CHAPTER 2

Forces, Vectors, and Resultants

OBJECTIVES

Upon completion of this chapter the student will be able to:

1. Determine the resultant of two vectors at right angles to each other by drawing a vector triangle and using the trigonometric functions of sine, cosine, and tangent.
2. Determine the resultant of any two vectors by drawing a vector triangle and using either the sine law or cosine law.
3. Resolve any vector quantity into components in the horizontal and vertical directions.
4. Resolve any vector quantity into components along any two axes.
5. Determine the resultant of several vectors by the method of components.

2–1 VECTORS

Everyone feels that he or she knows what a force is and would probably define it as a push or pull. Although this is true, there are further classifications. However, all forces do have one property in common: They can be represented by vectors.

We must first distinguish between a *vector* quantity and a *scalar* quantity. You are no doubt familiar with scalar quantities. A board 10 ft long, a 2-hour time interval, a floor area of 20 m^2, and a 60-W light bulb all tell us "how much." These are scalar quantities; they indicate size or magnitude.

Vector quantities have the additional property of direction. Some vector quantities are a force of 15 N vertically downward, a distance of 20 km north, a velocity of 20 km/h east, and an acceleration of 7 ft/sec^2 upward. A vector quantity is therefore represented by an arrow; the arrowhead indicates the direction, and the length of the arrow indicates the magnitude.

If we arbitrarily choose a scale of 1 cm = 2 N, a 15-N vertical force would be that shown in Figure 2–1. Similarly, the other vector quantities referred to previously would be those shown in Figures 2–2, 2–3, and 2–4. Drawing vectors to scale is only used for

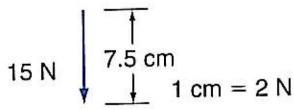

FIGURE 2–1

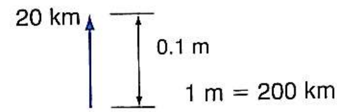

FIGURE 2–2

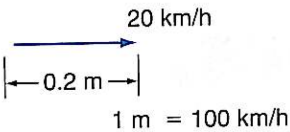

FIGURE 2–3

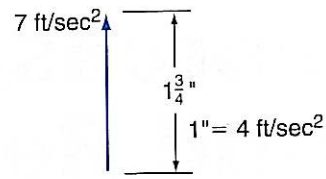

FIGURE 2–4

graphical solutions, but when drawing vectors for analytical solutions, draw them approximately to scale for easier visualization of the problem solution.

To be complete, both direction and magnitude must be labeled for each vector quantity. A vector representing a force has a point of application and a line of action. In Figure 2–5, a force of 20 N is applied to a cart. The vector shown indicates magnitude (20 N), a point of application (A), and the direction along the line of action. If the 20-N force is not sufficient to move the cart, the cart has a balance of external forces acting on it and is said to be in *static equilibrium*.

The principle of *transmissibility* states that a force acting on a body can be applied anywhere along the force's line of action without changing its effect on the body. Thus, the 20-N force can also be applied at point B, as shown in Figure 2–6. Whether the point of application is A or B, the 20-N force has the same effect on the cart. Later in the book, this principle will be used quite frequently in dealing with the subject of *moments*.

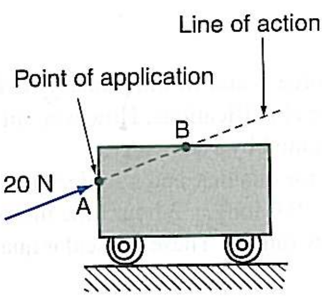

FIGURE 2–5

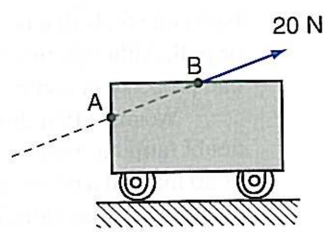

FIGURE 2–6

Vector quantities are not always vertical or horizontal. They may be at some angle or slope. The slope is indicated by reference to a horizontal line. This is done by giving either the angle in degrees (Figure 2–7a) or the rise and run of the slope (Figure 2–7b).

