rate option

[42] It is important to note here that the Black model requires that all inputs be in continuous compounding format. Therefore, the forward rate and risk-free rate would need to be the continuously compounded analogs to the discrete rates. Because the underlying is usually LIBOR, which is a discrete rate quoted on the basis of a 360-day year, some adjustments must be made to convert to a continuous rate quoted on the basis of a 365-day year. We will not address these adjustments here.

[43] Be very careful in this discounting procedure. The exponent in the exponential should have a time factor of the number of days between the option expiration and its payoff. Because there are 90 days between days 90 and 180, we use 90/365 = 0.2466. This value is not quite the same as the time until the option expiration, which today is 90 but which will count down to zero.

prices are always quoted as periodic rates (which are prices for $1 notional principal). We would adjust this rate by multiplying by 90/360.[44] The price would thus be

$$0.00242988(90/360) = 0.00060747$$

Finally, we should note that this price is valid for a $1 notional principal option. If the notional principal were $1 million, the option price would be

$$\$1{,}000{,}000(0.00060747) = \$607.47$$

We have just priced the first caplet of this cap. To price the second caplet, we need the forward rate for the period 180 days to 270 days, we would use 180/365 = 0.4932 as the time to expiration, and we need the risk-free rate for 180 days. To price the third caplet, we need the forward rate for the period 270 days to 360 days, we would use 270/365 = 0.7397 as the time to expiration, and we need the risk-free rate for 270 days. The price of the cap would be the sum of the prices of the three component caplets. If we were pricing a floor, we would price the component floorlets using the Black model for puts.

Although the Black model is frequently used to price interest rate options, binomial models, as we illustrated earlier, are somewhat more widely used in this area. These models are more attuned to deriving prices that prohibit arbitrage opportunities using any of the diverse instruments whose prices are given by the term structure. When you use the Black model to price interest rate options, there is some risk, perhaps minor, of having a counterparty be able to do arbitrage against you. Yet somehow the Black model is used often, and professionals seem to agree that it works remarkably well.

PRACTICE PROBLEM 12

Use the Black model to price an interest rate put that expires in 280 days. The forward rate is currently 6.8 percent, the 280-day continuously compounded risk-free rate is 6.25 percent, the exercise rate is 7 percent, and the volatility is 0.02. The option is based on a 180-day underlying rate, and the notional principal is $10 million.

SOLUTION

The time to expiration is T = 280/365 = 0.7671. Calculate the value of d_1, d_2, and $N(d_1)$, $N(d_2)$, and p_0 using the Black model:

$$d_1 = \frac{\ln(0.068/0.07) + [(0.02)^2/2]0.7671}{0.02\sqrt{0.7671}} = 1.6461$$

$$d_2 = -1.6461 - 0.02\sqrt{0.7671} = -1.6636$$

$$N(-1.65) = 1 - N(1.65) = 1 - 0.9505 = 0.0495$$

$$N(-1.66) = 1 - N(1.66) = 1 - 0.9515 = 0.0485$$

$$p_0 = e^{-0.0625(0.7671)}[0.07(1 - 0.0485) - 0.068(1 - 0.0495)] = 0.00187873$$

[44] It is customary in the interest rate options market to use 360 in the denominator to make this adjustment, even though we have used 365 in other places.

This formula assumes the option payoff is made at expiration. For an interest rate option, that assumption is false. This is a 180-day rate, so the payoff is made 180 days later. Therefore, we discount the payoff over 180 days using the forward rate:

$$e^{-0.068(180/365)}(0.00187873) = 0.00181677$$

Interest rate option prices must reflect the fact that the rate used in the formula is quoted as an annual rate. So, we must multiply by 180/360 because the transaction is based on a 180-day rate:

$$0.00181677(180/360) = 0.00090839$$

Then we multiply by the notional principal:

$$\$10{,}000{,}000(0.00090839) = \$9{,}084$$

9 THE ROLE OF OPTIONS MARKETS

As we did with futures markets, we conclude the chapter by looking at the important role options markets play in the financial system. Recall from Chapter 1 that we looked at the purposes of derivative markets. We noted that derivative markets provide price discovery and risk management, make the markets for the underlying assets more efficient, and permit trading at low transaction costs. These features are also associated with options markets. Yet, options offer further advantages that some other derivatives do not offer.

For example, forward and futures contracts have bidirectional payoffs. They have the potential for a substantial gain in one direction and a substantial loss in the other direction. The advantage of taking such a position lies in the fact that one need pay no cash up front. In contrast, options offer the feature that, if one is willing to pay cash up front, one can limit the loss in a given direction. In other words, options have unidirectional payoffs. This feature can be attractive to the holder of an option. To the writer, options offer the opportunity to be paid cash up front for a willingness to assume the risk of the unidirectional payoff. An option writer can assume the risk of potentially a large loss unmatched by the potential for a large gain. In fact, the potential gain is small. But for this risk, the option writer receives money up front.

Options also offer excellent devices for managing the risk of various exposures. An obvious one is the protective put, which we saw earlier and which can protect a position against loss by paying off when the value of the underlying is down. We shall see this and other such applications in Chapter 7.

Recall that futures contracts offer price discovery, the revelation of the prices at which investors will contract today for transactions to take place later. Options, on the other hand, provide volatility discovery. Through the implied volatility, investors can determine the market's assessment of how volatile it believes the underlying asset is. This valuable information can be difficult to obtain from any other source.

Futures offer advantages over forwards, in that futures are standardized, tend to be actively traded in a secondary market, and are protected by the exchange's clearinghouse against credit risk. Although some options, such as interest rate options, are available only in over-the-counter forms, many options exist in both over-the-counter and exchange-listed forms. Hence, one can often customize an option if necessary or trade it on an exchange.

In Chapter 2, we covered forward contracts; in Chapter 3, we covered futures contracts; and in this chapter we covered option contracts. We have one more major class of derivative instruments, swaps, which we now turn to in Chapter 5. We shall return to options in Chapter 7, where we explore option trading strategies.

KEY POINTS

- Options are rights to buy or sell an underlying at a fixed price, the exercise price, for a period of time. The right to buy is a call; the right to sell is a put. Options have a definite expiration date. Using the option to buy or sell is the action of exercising it. The buyer or holder of an option pays a price to the seller or writer for the right to buy (a call) or sell (a put) the underlying instrument. The writer of an option has the corresponding potential obligation to sell or buy the underlying.
- European options can be exercised only at expiration; American options can be exercised at any time prior to expiration. Moneyness refers to the characteristic that an option has positive intrinsic value. The payoff is the value of the option at expiration. An option's intrinsic value is the value that can be captured if the option is exercised. Time value is the component of an option's price that reflects the uncertainty of what will happen in the future to the price of the underlying.
- Options can be traded as standardized instruments on an options exchange, where they are protected from default on the part of the writer, or as customized instruments on the over-the-counter market, where they are subject to the possibility of the writer defaulting. Because the buyer pays a price at the start and does not have to do anything else, the buyer cannot default.
- The underlying instruments for options are individual stocks, stock indices, bonds, interest rates, currencies, futures, commodities, and even such random factors as the weather. In addition, a class of options called real options is associated with the flexibility in capital investment projects.
- Like FRAs, which are forward contracts in which the underlying is an interest rate, interest rate options are options in which the underlying is an interest rate. However, FRAs are commitments to make one interest payment and receive another, whereas interest rate options are rights to make one interest payment and receive another.
- Option payoffs, which are the values of options when they expire, are determined by the greater of zero or the difference between underlying price and exercise price, if a call, or the greater of zero or the difference between exercise price and underlying price, if a put. For interest rate options, the exercise price is a specified rate and the underlying price is a variable interest rate.
- Interest rate options exist in the form of caps, which are call options on interest rates, and floors, which are put options on interest rates. Caps consist of a series of call options, called caplets, on an underlying rate, with each option expiring at a different time. Floors consist of a series of put options, called floorlets, on an underlying rate, with each option expiring at a different time.
- The minimum value of European and American calls and puts is zero. The maximum value of European and American calls is the underlying price. The maximum value of a European put is the present value of the exercise price. The maximum value of an American put is the exercise price.

