7.1 MULTIPLICATIVE DECOMPOSITION

Consider a time series that exhibits increasing or decreasing seasonal variation. When the parameters describing the series are not changing over time, the time series sometimes can be modeled adequately by using what is called the **multiplicative decomposition model.** This model can be stated as follows.

The **multiplicative decomposition model** is

$$y_t = TR_t \times SN_t \times CL_t \times IR_t$$

where

y_t = the observed value of the time series in time period t

TR_t = the **trend component** (or factor) in time period t

SN_t = the **seasonal component** (or factor) in time period t

CL_t = the **cyclical component** (or factor) in time period t

IR_t = the **irregular component** (or factor) in time period t

We previously discussed the nature of trend effects, seasonal variations, and irregular fluctuations. The cyclical component, CL_t, refers to recurring up and down movements around trend levels as caused, for example, by the business cycle. These fluctuations can last anywhere from two to longer than ten years as measured from peak to peak or trough to trough. In business, a peak would mark the end of an expansion in business activity, and a trough would mark the end of a contraction.

Notice that this decomposition model employs a **multiplicative seasonal factor.** That is, the seasonal factor is multiplied by the trend (rather than, for example, added to the trend as in dummy variable regression; see Section 6.4). To see how the multiplicative seasonal factor can model increasing seasonal variation, suppose, for instance, that sales of outboard motors by the Power Drive Corporation are seasonal. Also suppose that sales are lowest in the first quarter, highest in the second quarter, moderately high in the third quarter, and moderately low in the fourth quarter. Furthermore, assume that sales exhibit a linear trend given by

$$TR_t = 500 + 50t$$

where $t = 0$ is considered to be the fourth quarter of 2002. If trend alone is considered, outboard motor sales in the four quarters of 2003 are expected to be

$$TR_1 = 500 + 50(1) = 550 \quad \text{(quarter 1)}$$
$$TR_2 = 500 + 50(2) = 600 \quad \text{(quarter 2)}$$
$$TR_3 = 500 + 50(3) = 650 \quad \text{(quarter 3)}$$
$$TR_4 = 500 + 50(4) = 700 \quad \text{(quarter 4)}$$

However, sales are seasonal. Therefore, we can model the seasonal behavior of sales by defining seasonal factors. Suppose that the seasonal factors for quarters 1, 2, 3, and 4 are $SN_{Q1} = .4$, $SN_{Q2} = 1.6$, $SN_{Q3} = 1.2$, and $SN_{Q4} = .8$. If we assume that these seasonal factors are multiplicative, then when we consider both trend and seasonal effects, sales in the four quarters of 2003 are expected to be

$$TR_1 \times SN_{Q1} = [500 + 50(1)](.4) = 220$$
$$TR_2 \times SN_{Q2} = [500 + 50(2)](1.6) = 960$$
$$TR_3 \times SN_{Q3} = [500 + 50(3)](1.2) = 780$$
$$TR_4 \times SN_{Q4} = [500 + 50(4)](.8) = 560$$

Multiplying the trend TR_t by the appropriate seasonal factors models the seasonal pattern of sales. This is illustrated in Figure 7.1.

If multiplicative seasonal factors remain constant over time, they allow us to model increasing seasonal variation. For example, in the outboard motor sales situation, when we consider both trend and seasonal effects, sales in the four quarters of 2004 are expected to be