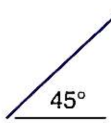

FIGURE 2–7a

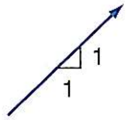

FIGURE 2–7b

2–2 FORCE TYPES, CHARACTERISTICS, AND UNITS

Realizing that forces can be represented by vectors, consider now the various types of forces. For the sake of easier discussion in a subject such as mechanics, several classifications are used. One such classification is that of *applied* and *nonapplied forces*. An applied force is a very real and noticeable force applied directly to an object. The force that you would apply to a book (Figure 2–8) to slide it across a table is an applied force. The nonapplied force acting on this same book (Figure 2–9) may not be as readily apparent since it is the force of gravity, or the weight of the book. In Figure 2–9a the weight is shown as a concentrated force acting at the center of gravity of the book. It could also have been shown as a *distributed* load (Figure 2–9b), consisting of many smaller forces distributed over the

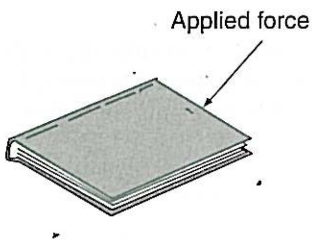

FIGURE 2–8

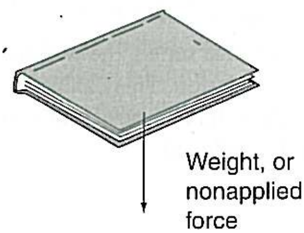

FIGURE 2–9a

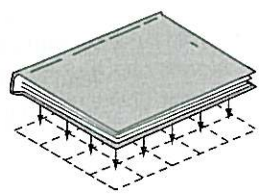

FIGURE 2–9b

entire surface of the book. Other examples of nonapplied forces are the force of magnetic attraction or repulsion and the force due to inertia. In analyzing various force systems later, keep in mind that forces such as weight and inertia are always present and may have to be included in your calculations.

Another classification categorizes forces as *internal* and *external*. The distinction here is very important, since it is often the source of incorrectly drawn free-body diagrams (Section 4–2). Internal forces are often included where they should not be.

An internal force is a force inside a structure, and an external force is a force outside the structure. The pin-connected structure in Figure 2–10 has an external force of 30 lb. Knowing that connection C is on rollers and free to move horizontally, one can visualize that the horizontal member AC is *in tension;* that is, there is a force tending to stretch it. The tensile force in AC is an internal force. There are also internal forces of compression in members AB and BC.

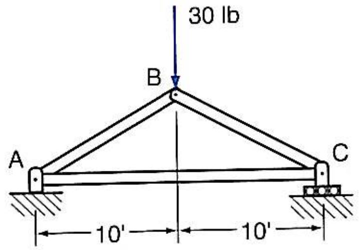

FIGURE 2–10

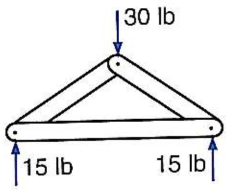

FIGURE 2–11

If we replace the actual supports at points A and C with equivalent supporting forces of 15 lb at each end (Figure 2–11), we have a total of three external forces. These external forces can be further subdivided into *acting* and *reacting forces*. The 15-lb forces are present because of the 30-lb force; thus, they are a reaction to the application of the 30-lb force. Therefore, the 30-lb force is an *acting* force, and the 15-lb forces are *reacting* forces. You will be asked to solve for the reactions on various structures. *Reactions* are simply the reacting forces that are necessary to support the structure when its given method of support is removed. As shown in Figure 2–11, the reaction at A is 15 lb vertically upward or R_A = 15 lb ↑, and similarly, R_B = 15 lb ↑. (Expressing a quantity in italic type will indicate that it is a vector quantity.)

Unless otherwise stated, the weight of all members or structures will be neglected to simplify problem solution. This can be done without appreciable error when the supported load is much greater than the structure's weight. When weights are significant in later problems, they will be included.

2–3 RESULTANTS

Scalar quantities such as 4 m^2 and 3 m^2 can be added to equal 7 m^2. But if we add vector quantities of 4 km and 3 km, their directions must be considered. This is known as *adding vectorially* or *vector addition*. The answer obtained is the *resultant;* it is a single vector giving the result of the addition of the original two or more vectors.