- The lower bound of a European call is established by constructing a portfolio consisting of a long call and risk-free bond and a short position in the underlying asset. This combination produces a non-negative value at expiration, so its current value must be non-negative. For this situation to occur, the call price has to be worth at least the underlying price minus the present value of the exercise price. The lower bound of a European put is established by constructing a portfolio consisting of a long put, a long position in the underlying, and the issuance of a zero-coupon bond. This combination produces a non-negative value at expiration so its current value must be non-negative. For this to occur, the put price has to be at least as much as the present value of the exercise price minus the underlying price. For both calls and puts, if this lower bound is negative, we invoke the rule that an option price can be no lower than zero.
- The lowest price of a European call is referred to as the lower bound. The lowest price of an American call is also the lower bound of a European call. The lowest price of a European put is also referred to as the lower bound. The lowest price of an American put, however, is its intrinsic value.
- Buying a call with a given exercise price and selling an otherwise identical call with a higher exercise price creates a combination that always pays off with a non-negative value. Therefore, its current value must be non-negative. For this to occur, the call with the lower exercise price must be worth at least as much as the other call. A similar argument holds for puts, except that one would buy the put with the higher exercise price. This line of reasoning shows that the put with the higher exercise price must be worth at least as much as the one with the lower exercise price.
- A longer-term European or American call must be worth at least as much as a corresponding shorter-term European or American call. A longer-term American put must be worth at least as much as a shorter-term American put. A longer-term European put, however, can be worth more or less than a shorter-term European put.
- A fiduciary call, consisting of a European call and a zero-coupon bond, produces the same payoff as a protective put, consisting of the underlying and a European put. Therefore, their current values must be the same. For this equivalence to occur, the call price plus bond price must equal the underlying price plus put price. This relationship is called put–call parity and can be used to identify combinations of instruments that synthesize another instrument by rearranging the equation to isolate the instrument you are trying to create. Long positions are indicated by positive signs, and short positions are indicated by negative signs. One can create a synthetic call, a synthetic put, a synthetic underlying, and a synthetic bond, as well as synthetic short positions in these instruments for the purpose of exploiting mispricing in these instruments.
- Put–callparity violations exist when one side of the equation does not equal the other. An arbitrageur buys the lower-priced side and sells the higher-priced side, thereby earning the difference in price, and the positions offset at expiration. The combined actions of many arbitrageurs performing this set of transactions would increase the demand and price for the underpriced instruments and decrease the demand and price for the overpriced instruments, until the put–call parity relationship is upheld.
- American option prices must always be no less than those of otherwise equivalent European options. American call options, however, are never exercised early unless there is a cash flow on the underlying, so they can sell for the same as their European counterparts in the absence of such a cash flow. American put options nearly always have a possibility of early exercise, so they ordinarily sell for more than their European counterparts.

- Cash flows on the underlying affect an option's boundary conditions and put–call parity by lowering the underlying price by the present value of the cash flows over the life of the option.
- A higher interest rate increases a call option's price and decreases a put option's price.
- In a one-period binomial model, the underlying asset can move up to one of two prices. A portfolio consisting of a long position in the underlying and a short position in a call option can be made risk-free and, therefore, must return the risk-free rate. Under this condition, the option price can be obtained by inferring it from a formula that uses the other input values. The option price is a weighted average of the two option prices at expiration, discounted back one period at the risk-free rate.
- If an option is trading for a price higher than that given in the binomial model, one can sell the option and buy a specific number of units of the underlying, as given by the model. This combination is risk free but will earn a return higher than the risk-free rate. If the option is trading for a price lower than the price given in the binomial model, a short position in a specific number of units of the underlying and a long position in the option will create a risk-free loan that costs less than the risk-free rate.
- In a two-period binomial model, the underlying can move to one of two prices in each of two periods; thus three underlying prices are possible at the option expiration. To price an option, start at the expiration and work backward, following the procedure in the one-period model in which an option price at any given point in time is a weighted average of the next two possible prices discounted at the risk-free rate.
- To calculate the price of an option on a zero-coupon bond or a coupon bond, one must first construct a binomial tree of the price of the bond over the life of the option. To calculate the price of an option on an interest rate, one should use a binomial tree of interest rates. Then the option price is found by starting at the option expiration, determining the payoff and successively working backwards by computing the option price as the weighted average of the next two option prices discounted back one period. For the case of options on bonds or interest rates, a different discount rate is used at different parts of the tree.
- For an option of a given expiration, a greater pricing accuracy is obtained by dividing the option's life into a greater number of time periods in a binomial tree. As more time periods are added, the discrete-time binomial price converges to a stable value as though the option is being modeled in a continuous-time world.
- The assumptions under which the Black–Scholes–Merton model is derived state that the underlying asset follows a geometric lognormal diffusion process, the risk-free rate is known and constant, the volatility of the underlying asset is known and constant, there are no taxes or transaction costs, there are no cash flows on the underlying, and the options are European.
- To calculate the value of an option using the Black–Scholes–Merton model, enter the underlying price, exercise price, risk-free rate, volatility, and time to expiration into a formula. The formula will require you to look up two normal probabilities, obtained from either a table or preferably a computer routine.
- The change in the option price for a change in the price of the underlying is called the delta. The change in the option price for a change in the risk-free rate is called the rho. The change in the option price for a change in the time to expiration is called the theta. The change in the option price for a change in the volatility is called the vega.
- The delta is defined as the change in the option price divided by the change in the underlying price. The option price change can be approximated by the delta times the

change in the underlying price. To construct a delta-hedged position, a short (long) position in each option is matched with a long (short) position in delta units of the underlying. Changes in the underlying price will generate offsetting changes in the value of the option position, provided the changes in the underlying price are small and occur over a short time period. A delta-hedged position should be adjusted as the delta changes and time passes.

- If changes in the price of the underlying are large or the delta hedge is not adjusted over a longer time period, the hedge may not be effective. This effect is due to the instability of the delta and is called the gamma effect. If the gamma effect is large, option price changes will not be very close to the changes as approximated by the delta times the underlying price change.
- Cash flows on the underlying are accommodated in option pricing models by reducing the price of the underlying by the present value of the cash flows over the life of the option.
- Volatility can be estimated by calculating the standard deviation of the continuously compounded returns from a sample of recent data for the underlying. This is called the historical volatility. An alternative measure, called the implied volatility, can be obtained by setting the Black–Scholes–Merton model price equal to the market price and inferring the volatility. The implied volatility is a measure of the volatility the market is using to price the option.
- The payoffs of a call on a forward contract and an appropriately chosen zero-coupon bond are equivalent to the payoffs of a put on the forward contract and the forward contract. Thus, their current values must be the same. For this equality to occur, the call price plus the bond price must equal the put price. The appropriate zero-coupon bond is one with a face value equal to the exercise price minus the forward price. This relationship is called put–call–forward (or futures) parity.
- There is no justification for exercising American options on forward contracts early, so they are equivalent to European options on forwards. American options on futures, both calls and puts, can sometimes be exercised early, so they are different from European options on futures and carry a higher price.
- The Black model can be used to price European options on forwards or futures by entering the forward price, exercise price, risk-free rate, time to expiration, and volatility into a formula that will also require the determination of two normal probabilities.
- The Black model can be used to price European options on interest rates by entering the forward interest rate into the model for the forward or futures price and the exercise rate for the exercise price.
- Options are useful in financial markets because they provide a way to limit losses to the premium paid while permitting potentially large gains. They can be used for hedging purposes, especially in the case of puts, which can be used to limit the loss on a long position in an asset. Options also provide information on the volatility of the underlying asset. Options can be standardized and exchange-traded or customized in the over-the-counter market.

APPENDIX 4A Cumulative Probabilities for a Standard Normal Distribution
$P(X \leq x) = N(x)$ for $x \geq 0$ or $1 - N(-x)$ for $x < 0$

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

PROBLEMS

1. A. Calculate the payoff at expiration for a call option on the S&P 100 stock index in which the underlying price is 579.32 at expiration, the multiplier is 100, and the exercise price is
   i. 450
   ii. 650

   B. Calculate the payoff at expiration for a put option on the S&P 100 in which the underlying is at 579.32 at expiration, the multiplier is 100, and the exercise price is
   i. 450
   ii. 650

2. A. Calculate the payoff at expiration for a call option on a bond in which the underlying is at \$0.95 per \$1 par at expiration, the contract is on \$100,000 face value bonds, and the exercise price is
   i. \$0.85
   ii. \$1.15

   B. Calculate the payoff at expiration for a put option on a bond in which the underlying is at \$0.95 per \$1 par at expiration, the contract is on \$100,000 face value bonds, and the exercise price is
   i. \$0.85
   ii. \$1.15

3. A. Calculate the payoff at expiration for a call option on an interest rate in which the underlying is a 180-day interest rate at 6.53 percent at expiration, the notional principal is \$10 million, and the exercise price is
   i. 5 percent
   ii. 8 percent

   B. Calculate the payoff at expiration for a put option on an interest rate in which the underlying is a 180-day interest rate at 6.53 percent at expiration, the notional principal is \$10 million, and the exercise price is
   i. 5 percent
   ii. 8 percent

4. A. Calculate the payoff at expiration for a call option on the British pound in which the underlying is at \$1.438 at expiration, the options are on 125,000 British pounds, and the exercise price is
   i. \$1.35
   ii. \$1.55

   B. Calculate the payoff at expiration for a put option on the British pound where the underlying is at \$1.438 at expiration, the options are on 125,000 British pounds, and the exercise price is
   i. \$1.35
   ii. \$1.55