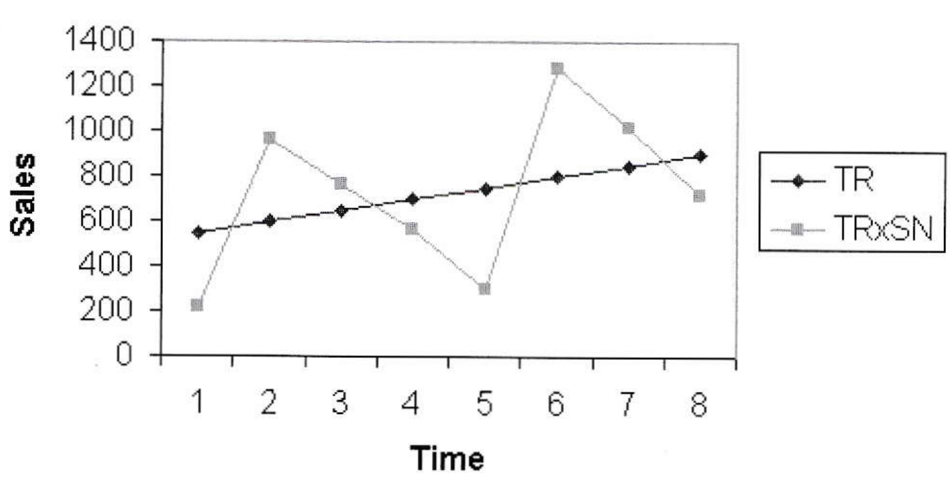
$$TR_5 \times SN_{Q1} = [500 + 50(5)](.4) = 300$$
$$TR_6 \times SN_{Q2} = [500 + 50(6)](1.6) = 1280$$
$$TR_7 \times SN_{Q3} = [500 + 50(7)](1.2) = 1020$$
$$TR_8 \times SN_{Q4} = [500 + 50(8)](.8) = 720$$

Multiplication of the trend by the seasonal factors implies that the size of the seasonal swing is proportional to the trend. Therefore, since the trend

$$TR_t = 500 + 50t$$

is increasing, the size of the seasonal swing is increasing (increasing seasonal variation). Note again that here we are assuming that the seasonal factors remain constant over time. Sometimes the seasonal factors instead change over time; then the methods in Chapters 8 to 12 should be considered.

FIGURE 7.1
An illustration in Excel of multiplicative seasonal factors $SN_{Q1} = .4$, $SN_{Q2} = 1.6$, $SN_{Q3} = 1.2$ and $SN_{Q4} = .8$



The seasonal factor SN_t models cyclical patterns in a time series that are completed within one calendar year. If a time series displays a cycle that has a longer duration, a cyclical factor CL_t can be defined. For instance, in the outboard motor sales situation, suppose that all four quarters of 2003 are included in a "boom period" of the business cycle. Assume that the cyclical factors describing the increased economic activity in the four quarters of 2003 are $CL_1 = 1.08$, $CL_2 = 1.09$, $CL_3 = 1.09$, and $CL_4 = 1.10$. If trend, seasonal, and cyclical effects are considered, sales in the four quarters of 2003 are expected to be

$$TR_1 \times SN_{Q1} \times CL_1 = 220(1.08) = 238$$
$$TR_2 \times SN_{Q2} \times CL_2 = 960(1.09) = 1046$$
$$TR_3 \times SN_{Q3} \times CL_3 = 780(1.09) = 850$$
$$TR_4 \times SN_{Q4} \times CL_4 = 560(1.10) = 616$$

Thus the cyclical factors increase expected sales above levels that would be expected if only trend and seasonal effects were considered. This reflects the boom in economic activity.

The **multiplicative decomposition method** can be used to obtain point estimates—denoted tr_t, sn_t, cl_t, and ir_t—of the factors TR_t, SN_t, CL_t, and IR_t. The following example illustrates the procedure.

EXAMPLE 7.1

The Discount Soda Shop, Inc., owns and operates ten drive-in soft drink stores. Discount Soda has been selling Tasty Cola, a soft drink that was introduced on the market just three years ago and has been gaining in popularity. Periodically, Discount Soda orders a supply of Tasty Cola from the regional distributor. The company uses an inventory policy that attempts to meet practically all of the demand for Tasty Cola, while at the same time ensuring that the company does not tie up its money needlessly by ordering much more Tasty Cola than it can reasonably expect to sell. In order to implement its inventory policy, Discount Soda needs to forecast monthly Tasty Cola sales (in hundreds of cases). At the end of each month, Discount Soda desires point forecasts and prediction interval forecasts of Tasty Cola sales in future months.