What is the result of walking 4 km east and then 3 km west? You have walked a total distance of 7 km, but how far are you from your original location? Let each distance be represented by a vector (Figure 2–12a). When adding 4 + 3 vectorially, place the vectors tip to tail. The resultant is a vector from the original point to the final point. In this case, R = 1 km in an easterly direction.

Suppose that you had walked 4 km east and 3 km north. Again, the distances would be represented by vectors and added by being placed tip to tail (Figure 2–12b). The resultant vector drawn from the original point to the final point is the hypotenuse of a right-angle triangle. For this case, R = 5 km in a northeast direction.

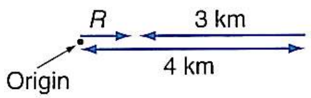

FIGURE 2–12a

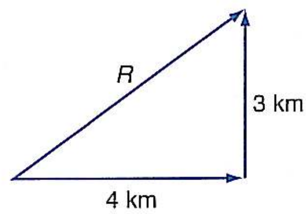

FIGURE 2–12b

2–4 VECTOR ADDITION: GRAPHICAL (TIP TO TAIL)

As was described in Section 2–3, the sum of two or more vectors is a resultant. The vector quantity used there was *distance;* other vector quantities often encountered are *force, velocity,* and *acceleration.* Force vectors are going to be our main concern in static mechanics.

Graphical vector addition requires the drawing of the vectors to some scale in their given direction. The resultant can then be measured or scaled from the drawing. This method of scale drawings will not be used to any extent in this book since drafting equipment is required and the accuracy of the solution is somewhat dependent on the scale used. However, graphical vector addition is still important, because the same drawings or sketches are required for the analytical method. When employing the analytical method, one uses the same sketches drawn roughly to scale and then solves mathematically.

The three methods or rules of vector addition are the *triangle, parallelogram,* and *vector polygon* methods. The triangle and parallelogram methods are simplified versions of the vector polygon method. The triangle method of vector addition can be used for a right-angle triangle (Figure 2–13a) or any other triangle (Figure 2–13b). Constructing A and B (Figure 2–13a) to a scale of 1 cm = 1 N, we can calculate the length of R to be 13 cm. R is therefore equal to 13 N. (Vectors are drawn tip to tail in any sequence.) Using a scale of 1 cm = 10 N (Figure 2–13b) and drawing B at the correct slope, we find the length of R to be 5.2 cm or R = 52 N. (Vector triangles are not always right-angle triangles and will therefore require different mathematics when being solved analytically.)

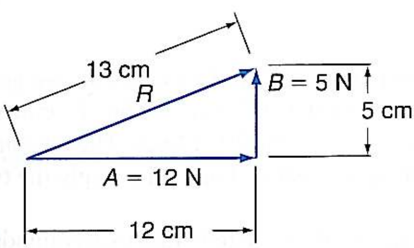

FIGURE 2–13a

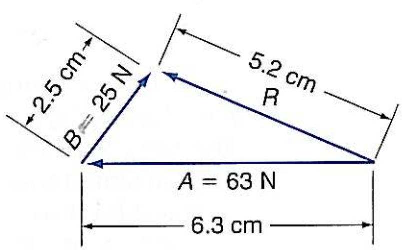

FIGURE 2–13b

EXAMPLE 2–1

Forces of 20 N and 30 N are pulling on a ring (Figure 2–14a). Determine the resultant using the triangle rule.

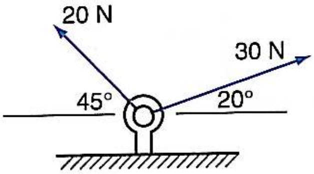

FIGURE 2–14a

Employing an appropriate scale such as 1 cm = 2 N and the angles of 20° and 45° as given, one can construct a vector triangle (Figure 2–14b or c), giving an answer of R = 28.2 N ∠60°. **(Note that it is immaterial whether 20 N is added to 30 N or 30 N is added to 20 N.)**

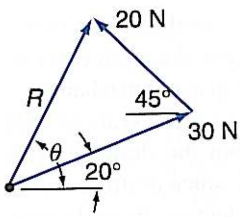

FIGURE 2–14b

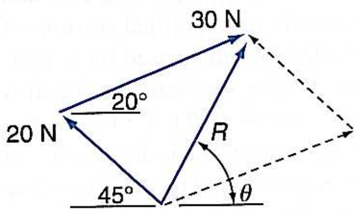

FIGURE 2–14c

Applying the parallelogram rule to the forces in Example 2–1, we get the parallelogram shown in Figure 2–15. From the top of the 20-N force, a line is drawn parallel to the 30-N force. Similarly, a line is drawn parallel to the 20-N force. The diagonal of the parallelogram formed represents the resultant. The parallelogram is simply the two triangles of the triangle method.