5. A. Calculate the payoff at expiration for a call option on a futures contract in which the underlying is at 1136.76 at expiration, the options are on a futures contract for \$1,000, and the exercise price is
   i. 1130
   ii. 1140

   B. Calculate the payoff at expiration for a put option on a futures contract in which the underlying is at 1136.76 at expiration, the options are on a futures contract for \$1000, and the exercise price is
   i. 1130
   ii. 1140

6. Consider a stock index option that expires in 75 days. The stock index is currently at 1240.89 and makes no cash payments during the life of the option. Assume that the stock index has a multiplier of 1. The risk-free rate is 3 percent.
   - **A.** Calculate the lowest and highest possible prices for European-style call options on the above stock index with exercise prices of
     - **i.** 1225
     - **ii.** 1255
   - **B.** Calculate the lowest and highest possible prices for European-style put options on the above stock index with exercise prices of
     - **i.** 1225
     - **ii.** 1255
7. **A.** Consider American-style call and put options on a bond. The options expire in 60 days. The bond is currently at $1.05 per $1 par and makes no cash payments during the life of the option. The risk-free rate is 5.5 percent. Assume that the contract is on $1 face value bonds. Calculate the lowest and highest possible prices for the calls and puts with exercise prices of
     - **i.** $0.95
     - **ii.** $1.10
   - **B.** Consider European style call and put options on a bond. The options expire in 60 days. The bond is currently at $1.05 per $1 par and makes no cash payments during the life of the option. The risk-free rate is 5.5 percent. Assume that the contract is on $1 face value bonds. Calculate the lowest and highest possible prices for the calls and puts with exercise prices of
     - **i.** $0.95
     - **ii.** $1.10
8. You are provided with the following information on put and call options on a stock:
   Call price, c_0 = $6.64
   Put price, p_0 = $2.75
   Exercise price, X = $30
   Days to option expiration = 219
   Current stock price, S_0 = $33.19
   Put–call parity shows the equivalence of a call/bond portfolio (fiduciary call) and a put/underlying portfolio (protective put). Illustrate put–call parity assuming stock prices at expiration (S_T) of $20 and of $40. Assume that the risk-free rate, r, is 4 percent.
9. Consider the following information on put and call options on a stock:
   Call price, c_0 = $4.50
   Put price, p_0 = $6.80
   Exercise price, X = $70
   Days to option expiration = 139
   Current stock price, S_0 = $67.32
   Risk-free rate, r = 5 percent
   - **A.** Use put–call parity to calculate prices of the following:
     - **i.** Synthetic call option
     - **ii.** Synthetic put option
     - **iii.** Synthetic bond
     - **iv.** Synthetic underlying stock
   - **B.** For each of the synthetic instruments in Part A, identify any mispricing by comparing the actual price with the synthetic price.
   - **C.** Based on the mispricing in Part B, illustrate an arbitrage transaction using a synthetic call.

D. Based on the mispricing in Part B, illustrate an arbitrage transaction using a synthetic put.

10. A stock currently trades at a price of \$100. The stock price can go up 10 percent or down 15 percent. The risk-free rate is 6.5 percent.

A. Use a one-period binomial model to calculate the price of a call option with an exercise price of \$90.

B. Suppose the call price is currently \$17.50. Show how to execute an arbitrage transaction that will earn more than the risk-free rate. Use 100 call options.

C. Suppose the call price is currently \$14. Show how to execute an arbitrage transaction that replicates a loan that will earn less than the risk-free rate. Use 100 call options.

11. Suppose a stock currently trades at a price of \$150. The stock price can go up 33 percent or down 15 percent. The risk-free rate is 4.5 percent.

A. Use a one-period binomial model to calculate the price of a put option with exercise price of \$150.

B. Suppose the put price is currently \$14. Show how to execute an arbitrage transaction that will earn more than the risk-free rate. Use 10,000 put options.

C. Suppose the put price is currently \$11. Show how to execute an arbitrage transaction that will earn more than the risk-free rate. Use 10,000 put options.

12. Consider a two-period binomial model in which a stock currently trades at a price of \$65. The stock price can go up 20 percent or down 17 percent each period. The risk-free rate is 5 percent.

A. Calculate the price of a call option expiring in two periods with an exercise price of \$60.

B. Based on your answer in Part A, calculate the number of units of the underlying stock that would be needed at each point in the binomial tree to construct a risk-free hedge. Use 10,000 calls.

C. Calculate the price of a call option expiring in two periods with an exercise price of \$70.

D. Based on your answer in Part C, calculate the number of units of the underlying stock that would be needed at each point in the binomial tree to construct a risk-free hedge. Use 10,000 calls.

13. Consider a two-period binomial model in which a stock currently trades at a price of \$65. The stock price can go up 20 percent or down 17 percent each period. The risk-free rate is 5 percent.

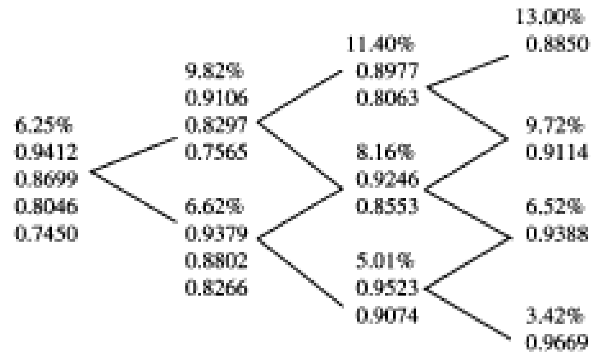
A. Calculate the price of a put option expiring in two periods with exercise price of \$60.

B. Based on your answer in Part A, calculate the number of units of the underlying stock that would be needed at each point in the binomial tree in order to construct a risk-free hedge. Use 10,000 puts.

C. Calculate the price of a put option expiring in two periods with an exercise price of \$70.

D. Based on your answer in Part C, calculate the number of units of the underlying stock that would be needed at each point in the binomial tree in order to construct a risk-free hedge. Use 10,000 puts.

14. The three-period binomial interest rate tree provided below gives one-period interest rates and prices of zero-coupon bonds. Starting at t = 0, you are provided with the one-period interest rate and prices of zero-coupon bonds with maturities of one period, two periods, three periods, and four periods. At t = 1, you are provided with the one-period interest rate and prices of zero-coupon bonds with maturities of one period, two periods, and three periods. At $t = 2$, you are provided with the one-period interest rate and prices of zero-coupon bonds with maturities of one period and two periods. At $t = 3$, you are provided with the one-period interest rate and prices of zero-coupon bonds with maturity of one period.



A. Calculate the value of a European call option on a four-period zero-coupon bond. The call option expires in two periods and has an exercise price of 0.85.

B. Calculate the value of a European call option on a four-period 8 percent coupon bond with a 1.0 face value. The call option expires in two periods and has an exercise price of 0.95.

C. Calculate the value of a European put option on a four-period zero-coupon bond. The call option expires in two periods and has an exercise price of 0.85.

D. Calculate the value of a European put option on a four-period 8 percent coupon bond with a 1.0 face value. The put option expires in two periods and has an exercise price of 1.

15. Assume that the evolution of one-period spot rates is still the same as in Problem 14; that is, the same three-period binomial interest rate tree still applies.

A. Calculate the price of a two-period cap with an exercise rate of 8 percent. The underlying is the one-period rate.

B. Calculate the price of a two-period floor with an exercise rate of 9 percent. The underlying is the one-period rate.

16. Call and put options on an asset are available with an exercise price of \$30. The options expire in 75 days, and the volatility is 0.40. The continuously compounded risk-free rate is 3.5 percent, and there are no cash flows on the underlying. The precise Black–Scholes–Merton values of these options at different underlying prices are as follows.

Asset Price	Call Price	Put Price
\$28	1.3144	3.0994
\$29	1.7538	2.5388
\$33	4.2270	1.0120

A. From the Black–Scholes–Merton model, obtain the approximate values of the call delta and put delta if the underlying asset price is $28.

B. Using the call delta and put delta obtained in Part A and the call and put prices given in the problem for the asset price of $28, calculate the approximate new call and put prices for a
   i. $1 increase in the price of the underlying asset
   ii. $5 increase in the price of the underlying asset

C. Based on a comparison of your answers in Part B with the actual call and put prices given in the problem, what can you say about the approximations based on delta?

17. Consider an asset that trades at $100 today. Call and put options on this asset are available with an exercise price of $100. The options expire in 275 days, and the volatility is 0.45. The continuously compounded risk-free rate is 3 percent.

A. Calculate the value of European call and put options using the Black–Scholes–Merton model. Assume that the present value of cash flows on the underlying asset over the life of the options is $4.25.

B. Calculate the value of European call and put options using the Black–Scholes–Merton model. Assume that the continuously compounded dividend yield is 1.5 percent.