Discount Soda has recorded monthly Tasty Cola sales for the previous three years, which we will call year 1, year 2, and year 3. This time series is given in Table 7.1 and is plotted in Figure 7.2. Notice that in addition to having a linear trend, the Tasty Cola sales time series possesses seasonal variation, with sales of the soft drink being greatest in the summer and early fall months and lowest in the winter months. We show later in this example that it is reasonable to conclude that y_t, the sales of Tasty Cola in period t, is adequately described by the model

$$y_t = TR_t \times SN_t \times CL_t \times IR_t$$

Therefore, we summarize in Table 7.2 the calculations needed to find estimates—denoted tr_t, sn_t, cl_t, and ir_t—of the components TR_t, SN_t, CL_t, and IR_t of this model.

To begin our consideration of the calculations, we will explain the calculation of **moving averages** and **centered moving averages** (the centered moving averages are denoted CMA_t). The purpose behind computing these averages is to eliminate seasonal

TABLE 7.1 Monthly Sales of Tasty Cola (In Hundreds of Cases)

Year	Month	t	Sales y_t	Year	Month	t	Sales y_t
1	1 (Jan.)	1	189	2	7	19	831
	2 (Feb.)	2	229		8	20	960
	3 (Mar.)	3	249		9	21	1152
	4 (Apr.)	4	289		10	22	759
	5 (May)	5	260		11	23	607
	6 (June)	6	431		12	24	371
	7 (July)	7	660	3	1	25	298
	8 (Aug.)	8	777		2	26	378
	9 (Sept.)	9	915		3	27	373
	10 (Oct.)	10	613		4	28	443
	11 (Nov.)	11	485		5	29	374
	12 (Dec.)	12	277		6	30	660
2	1	13	244		7	31	1004
	2	14	296		8	32	1153
	3	15	319		9	33	1388
	4	16	370		10	34	904
	5	17	313		11	35	715
	6	18	556		12	36	441

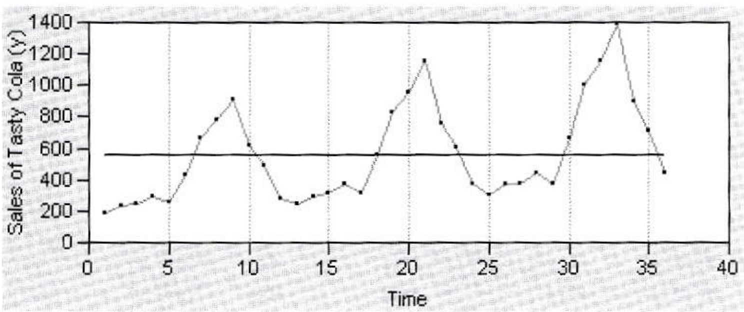


FIGURE 7.2 JMP IN plot of monthly sales of Tasty Cola (in hundreds of cases)

variations and irregular fluctuations from the data. The first moving average is the average of the first 12 Tasty Cola sales values

$$\frac{189 + 229 + 249 + 289 + 260 + 431 + 660 + 777 + 915 + 613 + 485 + 277}{12}$$

$$= 447.833$$

Here we use a "12-period moving average" because the Tasty Cola time series data are monthly ($L = 12$ time periods or "seasons" per year). If the data were quarterly, we would compute a "four-period moving average." The second moving average is obtained by dropping the first sales value (y_1) from the average and by including the next sales value (y_{13})

TABLE 7.2 Analysis of the Historical Tasty Cola Sales Time Series Using Multiplicative Decomposition