The vector polygon is a continuation of the triangle rule to accommodate more than two forces. Several vectors are added tip to tail—the sequence of the addition is not important. The resultant R is a vector from the origin of the polygon to the tip of the last vector (Figure 2–16).

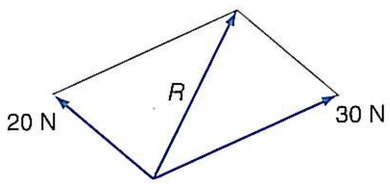

FIGURE 2–15

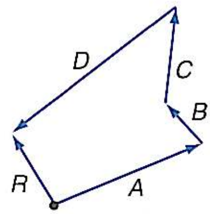

FIGURE 2–16

2–5 VECTOR ADDITION: ANALYTICAL

The solutions to mechanics problems are no exception to the old saying, "A picture is worth a thousand words." Sketches are vitally important in many cases; calculations should be accompanied by a sketch drawn as closely to scale as a little care will allow. A calculated answer can be visually checked for an obvious error in direction or magnitude. Analytical vector addition consists of two main methods:

1. Construction of a triangle and use of the cosine law or other simple trigonometric functions.
2. Addition of the components of vectors (Section 2–7).

EXAMPLE 2–2

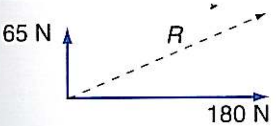

FIGURE 2–17a

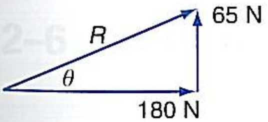

FIGURE 2–17b

Determine the resultant of the vectors shown in Figure 2–17a. The resultant is easily obtained since the vectors, when added, form a right-angle triangle, as shown in Figure 2–17b.

$$R^2 = (180\text{ N})^2 + (65\text{ N})^2$$

$$R = \sqrt{32{,}400 + 4225}$$

$$= \sqrt{36{,}625}$$

$$R = 191\text{ N}$$

$$\tan\theta = \frac{65}{180} = 0.361$$

$$\theta = 19.8°$$

The final answer is expressed as

$$\underline{R = 191\text{ N} \nearrow 19.8°}$$

An alternative method would be

$$\tan\theta = 0.361$$
$$\theta = 19.8^\circ$$
$$\sin\theta = \frac{65\text{ N}}{R}$$
$$R = \frac{65}{\sin 19.8^\circ}$$
$$\underline{R = 191\text{ N} \nearrow 19.8^\circ}$$

Note that with this method, the value of R depends on a correct initial calculation of the value of θ. In the first solution, neither R nor θ was dependent on the other. **Thus, the first method is preferred so that one mistake at the beginning does not make remaining calculations incorrect.**

EXAMPLE 2–3

Solve for the resultant of the force system shown in Figure 2–18a acting on point A.

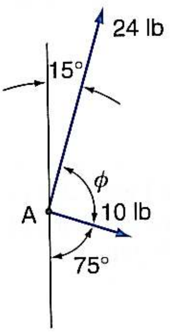

FIGURE 2–18a

$$\phi = 180 - 75 - 15 = 90^\circ$$

Therefore, the vector triangle is a right-angle triangle and

$$R = \sqrt{(24\text{ lb})^2 + (10\text{ lb})^2}$$
$$R = 26\text{ lb}$$
$$\tan\theta = \frac{10}{24} = 0.416$$
$$\theta = 22.6^\circ$$

Now find the angle between R and the horizontal plane.