18. Consider the following information on put and call options on an asset:

Call price, c_0 = $3.50
Put price, p_0 = $9
Exercise price, X = $50
Forward price, F(0,T) = $45
Days to option expiration = 175
Risk-free rate, r = 4 percent

A. Use put–call–forward parity to calculate prices of the following:
   i. Synthetic call option
   ii. Synthetic put option
   iii. Synthetic forward contract

B. For each of the synthetic instruments in Part A, identify any mispricing by comparing the actual price with the synthetic price.

C. Based on the mispricing in Part B, illustrate how to earn a risk-free profit using a synthetic call.

D. Based on the mispricing in Part B, illustrate how to earn a risk-free profit using a synthetic put.

19. A forward contract is priced at 145. European options on the forward contract have an exercise price of 150 and expire in 65 days. The continuously compounded risk-free rate is 3.75 percent, and volatility is 0.33.

A. Calculate the price of a call option on the forward contract using the Black model.

B. Calculate the price of the underlying asset. Calculate the price of a call option on the underlying asset using the Black–Scholes–Merton model. Compare your answer here with the answer in Part A.

C. Calculate the price of a put option on the forward contract using the Black model.

D. Now calculate the price of a put option on the underlying asset using the Black–Scholes–Merton model. Compare your answer here with the answer in Part C.

20. A. An interest rate call option based on a 90-day underlying rate has an exercise rate of 7.5 percent and expires in 180 days. The forward rate is 7.25 percent, and the volatility is 0.04. The continuously compounded risk-free rate is 5 percent. Calculate the price of the interest rate call option using the Black model.

B. An interest rate put option based on a 90-day underlying rate has an exercise rate of 7.5 percent and expires in 180 days. The forward rate is 7.25 percent, and the volatility is 0.04. The continuously compounded risk-free rate is 5 percent. Calculate the price of the interest rate put option using the Black model.

SOLUTIONS

1. A. $S_T = 579.32$
   i. Call payoff, X = 450: Max(0,579.32 − 450) × 100 = $12,932
   ii. Call payoff, X = 650: Max(0,579.32 − 650) × 100 = 0

   B. $S_T = 579.32$
   i. Put payoff, X = 450: Max(0,450 − 579.32) × 100 = 0
   ii. Put payoff, X = 650: Max(0,650 − 579.32) × 100 = $7,068

2. A. $S_T = \$0.95$
   i. Call payoff, X = 0.85: Max(0,0.95 − 0.85) × 100,000 = $10,000
   ii. Call payoff, X = 1.15: Max(0,0.95 − 1.15) × 100,000 = $0

   B. $S_T = \$0.95$
   i. Put payoff, X = 0.85: Max(0,0.85 − 0.95) × 100,000 = $0
   ii. Put payoff, X = 1.15: Max(0,1.15 − 0.95) × 100,000 = $20,000

3. A. $S_T = 0.0653$
   i. Call payoff, X = 0.05: Max(0,0.0653 − 0.05) × (180/360) × 10,000,000 = $76,500
   ii. Call payoff, X = 0.08: Max(0,0.0653 − 0.08) × (180/360) × 10,000,000 = 0

   B. $S_T = 0.0653$
   i. Put payoff, X = 0.05: Max(0,0.05 − 0.0653) × (180/360) × 10,000,000 = 0
   ii. Put payoff, X = 0.08: Max(0,0.08 − 0.0653) × (180/360) × 10,000,000 = $73,500

4. A. $S_T = \$1.438$
   i. Call payoff, X = 1.35: Max(0,1.438 − 1.35) × 125,000 = $11,000
   ii. Call payoff, X = 1.55: Max(0,1.438 − 1.55) × 125,000 = $0

   B. $S_T = \$1.438$
   i. Put payoff, X = 1.35: Max(0,1.35 − 1.438) × 125,000 = $0
   ii. Put payoff, X = 1.55: Max(0,1.55 − 1.438) × 125,000 = $14,000

5. A. $S_T = 1136.76$
   i. Call payoff, X = 1130: Max(0,1136.76 − 1130) × 1,000 = $6,760
   ii. Call payoff, X = 1140: Max(0,1136.76 − 1140) × 1,000 = 0

   B. $S_T = 1136.76$
   i. Put payoff, X = 1130: Max(0,1130 − 1136.76) × 1,000 = 0
   ii. Put payoff, X = 1140: Max(0,1140 − 1136.76) × 1,000 = $3,240

6. A. $S_0 = 1240.89$, T = 75/365 = 0.2055, X = 1225 or 1255, call options
   i. X = 1225
   Maximum value for the call: $c_0 = S_0 = 1240.89$
   Lower bound for the call: $c_0 = \text{Max}[0,1240.89 - 1225/(1.03)^{0.2055}] = 23.31$
   ii. X = 1255
   Maximum value for the call: $c_0 = S_0 = 1240.89$
   Lower bound for the call: $c_0 = \text{Max}[0,1240.89 - 1255/(1.03)^{0.2055}] = 0$

   B. $S_0 = 1240.89$, T = 75/365 = 0.2055, X = 1225 or 1255, put options
   i. X = 1225
   Maximum value for the put: $p_0 = 1225/(1.03)^{0.2055} = 1217.58$
   Lower bound for the put: $p_0 = \text{Max}[0,1225/(1.03)^{0.2055} - 1240.89] = 0$
   ii. X = 1255
   Maximum value for the put: $p_0 = 1255/(1.03)^{0.2055} = 1247.40$
   Lower bound for the put: $p_0 = \text{Max}[0,1255/(1.03)^{0.2055} - 1240.89] = 6.51$

7. A. $S_0 = 1.05$, T = 60/365 = 0.1644, X = 0.95 or 1.10, American-style options
   i. X = $0.95

Maximum value for the call: $C_0 = S_0 = \$1.05$
Lower bound for the call: $C_0 = \text{Max}[0, 1.05 - 0.95/(1.055)^{0.1644}] = \$0.11$
Maximum value for the put: $P_0 = X = \$0.95$
Lower bound for the put: $P_0 = \text{Max}(0, 0.95 - 1.05) = \$0$

ii. $X = \$1.10$
Maximum value for the call: $C_0 = S_0 = \$1.05$
Lower bound for the call: $C_0 = \text{Max}[0, 1.05 - 1.10/(1.055)^{0.1644}] = \$0$
Maximum value for the put: $P_0 = X = \$1.10$
Lower bound for the put: $P_0 = \text{Max}(0, 1.10 - 1.05) = \$0.05$

B. $S_0 = 1.05$, $T = 60/365 = 0.1644$, $X = 0.95$ or 1.10, European-style options

i. $X = \$0.95$
Maximum value for the call: $c_0 = S_0 = \$1.05$
Lower bound for the call: $c_0 = \text{Max}[0, 1.05 - 0.95/(1.055)^{0.1644}] = \$0.11$
Maximum value for the put: $p_0 = 0.95/(1.055)^{0.1644} = \$0.94$
Lower bound for the put: $p_0 = \text{Max}[0, 0.95/(1.055)^{0.1644} - 1.05] = \$0$

ii. $X = \$1.10$
Maximum value for the call: $c_0 = S_0 = \$1.05$
Lower bound for the call: $c_0 = \text{Max}[0, 1.05 - 1.10/(1.055)^{0.1644}] = \$0$
Maximum value for the put: $p_0 = 1.10/(1.055)^{0.1644} = \$1.09$
Lower bound for the put: $p_0 = \text{Max}[0, 1.10/(1.055)^{0.1644} - 1.05] = \$0.04$

8. We can illustrate put–call parity by showing that for the fiduciary call and the protective put, the current values and values at expiration are the same.
Call price, $c_0 = \$6.64$
Put price, $p_0 = \$2.75$
Exercise price, $X = \$30$
Risk-free rate, $r = 4$ percent
Time to expiration $= 219/365 = 0.6$
Current stock price, $S_0 = \$33.19$
Bond price, $X/(1 + r)^T = 30/(1 + 0.04)^{0.6} = \$29.3$

		Value at Expiration	
Transaction	Current Value	$S_T = 20$	$S_T = 40$
Fiduciary call			
Buy call	6.64	0	40 − 30 = 10
Buy bond	29.30	30	30
Total	35.94	30	40
Protective put			
Buy put	2.75	30 − 20 = 10	0
Buy stock	33.19	20	40
Total	35.94	30	40

The values in the table above show that the current values and values at expiration for the fiduciary call and the protective put are the same. That is, $c_0 + X/(1 + r)^T = p_0 + S_0$.