(a) The Values of sn_t, d_t, and tr_t

t	y_t	12-Period Moving Average	$CMA_t = tr_t \times cl_t$	$sn_t \times ir_t = y_t/(tr_t \times cl_t)$	sn_t	$d_t = \frac{y_t}{sn_t}$	$tr_t = 380.163 + 9.489t$	$\hat{y}_t = tr_t \times sn_t$
1	189				.493	383.37	389.652	192.10
2	229				.596	384.23	399.141	237.89
3	249				.595	418.49	408.630	243.13
4	289				.680	425	418.119	284.32
5	260				.564	460.99	427.608	241.17
6	431				.986	437.12	437.097	430.98
7	660	447.833	450.1	1.466	1.467	449.9	446.586	655.14
8	777	452.417	455.2	1.707	1.693	458.95	456.075	772.13
9	915	458	460.9	1.985	1.990	459.79	465.564	926.47
10	613	463.833	467.2	1.312	1.307	469.01	475.053	620.89
11	485	470.583	472.8	1.026	1.029	471.33	489.542	498.59
12	277	475	480.2	.577	.600	461.67	494.031	296.42
13	244	485.417	492.5	.495	.493	494.97	503.520	248.24
14	296	499.667	507.3	.583	.596	496.64	513.009	305.75
15	319	514.917	524.8	.608	.595	536.13	522.498	310.89
16	370	534.667	540.7	.684	.680	544.12	531.987	361.75
17	313	546.833	551.9	.567	.564	554.97	541.476	305.39
18	556	557	560.9	.991	.986	563.89	550.965	543.25
19	831	564.833	567.1	1.465	1.467	566.46	560.454	822.19
20	960	569.333	572.7	1.676	1.693	567.04	569.943	964.91
21	1152	576.167	578.4	1.992	1.990	578.89	579.432	1153.07
22	759	580.667	583.7	1.300	1.307	580.72	588.921	769.72
23	607	586.75	589.3	1.030	1.029	589.89	598.410	615.76
24	371	591.833	596.2	.622	.600	618.33	607.899	364.74
25	298	600.5	607.7	.490	.493	604.46	617.388	304.37
26	378	614.917	623.0	.607	.596	634.23	626.877	373.62
27	373	631	640.8	.582	.595	626.89	636.366	378.64
28	443	650.667	656.7	.675	.680	651.47	645.855	439.18
29	374	662.75	667.3	.561	.564	663.12	655.344	369.61
30	660	671.75	674.7	.978	.986	669.37	664.833	655.53
31	1004	677.583			1.467	684.39	674.322	989.23
32	1153				1.693	681.04	683.811	1157.69
33	1388				1.990	697.49	693.300	1379.67
34	904				1.307	691.66	702.789	918.55
35	715				1.029	694.85	712.278	732.93
36	441				.600	735	721.707	433.06

in the average. Thus we obtain

$$\frac{229 + 249 + 289 + 260 + 431 + 660 + 777 + 915 + 613 + 485 + 277 + 244}{12}$$

$$= 452.417$$

TABLE 7.2 (Continued)

(b) The Values of cl_t and ir_t

t	y_t	$tr_t \times sn_t$	$cl_t \times ir_t = \dfrac{y_t}{tr_t \times sn_t}$	$cl_t = \dfrac{cl_{t-1}ir_{t-1} + cl_t ir_t + cl_{t+1}ir_{t+1}}{3}$	$ir_t = \dfrac{cl_t \times ir_t}{cl_t}$
1	189	192.10	.9839		
2	229	237.89	.9626	.9902	.9721
3	249	243.13	1.0241	1.0010	1.0231
4	289	284.32	1.0165	1.0396	.9778
5	260	241.17	1.0781	1.0315	1.0452
6	431	430.98	1.0000	1.0285	.9723
7	660	655.14	1.0074	1.0046	1.0028
8	777	772.13	1.0063	1.0004	1.0059
9	915	926.47	.9876	.9937	.9939
10	613	620.89	.9873	.9825	1.0063
11	485	498.59	.9727	.9648	1.0082
12	277	296.42	.9345	.9634	.9700
13	244	248.24	.9829	.9618	1.0219
14	296	305.75	.9681	.9924	.9755
15	319	310.89	1.0261	1.0567	.9710
16	370	361.75	1.0228	1.0246	.9982
17	313	305.39	1.0249	1.0237	1.0012
18	556	543.25	1.0235	1.0197	1.0037
19	831	822.19	1.0107	1.0097	1.0010
20	960	964.91	.9949	1.0016	.9933
21	1152	1153.07	.9991	.9934	1.0057
22	759	769.72	.9861	.9903	.9958
23	607	615.76	.9858	.9964	.9894
24	371	364.74	1.0172	.9940	1.0233
25	298	304.37	.9791	1.0027	.9765
26	378	373.62	1.0117	.9920	1.0199
27	373	378.64	.9851	1.0018	.9833
28	443	439.18	1.0087	1.0030	1.0057
29	374	369.61	1.0119	1.0091	1.0028
30	660	655.53	1.0068	1.0112	.9956
31	1004	989.23	1.0149	1.0059	1.0089
32	1153	1157.69	.9959	1.0053	.9906
33	1388	1379.67	1.0060	.9954	1.0106
34	904	918.55	.9842	.9886	.9955
35	715	732.93	.9755	.9927	.9827
36	441	433.06	1.0183		