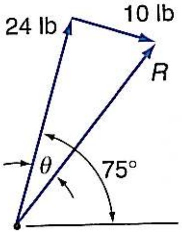

FIGURE 2–18b

$$75 - \theta = 75 - 22.6 \quad \text{(Figure 2–18b)}$$
$$= 52.4^\circ$$
$$\underline{R = 26\text{ lb} \angle 52.4^\circ}$$

EXAMPLE 2–4

Find the resultant in Figure 2–19a.

Sketch a vector triangle (Figure 2–19b), adding the vectors tip to tail and labeling all forces and angles as you construct the triangle. Use the cosine law.

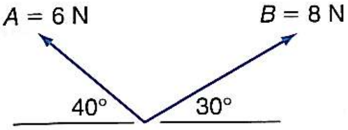

FIGURE 2–19a

$$R^2 = A^2 + B^2 - 2AB\cos\alpha$$
$$= (6\text{ N})^2 + (8\text{ N})^2 - 2(6\text{ N})(8\text{ N})(\cos 70^\circ)$$
$$= 36 + 64 - 96(0.342)$$
$$= 100 - 32.8$$
$$R = \sqrt{67.2}$$
$$R = 8.2\text{ N}$$

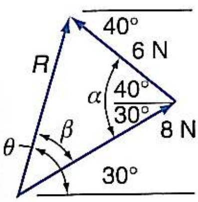

FIGURE 2–19b

To show a direction for R, we must solve for $\theta = \beta + 30^\circ$. By the sine law

$$\frac{6\text{ N}}{\sin\beta} = \frac{8.2\text{ N}}{\sin 70^\circ}$$
$$\sin\beta = \frac{6(0.94)}{8.2}$$
$$= 0.688$$
$$\beta = 43.5^\circ$$
$$\theta = 43.5 + 30 = 73.5^\circ$$

Our final answer is

$$\underline{R = 8.2\text{ N} \nearrow 73.5^\circ}$$

2–6 COMPONENTS

Previously, our main concern was the addition of two or more vectors to obtain a single vector, the resultant. *Resolution* of a vector into its components is the reverse of adding to get the resultant. A single force can be broken up into two separate forces. This is known as resolution of a force into its components. It is often convenient in problem solutions to be concerned only with forces in either the vertical or the horizontal direction. Therefore, the x-y axis system is used: A component in the horizontal direction has a subscript x, and a component in the vertical direction has a subscript y.

EXAMPLE 2–5

Determine the horizontal and vertical components of P (Figure 2–20a).

By constructing a parallelogram or rectangle, we are left with a right-angle triangle in which

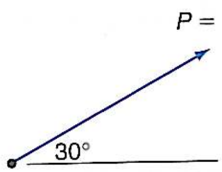

FIGURE 2–20a

$$\sin\theta = \frac{P_y}{P} \qquad \text{(Figure 2–20b)}$$

$$P_y = (200\text{ N})(\sin 30°)$$
$$= 200(0.5)$$
$$\underline{P_y = 100\text{ N}\uparrow}$$

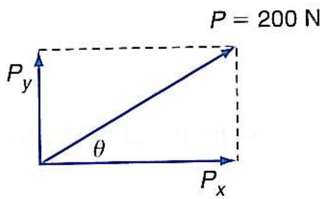

FIGURE 2–20b

$$\cos\theta = \frac{P_x}{P}$$

$$P_x = (200\text{ N})(\cos 30°)$$
$$= 200(0.866)$$
$$\underline{P_x = 173\text{ N}\rightarrow}$$

EXAMPLE 2–6

Determine the horizontal and vertical components when the direction of P is shown as a slope (Figure 2–21a).

Construct the vector triangle (Figure 2–21b). Notice that we have two similar triangles. The hypotenuse of the small triangle $= \sqrt{(4)^2 + (3)^2} = 5$. Therefore,

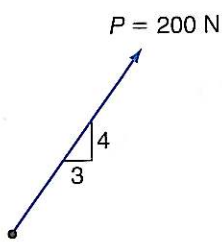

FIGURE 2–21a

$$\frac{P_y}{4} = \frac{P}{5} \quad \text{and} \quad \frac{P_x}{3} = \frac{P}{5}$$

or

$$P_y = \frac{4}{5}P \qquad P_x = \frac{3}{5}P$$

$$= \frac{4}{5}(200\text{ N}) \qquad = \frac{3}{5}(200\text{ N})$$

$$\underline{P_y = 160\text{ N}\uparrow} \qquad \underline{P_x = 120\text{ N}\rightarrow}$$