9. Call price, $c_0 = \$4.50$
Put price, $p_0 = \$6.80$
Exercise price, $X = \$70$
Risk-free rate, $r = 5$ percent

Time to expiration = 139/365 = 0.3808
Current stock price, S_0 = \$67.32
Bond price = $X/(1 + r)^T = 70/(1 + 0.05)^{0.3808}$ = \$68.71

A. Synthetic call = $p_0 + S_0 - X/(1 + r)^T = 6.8 + 67.32 - 68.71$ = \$5.41
Synthetic put = $c_0 + X/(1 + r)^T - S_0 = 4.5 + 68.71 - 67.32$ = \$5.89
Synthetic bond = $p_0 + S_0 - c_0 = 6.8 + 67.32 - 4.5$ = \$69.62
Synthetic underlying = $c_0 + X/(1 + r)^T - p_0 = 4.5 + 68.71 - 6.8$ = \$66.41

B.

Instrument	Actual Price	Synthetic Price	Mispricing/Profit
Call	4.50	5.41	0.91
Put	6.80	5.89	0.91
Bond	68.71	69.62	0.91
Stock	67.32	66.41	0.91

Thus, the mispricing is the same regardless of the instrument used to look at it.

C. The actual call is cheaper than the synthetic call. Therefore, an arbitrage transaction where you buy the call (underpriced) and sell the synthetic call (overpriced) will yield a risk-free profit of \$5.41 − \$4.50 = \$0.91.

As shown below, at expiration no cash will be received or paid out.

	Value at Expiration	
Transaction	$S_T < 70$	$S_T > 70$
Buy call	0	$S_T - 70$
Sell synthetic call		
Short put	$-(70 - S_T)$	0
Short stock	$-S_T$	$-S_T$
Long bond	70	70
Total	0	0

D. The actual put is more expensive than the synthetic put. Therefore, an arbitrage transaction in which you buy the synthetic put (underpriced) and sell the put (overpriced) will yield a risk-free profit of \$6.80 − \$5.89 = \$0.91. As shown below, at expiration no cash will be received or paid out.

	Value at Expiration	
Transaction	$S_T < 70$	$S_T > 70$
Sell put	$-(70 - S_T)$	0
Buy synthetic put		
Long call	0	$S_T - 70$
Long bond	70	70
Short stock	$-S_T$	$-S_T$
Total	0	0

10. Current stock price, S = \$100
Up move, u = 1.1
Down move, d = 0.85
Exercise price, X = \$90
Risk-free rate, r = 6.5 percent

A. Stock prices one period from now are

$$S^+ = Su = 100(1.1) = \$110$$

$$S^- = Sd = 100(0.85) = \$85$$

Call option values at expiration one period from now are

$$c^+ = \text{Max}(0,110 - 90) = \$20$$

$$c^- = \text{Max}(0,85 - 90) = \$0$$

The risk-neutral probability is

$$\pi = \frac{1.065 - 0.85}{1.1 - 0.85} = 0.86 \text{ and } 1 - \pi = 0.14$$

The call price today is

$$c = \frac{0.86(20) + 0.14(0)}{1.065} = 16.15$$

B. If the current call price is \$17.50, it is overpriced. Therefore, we should sell the call and buy the underlying stock. The hedge ratio is

$$n = \frac{20 - 0}{110 - 85} = 0.8$$

For every option sold we should purchase 0.8 shares of stock. If we sell 100 calls we should buy 80 shares of stock.

Sell 100 calls at 17.50 =	1,750
Buy 80 shares at 100 =	−8,000
Net cash flow =	−6,250

At expiration the value of this combination will be

$$80(110) - 100(20) = \$6,800 \text{ if } S_T = 110$$

$$80(85) - 100(0) = \$6,800 \text{ if } S_T = 85$$

We invested \$6,250 for a payoff of \$6,800. The rate of return is (6,800/6,250) − 1 = 0.088. This rate is higher than the risk-free rate of 0.065.

C. If the current call price is \$14, it is underpriced. Therefore, we should buy the call and sell the underlying stock. The hedge ratio is

$$n = \frac{20 - 0}{110 - 85} = 0.8$$

For every option purchased we should sell 0.8 shares of stock. If we buy 100 calls we should sell 80 shares of stock.

Buy 100 calls at 14 =	−1,400
Sell 80 shares at 100 =	8,000
Net cash flow =	6,600

Thus, we generate \$6,600 up front.

At expiration the value of this combination will be

$$100(20) - 80(110) = -\$6{,}800 \text{ if } S_T = 110$$

$$100(0) - 80(85) = -\$6{,}800 \text{ if } S_T = 85$$

We generated \$6,600 up front and pay back \$6,800. The rate of return is (6,800/6,600) − 1 = 0.0303. This borrowing rate is lower than the risk-free rate of 0.065.

11. Current stock price, S = \$150
Up move, u = 1.33
Down move, d = 0.85
Exercise price, X = \$150
Risk-free rate, r = 4.5 percent

A. Stock prices one period from now are

$$S^+ = Su = 150(1.33) = \$199.5$$

$$S^- = Sd = 150(0.85) = \$127.5$$

Put option values at expiration one period from now are

$$p^+ = \text{Max}(0, 150 - 199.5) = \$0$$

$$p^- = \text{Max}(0, 150 - 127.5) = \$22.5$$

The risk-neutral probability is

$$\pi = \frac{1.045 - 0.85}{1.33 - 0.85} = 0.4063, \text{ and } 1 - \pi = 0.5937$$

The put price today is

$$\pi = \frac{0.4063(0) + 0.5937(22.50)}{1.045} = 12.78$$

B. If the current put price is \$14, it is overpriced. In order to create a hedge portfolio, we should sell the put and short the underlying stock. The hedge ratio is

$$n = \frac{p^+ - p^-}{S^+ - S^-} = \frac{0 - 22.5}{199.5 - 127.5} = -0.3125$$

For every option sold, we should sell 0.3125 shares of stock. If we sell 10,000 puts, we should sell 3,125 shares of stock.

Sell 10,000 puts at 14	= 140,000
Sell 3,125 shares at 150	= 468,750
Net cash flow	= 608,750

Thus, we generate \$608,750 up front.

At expiration, the value of this combination will be

$$-3{,}125(199.5) - 10{,}000(0) = -\$623{,}437 \text{ if } S_T = \$199.5$$

$$-3{,}125(127.5) - 10{,}000(22.5) = -\$623{,}437 \text{ if } S_T = \$127.5$$

We generated \$608,750 up front and pay back \$623,437. The rate of return is

$$\frac{623{,}437}{608{,}750} - 1 = 0.0241$$

This borrowing rate is lower than the risk-free rate of 0.045.

C. If the current put price is \$11, it is underpriced. In order to create a hedge portfolio, we should buy the put and buy the underlying stock. The hedge ratio is

$$n = \frac{p^+ - p^-}{S^+ - S^-} = \frac{0 - 22.5}{199.5 - 127.5} = -0.3125$$

For every option purchased we should buy 0.3125 shares of stock. If we buy 10,000 puts we should buy 3,125 shares of stock.

Buy 10,000 puts at 11	= −110,000
Buy 3,125 shares at 150	= −<u>468,750</u>
Net cash flow	= −578,750

That is, we invest \$578,750.

At expiration, the value of this combination will be

$$3{,}125(199.5) + 10{,}000(0) = \$623{,}437 \text{ if } S_T = \$199.5$$

$$3{,}125(127.5) + 10{,}000(22.5) = \$623{,}437 \text{ if } S_T = \$127.5$$

We invested \$578,750 for a payoff of \$623,437. The rate of return is $\frac{623{,}437}{578{,}750} - 1 =$ 0.0772. This rate is higher than the risk-free rate of 0.045.

12. Current stock price, S = \$65
Up move, u = 1.20
Down move, d = 0.83
Risk-free rate, r = 5 percent

A. Exercise price, X = \$60

Stock prices in the binomial tree one and two periods from now are

$$S^+ = Su = 65(1.20) = \$78$$

$$S^- = Sd = 65(0.83) = \$53.95$$

$$S^{++} = Su^2 = 65(1.20)(1.20) = \$93.60$$

$$S^{+-} = Sud = 65(1.20)(0.83) = \$64.74$$

$$S^{--} = Sd^2 = 65(0.83)(0.83) = \$44.78$$

Call option values at expiration two periods from now are

$$c^{++} = \text{Max}(0, 93.60 - 60) = \$33.6$$

$$c^{+-} = \text{Max}(0, 64.74 - 60) = \$4.74$$

$$c^{--} = \text{Max}(0, 44.78 - 60) = \$0$$

The risk-neutral probability is

$$\pi = \frac{1.05 - 0.83}{1.20 - 0.83} = 0.5946, \text{ and } 1 - \pi = 0.4054$$

Now find the option prices at time 1:

$$c^+ = \frac{0.5946(33.6) + 0.4054(4.74)}{1.05} = \$20.86$$

$$c^- = \frac{0.5946(4.74) + 0.4054(0)}{1.05} = \$2.68$$

The call price today is

$$c = \frac{0.5946(20.86) + 0.4054(2.68)}{1.05} = \$12.85$$

B. The hedge ratios at each point in the binomial tree are calculated as follows:

At the current stock price of \$65,

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{20.86 - 2.68}{78 - 53.95} = 0.7559$$

Therefore, today at time 0, the risk-free hedge would consist of a short position in 10,000 calls and a long position in 7,559 shares of the underlying stock.