The third moving average is obtained by dropping y_2 from the average and by including y_{14} in the average. We obtain

$$\frac{249 + 289 + 260 + 431 + 660 + 777 + 915 + 613 + 485 + 277 + 244 + 296}{12}$$

$$= 458$$

Successive moving averages are computed similarly until we include y_{36} in the last moving average. Note that we use the term "moving average" here because as we calculate these averages, we move along by dropping the most remote observation in the previous average and by including the "next" observation in the new average.

The first moving average corresponds to a time that is midway between periods 6 and 7, the second moving average corresponds to a time that is midway between periods 7 and 8, and so forth. In order to obtain averages corresponding to time periods in the original Tasty Cola time series, we calculate **centered moving averages.** The centered moving averages are two-period moving averages of the previously computed 12-period moving averages. Thus the first centered moving average is

$$\frac{447.833 + 452.417}{2} = 450.1$$

The second centered moving average is

$$\frac{452.417 + 458}{2} = 455.2$$

Successive centered moving averages are calculated in a similar fashion. The 12-period moving averages and centered moving averages for the Tasty Cola sales time series are given in Table 7.2(a).

If the original moving averages had been computed using an odd number of time series values, the centering procedure would not have been necessary. For example, if we had three seasons per year, we would compute three-period moving averages. Then the first moving average would correspond to period 2, the second moving average would correspond to period 3, and so on. However, most seasonal time series are quarterly, monthly, or weekly, and the centering procedure is necessary.

The centered moving average in time period t, CMA_t, is considered to be equal to $tr_t \times cl_t$, the estimate of $TR_t \times CL_t$. This is because the averaging procedure is assumed to have removed (1) seasonal variations (note that each moving average is computed using exactly one observation from each season) and (2) short-term irregular fluctuations. The (longer-term) trend effects and cyclical effects, that is, $tr_t \times cl_t$, remain.

Since the model

$$y_t = TR_t \times SN_t \times CL_t \times IR_t$$

implies that

$$SN_t \times IR_t = \frac{y_t}{TR_t \times CL_t}$$

it follows that the estimate $sn_t \times ir_t$ of $SN_t \times IR_t$ is

$$sn_t \times ir_t = \frac{y_t}{tr_t \times cl_t} = \frac{y_t}{\text{CMA}_t}$$

Noting that the values of $sn_t \times ir_t$ are calculated in Table 7.2(a), we can find sn_t by grouping the values of $sn_t \times ir_t$ by months and calculating an average, $\overline{sn}_t$ for each month.

TABLE 7.3 Estimates of the Seasonal Factors of the Tasty Cola Sales Time Series

		$sn_t \times ir_t = y_t/(tr_t \times cl_t)$			
		Year 1	**Year 2**	$\overline{sn}_t$	$sn_t = 1.0008758\,(\overline{sn}_t)$
1	Jan.	.495	.490	.4925	.493
2	Feb.	.583	.607	.595	.596
3	Mar.	.608	.582	.595	.595
4	Apr.	.684	.675	.6795	.680
5	May	.567	.561	.564	.564
6	June	.991	.978	.9845	.986
7	July	1.466	1.465	1.4655	1.467
8	Aug.	1.707	1.676	1.6915	1.693
9	Sept.	1.985	1.992	1.9885	1.990
10	Oct.	1.312	1.300	1.306	1.307
11	Nov.	1.026	1.030	1.028	1.029
12	Dec.	.577	.622	.5995	.600

These seasonal factors are then normalized so that they add to $L = 12$, the number of periods in a year. This normalization is accomplished by multiplying each value of $\overline{sn}_t$ by the quantity

$$\frac{L}{\sum_{t=1}^{L} \overline{sn}_t} = \frac{12}{11.9895} = 1.0008758$$

This normalization process results in the estimate $sn_t = 1.0008758(\overline{sn}_t)$, which is the estimate of SN_t. These calculations are summarized in Table 7.3.