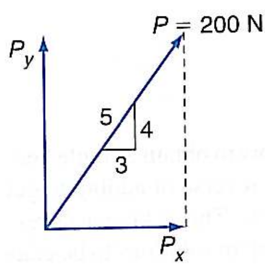

FIGURE 2–21b

There are three main combinations of slope numbers that produce a whole number for the hypotenuse of a right-angle triangle. These combinations are 3, 4, 5; 5, 12, 13; and 8, 15, 17, or multiples thereof.

The following figures (Figure 2–22a, 2–22b, and 2–22c) illustrate the use of these combinations in calculating components. Each component is a fraction or ratio of the total as given by the slope numbers.

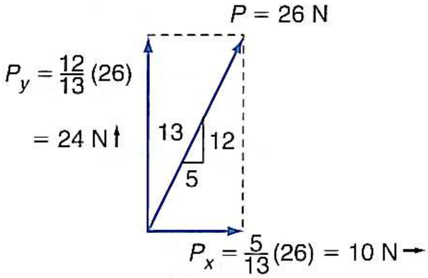

FIGURE 2–22a

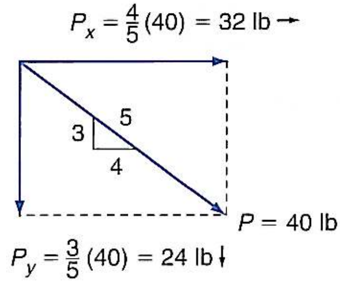

FIGURE 2–22b

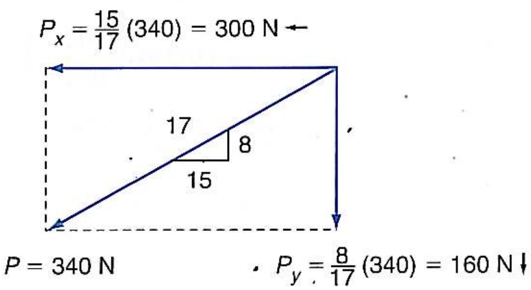

FIGURE 2–22c

The components are not always in the horizontal and vertical directions, nor are they always at right angles to one another—as illustrated in the following example.

EXAMPLE 2–7

Find the components of force Q for the axis system of A and B as shown in Figure 2–23a.

Construct a vector parallelogram by drawing lines, parallel to axes A and B, from the tip of Q (Figure 2–23b). Applying the sine law to the left half of the parallelogram, we obtain

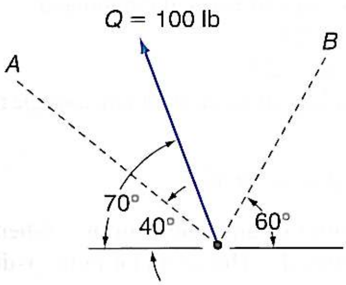

FIGURE 2–23a

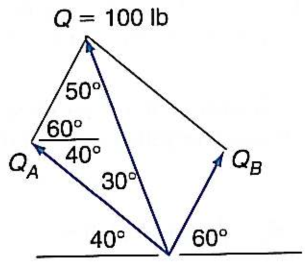

FIGURE 2–23b

$$\frac{Q_A}{\sin 50°} = \frac{Q}{\sin 100°}$$

$$Q_A = \frac{Q \sin 50°}{\sin 100°}$$

$$= \frac{Q \sin 50°}{\sin 80°}$$

$$= (100 \text{ lb})\left(\frac{0.766}{0.985}\right)$$

$$\underline{Q_A = 77.8 \text{ lb } 40°}$$

$$\frac{Q_B}{\sin 30°} = \frac{Q}{\sin 100°}$$

$$Q_B = \frac{Q \sin 30°}{\sin 80°}$$

$$= (100 \text{ lb})\left(\frac{0.5}{0.985}\right)$$

$$\underline{Q_B = 50.8 \text{ lb } 60°}$$

2–7 VECTOR ADDITION: COMPONENTS

In Section 2–4 a graphical solution was shown where several vectors were added by the use of a vector polygon. To add these vectors analytically using the method of components, proceed according to the following steps:

1. Resolve each vector into a horizontal and vertical component.
2. Add the vertical components, $R_y = \Sigma F_y$.
3. Add the horizontal components, $R_x = \Sigma F_x$.
4. Combine the horizontal and vertical components to obtain a single resultant vector.

$$R = \sqrt{(R_x)^2 + (R_y)^2}$$

Note: The Greek capital letter Σ (sigma) means "the sum of." When writing $R_y = \Sigma F_y$, it is recommended that you say to yourself, "The resultant in the y-direction equals the sum of the forces in the y-direction."