At a stock price of \$78,

$$n^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}} = \frac{33.6 - 4.74}{93.6 - 64.74} = 1$$

Now the risk-free hedge would consist of a short position in 10,000 calls and a long position in 10,000 shares of the underlying stock.

At a stock price of \$53.95,

$$\frac{n^- = c^{+-} - c^{--}}{S^{+-} - S^{--}} = \frac{4.74 - 0}{64.74 - 44.78} = 0.2375$$

Now the risk-free hedge would consist of a short position in 10,000 calls and a long position in 2,375 shares of the underlying stock.

C. Exercise price, X = \$70

Stock prices in the binomial tree one and two periods from now are

$$S^+ = Su = 65(1.20) = \$78$$

$$S^- = Sd = 65(0.83) = \$53.95$$

$$S^{++} = Su^2 = 65(1.20)(1.20) = \$93.6$$

$$S^{+-} = Sud = 65(1.20)(0.83) = \$64.74$$

$$S^{--} = Sd^2 = 65(0.83)(0.83) = \$44.78$$

Call option values at expiration two periods from now are

$$c^{++} = \text{Max}(0,93.6 - 70) = \$23.6$$

$$c^{+-} = \text{Max}(0,64.74 - 70) = \$0$$

$$c^{--} = \text{Max}(0,44.78 - 70) = \$0$$

The risk-neutral probability is

$$\pi = \frac{1.05 - 0.83}{1.20 - 0.83} = 0.5946, \text{ and } 1 - \pi = 0.4054$$

Now find the option prices at time 1:

$$c^+ = \frac{0.5946(23.6) + 0.4054(0)}{1.05} = \$13.36$$

$$c^- = \frac{0.5946(0) + 0.4054(0)}{1.05} = \$0$$

The call price today is

$$c = \frac{0.5946(13.36) + 0.4054(0)}{1.05} = \$7.57$$

D. The hedge ratios at each point in the binomial tree are calculated as follows.

At the current stock price of $65,

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{13.36 - 0}{78 - 53.95} = 0.5555$$

Therefore, today at time 0, the risk-free hedge would consist of a short position in 10,000 calls and a long position in 5,555 shares of the underlying stock.

At a stock price of $78,

$$n^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}} = \frac{23.6 - 0}{93.6 - 64.74} = 0.8177$$

Now, the risk-free hedge would consist of a short position in 10,000 calls and a long position in 8,177 shares of the underlying stock.

At a stock price of $53.95,

$$n^- = \frac{c^{+-} - c^{--}}{S^{+-} - S^{--}} = \frac{0 - 0}{64.74 - 44.78} = 0$$

Zero shares of the underlying stock are needed for the short position in calls.

13. Current stock price, $S_0 = \$65$
Up move, $u = 1.20$
Down move, $d = 0.83$
Risk-free rate, $r = 5$ percent

A. Exercise price, $X = \$60$

Stock prices in the binomial tree one and two periods from now are

$$S^+ = Su = 65(1.20) = \$78$$

$$S^- = Sd = 65(0.83) = \$53.95$$

$$S^{++} = Su^2 = 65(1.20)(1.20) = \$93.6$$

$$S^{+-} = Sud = 65(1.20)(0.83) = \$64.74$$

$$S^{--} = Sd^2 = 65(0.83)(0.83) = \$44.78$$

Put option values at expiration two periods from now are

$$p^{++} = \text{Max}(0, 60 - 93.6) = \$0$$

$$p^{+-} = \text{Max}(0, 60 - 64.74) = \$0$$

$$p^{--} = \text{Max}(0, 60 - 44.78) = \$15.22$$

The risk-neutral probability is

$$\pi = \frac{1.05 - 0.83}{1.20 - 0.83} = 0.5946 \text{ and } 1 - \pi = 0.4054$$

Now find the option prices at time 1:

$$p^+ = \frac{0.5946(0) + 0.4054(0)}{1.05} = \$0$$

$$p^- = \frac{0.5946(0) + 0.4054(15.22)}{1.05} = \$5.88$$

The put price today is

$$p = \frac{0.5946(0) + 0.4054(5.88)}{1.05} = \$2.27$$

B. Unlike the hedge portfolio for calls, which has the opposite positions in the two instruments (calls and underlying stock), the hedge portfolio for puts has the same positions in the two instruments. Therefore, the current value of the hedge portfolio for puts is

$$H = nS + p$$

The possible values of the hedge portfolio one period later are

$$H^+ = nS^+ + p^+$$

$$H^- = nS^- + p^-$$

Setting H^+ equal to H^- and solving for n,

$$n = \frac{p^- - p^+}{S^+ - S^-}$$

Note that the above formula is the same as that for the hedge portfolio for calls, except that the p^+ and p^- have switched positions in the numerator. Similarly, the hedge ratios for the next time point are

$$n^+ = \frac{p^{+-} - p^{++}}{S^{++} - S^{+-}}$$

$$n^- = \frac{p^{--} - p^{+-}}{S^{+-} - S^{--}}$$

At the current stock price of $65,

$$n = \frac{p^- - p^+}{S^+ - S^-} = \frac{5.88 - 0}{78 - 53.95} = 0.2445$$

Therefore, today at time 0, the risk-free hedge would consist of a long position in 10,000 puts and a long position in 2,445 shares of the underlying stock.

At a stock price of $78,

$$n^+ = \frac{p^{+-} - p^{++}}{S^{++} - S^{+-}} = \frac{0 - 0}{93.6 - 64.74} = 0$$

Zero shares of the underlying stock are needed for the long position in puts.

At a stock price of $53.95,

$$n^- = \frac{p^{--} - p^{+-}}{S^{+-} - S^{--}} = \frac{15.22 - 0}{64.74 - 44.78} = 0.7625$$

Now the risk-free hedge would consist of a long position in 10,000 puts and a long position in 7,625 shares of the underlying stock.

C. Exercise price, X = \$70

Stock prices in the binomial tree, one and two periods from now are

$S^+ = Su = 65(1.20) = \$78$

$S^- = Sd = 65(0.83) = \$53.95$

$S^{++} = Su^2 = 65(1.20)(1.20) = \$93.6$

$S^{+-} = Sud = 65(1.20)(0.83) = \$64.74$

$S^{--} = Sd^2 = 65(0.83)(0.83) = \$44.78$

Put option values at expiration two periods from now are

$p^{++} = \text{Max}(0,70 - 93.6) = \$0$

$p^{+-} = \text{Max}(0,70 - 64.74) = \$5.26$

$p^{--} = \text{Max}(0,70 - 44.78) = \$25.22$

The risk-neutral probability is

$$\pi = \frac{1.05 - 0.83}{1.20 - 0.83} = 0.5946, \text{ and } 1 - \pi = 0.4054$$

Now find the option prices at time 1:

$$p^+ = \frac{0.5946(0) + 0.4054(5.26)}{1.05} = \$2.03$$

$$p^- = \frac{0.5946(5.26) + 0.4054(25.22)}{1.05} = \$12.72$$

The put price today is

$$p = \frac{0.5946(2.03) + 0.4054(12.72)}{1.05} = \$6.06$$

D. The hedge ratios at each point in the binomial tree are calculated as follows:

At the current stock price of \$65,

$$n = \frac{p^- - p^+}{S^+ - S^-} = \frac{12.72 - 2.03}{78 - 53.95} = 0.4445$$

Therefore, today at time 0, the risk-free hedge would consist of a long position in 10,000 puts and a long position in 4,445 shares of the underlying stock.

At stock price \$78,

$$n^+ = \frac{p^{+-} - p^{++}}{S^{++} - S^{+-}} = \frac{5.26 - 0}{93.6 - 64.74} = 0.1823$$

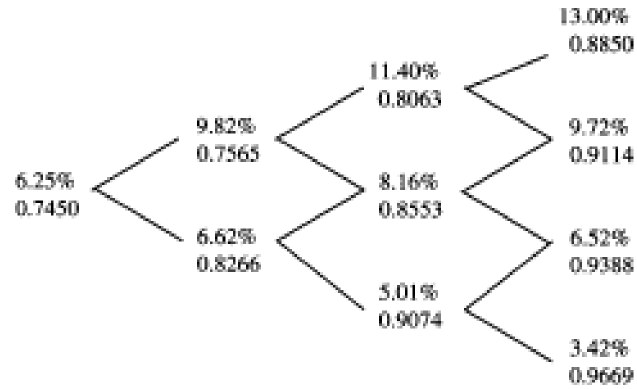
Therefore, today at time 0, the risk-free hedge would consist of a long position in 10,000 puts and a long position in 1,823 shares of the underlying stock.