Having calculated the values of sn_t and placed them in Table 7.2(a), we next define the deseasonalized observation in time period t to be

$$d_t = \frac{y_t}{sn_t}$$

Deseasonalized observations are computed in order to better estimate the trend component TR_t. Dividing y_t by the estimated seasonal factor removes the seasonality from the data and allows us to better understand the nature of the trend. The deseasonalized observations are calculated in Table 7.2(a) and are plotted in Figure 7.3. Since the deseasonalized observations have a straight-line appearance, it seems reasonable to assume a linear trend

$$TR_t = \beta_0 + \beta_1 t$$

We estimate TR_t by fitting a straight line to the deseasonalized data. That is, we compute the least squares point estimates of the parameters in the simple linear regression model relating the dependent variable d_t to the independent variable t:

$$d_t = \beta_0 + \beta_1 t + \varepsilon_t$$

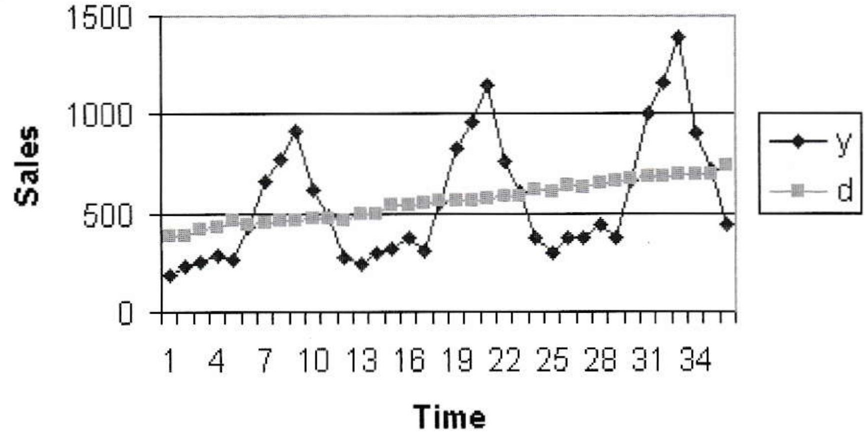


FIGURE 7.3 Excel plot of deseasonalized Tasty Cola sales d_t

Thus, we obtain tr_t, the estimate of TR_t, by computing

$$b_1 = \frac{SS_{td}}{SS_{tt}} = \frac{\sum_{t=1}^{36} td_t - \frac{\left(\sum_{t=1}^{36} t\right)\left(\sum_{t=1}^{36} d_t\right)}{36}}{\sum_{t=1}^{36} t^2 - \frac{\left(\sum_{t=1}^{36} t\right)^2}{36}} = 9.489$$

and

$$b_0 = \bar{d} - b_1\bar{t} = \frac{\sum_{t=1}^{36} d_t}{36} - b_1 \frac{\sum_{t=1}^{36} t}{36} = 380.163$$

Therefore,

$$tr_t = b_0 + b_1 t = 380.163 + 9.489t$$

The values of tr_t are calculated in Table 7.2(a). Note that, for example, although $y_{22} = 759$ (Tasty Cola sales in period 22) is larger than $tr_{22} = 588.921$ (the estimated trend in period 22), $d_{22} = 580.72$ is smaller than $tr_{22} = 588.921$. This implies that on a deseasonalized basis, Tasty Cola sales were slightly down in October of year 2. This might have been caused by a slightly colder October than usual.

Thus far we have found estimates sn_t and tr_t of SN_t and TR_t. Since the model

$$y_t = TR_t \times SN_t \times CL_t \times IR_t$$

implies that

$$CL_t \times IR_t = \frac{y_t}{TR_t \times SN_t}$$