EXAMPLE 2–8

Find the resultant of forces P and Q as shown in Figure 2–24. Using the x- and y-axes for the algebraic signs of the components, we have

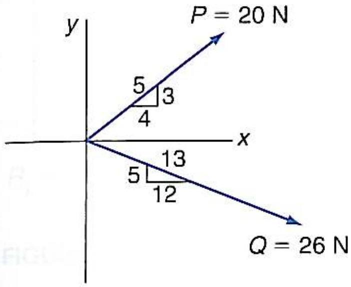

FIGURE 2–24

$$P_y = \frac{3}{5}(20\text{ N}) \qquad P_x = \frac{4}{5}(20\text{ N})$$

$$P_y = +12\text{ N} \qquad P_x = +16\text{ N}$$

$$Q_y = -\frac{5}{13}(26\text{ N}) \qquad Q_x = \frac{12}{13}(26\text{ N})$$

$$Q_y = -10\text{ N} \qquad Q_x = +24\text{ N}$$

Now sum the y and x components

$$R_y = 12\text{ N} - 10\text{ N} \qquad R_x = 16\text{ N} + 24\text{ N}$$

$$R_y = 2\text{ N} \qquad R_x = 40\text{ N}$$

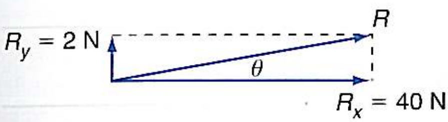

FIGURE 2–25

$$R = \sqrt{(40\text{ N})^2 + (2\text{ N})^2} \quad \text{(Figure 2–25)}$$

$$R = 40.1\text{ N}$$

$$\tan\theta = \frac{2}{40} = 0.05$$

$$\theta = 2.9^\circ$$

$$\underline{R = 40.1\text{ N} \nearrow 2.9^\circ}$$

When determining the resultant of several forces, one may find it more convenient to tabulate all the components as follows:

Force	X	Y
P	+16 N	+12 N
Q	+24 N	−10 N
	R_x = +40 N	R_y = +2 N

EXAMPLE 2–9

Determine the resultant of the forces shown in Figure 2–26.

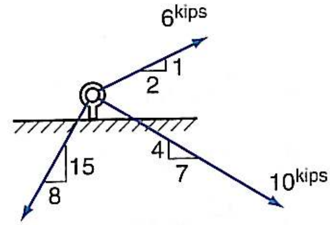

FIGURE 2–26

For easier visualization, the components of each force can be drawn as in Figure 2–27. Tabulate the components as follows:

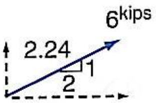

FIGURE 2–27a

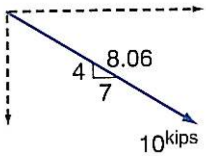

FIGURE 2–27b

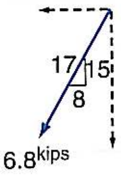

FIGURE 2–27c

Force (kips)	X	Y
6^{kips}	$\frac{2}{2.24} \times 6^{kips} = +5.35^{kips}$	$\frac{1}{2.24} \times 6^{kips} = +2.68^{kips}$
10^{kips}	$\frac{7}{8.06} \times 10^{kips} = 8.69^{kips}$	$\frac{4}{8.06} \times 10^{kips} = -4.97^{kips}$
6.8^{kips}	$\frac{8}{17} \times 6.8^{kips} = \underline{-3.2^{kips}}$	$\frac{15}{17} \times 6.8^{kips} = \underline{-6^{kips}}$
	$R_x = +10.84^{kips}$	$R_y = -8.29^{kips}$