At stock price \$53.95,

$$n^- = \frac{p^{--} - p^{+-}}{S^{+-} - S^{--}} = \frac{25.22 - 5.26}{64.74 - 44.78} = 1$$

Now, the risk-free hedge would consist of a long position in 10,000 puts and a long position in 10,000 shares of the underlying stock.

14. **A.** Because the underlying is a four-period bond, redraw the interest rate tree to include only the four-year maturity bond prices.



Because the call expires in two periods, first we must calculate the payoff values of the call at expiration in two periods using the bond prices at time 2 in the above tree.

$$c^{++} = \text{Max}(0,0.8063 - 0.85) = 0$$

$$c^{+-} = \text{Max}(0,0.8553 - 0.85) = 0.0053$$

$$c^{--} = \text{Max}(0,0.9074 - 0.85) = 0.0574$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1:

$$c^{+} = \frac{0.5(0) + 0.5(0.0053)}{1.0982} = 0.0024$$

$$c^{-} = \frac{0.5(0.0053) + 0.5(0.0574)}{1.0662} = 0.0294$$

The call price today is

$$c = \frac{0.5(0.0024) + 0.5(0.0294)}{1.0625} = 0.0150$$

B. Because the bond expires in two periods, first we have to calculate the prices of the 8 percent coupon bond with \$1 face value at $t = 2$. At $t = 2$, the bond is a two-period bond. Consider the top node at $t = 2$. The one-and two-period zero-coupon bond prices are 0.8977 and 0.8063, respectively. Therefore, the coupon bond price at the top node at $t = 2$ is

$$0.9426 = 0.08(0.8977) + 1.08(0.8063)$$

Similarly, the coupon bond prices at the middle and bottom nodes at $t = 2$ are

$$0.9977 = 0.08(0.9246) + 1.08(0.8553)$$

$$1.0562 = 0.08(0.9523) + 1.08(0.9074)$$

Now compute the option prices. At time 2, the prices are

$$c^{++} = \text{Max}(0,0.9426 - 0.95) = 0$$

$$c^{+-} = \text{Max}(0,0.9977 - 0.95) = 0.0477$$

$$c^{--} = \text{Max}(0,1.0562 - 0.95) = 0.1062$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1:

$$c^{+} = \frac{0.5(0) + 0.5(0.0477)}{1.0982} = 0.0217$$

$$c^{-} = \frac{0.5(0.0477) + 0.5(0.1062)}{1.0662} = 0.0722$$

The call price today is

$$c = \frac{0.5(0.0217) + 0.5(0.0722)}{1.0625} = 0.0442$$

C. First calculate the payoff values of the put at expiration in two periods.

$$p^{++} = \text{Max}(0,0.85 - 0.8063) = 0.0437$$

$$p^{+-} = \text{Max}(0,0.85 - 0.8553) = 0$$

$$p^{--} = \text{Max}(0,0.85 - 0.9074) = 0$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1:

$$p^{+} = \frac{0.5(0.0437) + 0.5(0)}{1.0982} = 0.0199$$

$$p^{-} = \frac{0.5(0) + 0.5(0)}{1.0662} = 0$$

The put price today is

$$p = \frac{0.5(0.0199) + 0.5(0)}{1.0625} = 0.0094$$

D. The prices at time 2 of the 8 percent coupon bond with $1 face value are the same as in Part B.

The payoff values of the put at expiration in two periods at $t = 2$ are

$$p^{++} = \text{Max}(0,1 - 0.9426) = 0.0574$$

$$p^{+-} = \text{Max}(0,1 - 0.9977) = 0.0023$$

$$p^{--} = \text{Max}(0,1 - 1.0562) = 0$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1:

$$p^{+} = \frac{0.5(0.0574) + 0.5(0.0023)}{1.0982} = 0.0272$$

$$p^{-} = \frac{0.5(0.0023) + 0.5(0)}{1.0662} = 0.0011$$

The put price today is

$$p = \frac{0.5(0.0272) + 0.5(0.0011)}{1.0625} = 0.0133$$

15. A. The two-period cap with an exercise rate of 8 percent consists of two caplets, a one-period call on the one-period interest rate with an exercise rate of 8 percent, and a two-period call on the one-period interest rate with an exercise rate of 8 percent. Price these caplets separately and sum them to obtain the price of the cap.

Pricing the two-period caplet
The option values at expiration at $t = 2$ are

$$c^{++} = \text{Max}(0, 0.114 - 0.08)/1.114 = 0.0305$$

$$c^{+-} = \text{Max}(0, 0.0816 - 0.08)/1.0816 = 0.0015$$

$$c^{--} = \text{Max}(0, 0.0501 - 0.08)/1.0501 = 0$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1

$$c^{+} = \frac{0.5(0.0305) + 0.5(0.0015)}{1.0982} = 0.0146$$

$$c^{-} = \frac{0.5(0.0015) + 0.5(0)}{1.0662} = 0.0007$$

The price of the two-period caplet today is

$$c = \frac{0.5(0.0146) + 0.5(0.0007)}{1.0625} = 0.0072$$

Pricing the one-period caplet
The option values at expiration at $t = 1$ are

$$c^{+} = \text{Max}(0, 0.0982 - 0.08)/1.0982 = 0.0166$$

$$c^{-} = \text{Max}(0, 0.0662 - 0.08)/1.0662 = 0$$

The price of the one-period caplet today is

$$c = \frac{0.5(0.0166) + 0.5(0)}{1.0625} = 0.0078$$

The price of the two-period cap = 0.0072 + 0.0078 = 0.0150

B. The two-period floor with an exercise rate of 9 percent consists of two floorlets, a one-period put on the one-period interest rate with an exercise rate of 9 percent, and a two-period put on the one-period interest rate with an exercise rate of 9 percent. Price these floorlets separately and sum them to obtain the price of the floor.

Pricing the two-period floorlet
The option values at expiration at t = 2 are

$$p^{++} = \text{Max}(0, 0.09 - 0.114)/1.114 = 0$$

$$p^{+-} = \text{Max}(0, 0.09 - 0.0816)/1.0816 = 0.0078$$

$$p^{--} = \text{Max}(0, 0.09 - 0.0501)/1.0501 = 0.0380$$

The risk-neutral probability is

$$\pi = 0.5 \text{ and } 1 - \pi = 0.5$$

Now find the option prices at time 1:

$$p^{+} = \frac{0.5(0) + 0.5(0.0078)}{1.0982} = 0.0036$$

$$p^{-} = \frac{0.5(0.0078) + 0.5(0.0380)}{1.0662} = 0.0215$$

The price of the two-period floorlet today is

$$p = \frac{0.5(0.0036) + 0.5(0.0215)}{1.0625} = 0.0118$$

Pricing the one-period floorlet
The option values at expiration at t = 1 are

$$p^{+} = \text{Max}(0, 0.09 - 0.0982)/1.0982 = 0$$

$$p^{-} = \text{Max}(0, 0.09 - 0.0662)/1.0662 = 0.0223$$

The price of the one-period floorlet today is

$$p = \frac{0.5(0) + 0.5(0.0223)}{1.0625} = 0.0105$$

The price of the two-period floor = 0.0118 + 0.0105 = 0.0223

16. A. The value of $N(d_1)$ is the approximate value of delta for calls and $N(d_1) - 1$ is the approximate value of delta for puts. So, first calculate the value of d_1. The time to expiration is T = 75/365 = 0.2055.

$$d_1 = \frac{\ln(28/30) + [0.035 + (0.40)^2/2](0.2055)}{0.40\sqrt{0.2055}} = -0.25$$

Using the normal probability table,

$$N(-0.25) = 1 - N(0.25) = 1 - 0.5987 = 0.4013 = N(d_1)$$

Therefore, the approximate value of delta for calls is 0.4013 and the approximate value of delta for puts is 0.4013 − 1 = − 0.5987.

B. Change in option price = Delta × Change in underlying asset price
Call delta = 0.4013
Put delta = −0.5987

i. For a $1 change in the price of the underlying:
Change in call price = 0.4013(1) = $0.4013
Change in put price = −0.5987(1) = −$0.5987
Approximate new call price = 1.3144 + 0.4013 = $1.7157
Approximate new put price = 3.0994 − 0.5987 = $2.5007

ii. For a \$5 change in the price of the underlying:
Change in call price = 0.4013(5) = \$2.0065
Change in put price = −0.5987(5) = −\$2.9935
Approximate new call price = 1.3144 + 2.0065 = \$3.3209
Approximate new put price = 3.0994 − 2.9935 = \$0.1059

C. For a \$1 change in the price of the underlying, the approximate new call price of \$1.7157 is different from but not too far off the actual call price of \$1.7538. Similarly, the approximate new put price of \$2.5007 is different from but not too far off the actual put price of \$2.5388. Therefore, the delta approximation is good but not perfect.

For a \$5 change in the price of the underlying, the approximate new call price of \$3.3209 is fairly far off the actual call price of \$4.2270. Similarly, the approximate new put price of \$0.1059 is quite far off the actual put price of \$1.0120. Therefore, the delta approximation is not particularly good.

The above comparisons indicate that the approximations based on delta are worse for larger moves in the underlying price.

17. A. First calculate the values of d_1 and d_2. The time to expiration is T = 275/365 = 0.7534. Adjust the price of the asset S_0 = \$100 − \$4.25 = \$95.75

$$d_1 = \frac{\ln(95.75/100) + [0.03 + (0.45)^2/2](0.7534)}{0.45\sqrt{0.7534}} = 0.1420$$

$$d_2 = 0.1420 - 0.45\sqrt{0.7534} = -0.2486$$

Using the normal probability table,

$$N(0.14) = 0.5557 = N(d_1)$$

$$N(-0.25) = 1 - N(0.25) = 1 - 0.5987 = 0.4013 = N(d_2)$$

The value of the call option is

$$c = 95.75(0.5557) - 100e^{-0.03(0.7534)}(0.4013) = 13.975$$

The value of the put option is

$$p = 100e^{-0.03(0.7534)}(1 - 0.4013) - 95.75(1 - 0.5557) = 15.990$$

B. First calculate the values of d_1 and d_2. The time to expiration is T = 275/365 = 0.7534. Adjust the price of the asset $S_0 = 100e^{-0.015(0.7534)} = 98.87626$

$$d_1 = \frac{\ln(98.876/100) + [0.03 + (0.45)^2/2](0.7534)}{0.45\sqrt{0.7534}} = 0.2242$$

$$d_2 = 0.22423 - 0.45\sqrt{0.7534} = -0.1664$$

Using the normal probability table,

$$N(0.22) = 0.5871 = N(d_1)$$

$$N(-0.17) = 1 - N(0.17) = 1 - 0.5675 = 0.4325 = N(d_2)$$

The value of the call option is

$$c = 98.876(0.5871) - 100e^{-0.03(0.7534)}(0.4325) = \$15.767$$

The value of the put option is

$$p = 100e^{-0.03(0.7534)}(1 - 0.4325) - 98.876(1 - 0.5871) = \$14.656$$

18. Call price, c_0 = \$3.50
Put price, p_0 = \$9
Exercise price, X = \$50
Forward price, F(0,T) = \$45
Days to option expiration = 175
Risk-free rate, r = 4 percent
Time to expiration = 175/365 = 0.4795
Bond price = $[X - F(0,T)]/(1 + r)^T = (50 - 45)/(1+.04)^{0.4795} = \$4.91$

A. Synthetic call = Long forward + $p_0 - [X - F(0,T]/(1 + r)^T = 0 + 9 - 4.91 = 4.09$
Synthetic put = c_0 + Short forward + $[X - F(0,T)]/(1 + r)^T = 3.5 + 0 + 4.91 = 8.41$
Synthetic forward = $c_0 - p_0 + [X - F(0,T)]/(1 + r)^T = 3.5 - 9 + 4.91 = -0.59$

B.

Instrument	Actual Price	Synthetic Price	Mispricing/Profit
Call	3.50	4.09	0.59
Put	9	8.41	0.59
Forward	0	−0.59	0.59

C. The actual call is cheaper than the synthetic call. Therefore, an arbitrage transaction in which you buy the actual call (underpriced) and sell the synthetic call (overpriced) will yield a risk-free profit up front of \$4.09 − \$3.50 = \$0.59. The initial up-front cash is generated as follows:

Transaction	
Buy call	−3.5
Sell synthetic call	
Short forward	0
Short put	9
Long bond	−4.91
Total	0.59

As shown below, at expiration no cash will be received or paid out.

	Value at Expiration	
Transaction	$S_T < 50$	$S_T > 50$
Buy call	0	$S_T - 50$
Sell synthetic call		
Short forward	$-(S_T - 45)$	$-(S_T - 45)$
Short put	$-(50 - S_T)$	0
Long bond	50 − 45	50 − 45
Total	0	0

D. The actual put is more expensive than the synthetic put. Therefore, an arbitrage transaction in which you buy the synthetic put (underpriced) and sell the actual put (overpriced) will yield a risk-free profit of \$9 − \$8.41 = \$0.59. The initial up-front cash is generated as follows:

Transaction	
Sell put	9
Buy synthetic put	
Long call	−3.5
Short forward	0
Long bond	−4.91
Total	0.59

As shown below, at expiration no cash will be received or paid out.

	Value at Expiration	
Transaction	$S_T < 50$	$S_T > 50$
Sell put	$-(50 - S_T)$	0
Buy synthetic put		
Long call	0	$S_T - 50$
Short forward	$-(S_T - 45)$	$-(S_T - 45)$
Long bond	50 − 45	50 − 45
Total	0	0

19. A. First calculate the values of d_1 and d_2. The time to expiration, T = 65/365 = 0.1781.

$$d_1 = \frac{\ln(145/150) + [(0.03)^2/2](0.1781)}{0.33\sqrt{0.1781}} = -0.1738$$

$$d_2 = -0.1738 - 0.33\sqrt{0.1781} = -0.3131$$

Using the normal probability table,

$$N(-0.17) = 1 - N(0.17) = 1 - 0.5675 = 0.4325 = N(d_1)$$

$$N(-0.31) = 1 - N(0.31) = 1 - 0.6217 = 0.3783 = N(d_2)$$

The value of the call option is

$$c = e^{-0.0375(0.1781)}[145(0.4325) - 150(0.3783)] = 5.928$$

B. With no cash flows on the underlying and continuously compounded interest, the price of the underlying asset can be calculated as

$$S_0 = 145e^{-0.0375(0.1781)} = 144.03$$

Now use the Black–Scholes–Merton model:

$$d_1 = \frac{\ln(144.03/150) + [0.0375 + (0.33)^2/2](0.1781)}{0.33\sqrt{0.1781}} = -0.1740$$

$$d_2 = -0.174 - 0.33\sqrt{0.1781} = -0.3133$$

Using the normal probability table,

$$N(-0.17) = 1 - N(0.17) = 1 - 0.5675 = 0.4325 = N(d_1)$$

$$N(-0.31) = 1 - N(0.31) = 1 - 0.6217 = 0.3783 = N(d_2)$$

The value of the call option is

$$c = 144.03(0.4325) - 150e^{-0.0375(0.1781)}(0.3783) = 5.926$$

Consistent with the equivalence of options on the underlying and options on the forward contract, this value is the same as the value of the call option on the forward contract in Part A (the slight difference comes from rounding).

C. The values for $N(d_1)$ and $N(d_2)$ are the same as in Part A. Putting these values into the formula for the put option in the Black model produces

$$p = e^{-0.0375(0.1781)}[150(1 - 0.3783) - 145(1 - 0.4325)] = 10.894$$

D. The values for $N(d_1)$ and $N(d_2)$ are the same as in Part B. Putting these values into the formula for the put option in the Black–Scholes–Merton model produces

$$p = 150e^{-0.0375(0.1781)}(1 - 0.3783) - 144.03(1 - 0.4325) = 10.897$$

Consistent with the equivalence of options on the underlying and options on the forward contract, this value is the same as the value of the put option on the forward contract in Part C (the slight difference comes from rounding).

20. A. Calculate the values of d_1 and d_2. The time to expiration is T = 180/365 = 0.4932.

$$d_1 = \frac{\ln(0.0725/0.075) + [(0.04)^2/2](0.4932)}{0.04\sqrt{0.4932}} = -1.1928$$

$$d_2 = -1.1928 - 0.04\sqrt{0.4932} = -1.2209$$

Using the normal probability table,

$$N(-1.19) = 1 - N(1.19) = 1 - 0.8830 = 0.1170 = N(d_1)$$

$$N(-1.22) = 1 - N(1.22) = 1 - 0.8888 = 0.1112 = N(d_2)$$

The value of the call option is

$$c = e^{-0.05(0.4932)}[0.0725(0.1170) - 0.075(0.1112)] = 0.00013903$$

This value is computed based on the assumption that the option payoff is made at expiration. For the 90-day interest rate option, however, the payment is made 90 days after expiration. So, discount back 90 days:

$$0.00013903e^{-0.0725(90/365)} = 0.00013657$$

Convert to periodic rate based on a 90-day rate and using the customary 360-day year.

$$0.00013657(90/360) = 0.00003414$$

B. The values of $N(d_1)$ and $N(d_2)$ are the same as in Part A. Plug these values into the Black model for the put option.

$$p = e^{-0.05(0.4932)}[0.075(1 - 0.1112) - 0.0725(1 - 0.1170)] = 0.00257813$$

Now discount back 90 days:

$$0.00257813e^{-0.0725(90/365)} = 0.00253245$$

Convert to periodic rate based on 90-day rate:

$$0.00253245(90/360) = 0.00063311$$