Tutorial Letter 203/0/2017

PRECALCULUS MATHEMICS

MAT1510

Year module

Department of Mathematical Sciences

This tutorial letter contains solutions to assignment 3

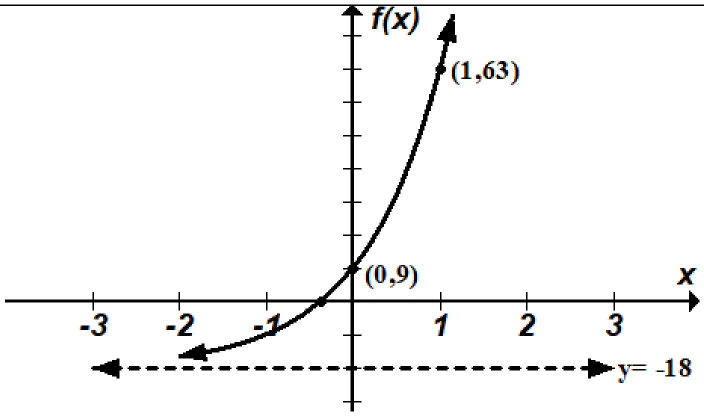
BARCODE

Define tomorrow.

UNISA university of south africa

2017 MAT1510 ASSIGNMENT 3: SOLUTIONS

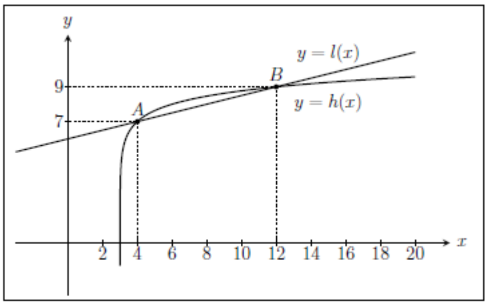
Q1.1	**(1.1)** Rewrite the expression $$\ln\sqrt{\frac{(x^3+1)^2(x+2)}{(x^2+3)^4}}$$ in a form with no logarithms of products, quotients, or powers. (4)
Solution	$\ln\sqrt{\dfrac{\left(x^3+1\right)^2\left(x+2\right)}{\left(x^2+3\right)^4}}$ $=\ln\dfrac{\left(x^3+1\right)\left(x+2\right)^{\frac{1}{2}}}{\left(x^2+3\right)^2}$ $=\ln\left(x^3+1\right)+\frac{1}{2}\ln\left(x+2\right)-2\ln\left(x^2+3\right)$
Q1.2	**(1.2)** Rewrite the expression (3) $$2\log(x+4)+4\log(x-3)-\frac{1}{2}\log(x+2)$$ as a single logarithm.
Solution	$2\log\left(x+4\right)+4\log\left(x-3\right)-\frac{1}{2}\log\left(x+2\right)$ $=\log\left(x+4\right)^2+\log\left(x-3\right)^4-\log\sqrt{\left(x+2\right)}$ $=\log\dfrac{\left(x+4\right)^2\left(x-3\right)^4}{\sqrt{\left(x+2\right)}}$

Q2	**Question 2: 19 Marks** Suppose the functions f and g are defined by $$y = f(x) = 3^{x+3} - 18 \quad \text{and} \quad y = g(x) = 3^x$$ respectively.
2.1	**(2.1)** Explain the steps of the transformation process that you would apply to the graph of g to obtain the graph of f. (2)
Solution	Step 1: Shift the graph 3 units horizontally to the left Step 2: Shift the resulting graph vertically 18 units down.
2.2	**(2.2)** Write down the sets that represent the domain and the range of the function f, and the equation of the asymptote of the graph of f. (3)
Solution	$Domain_f: \; x \in \mathbb{R}$ $\qquad Range_f: \; y \in \left(-18, \infty\right)$ $\qquad asymptote_f: \; y = -18$
2.3	**(2.3)** Sketch the graph of f, including the asymptote represented by a broken line. Label the horizontal and vertical axes, and the $x-$ and $y-$intercepts (if any exist). (**Hint:** The scale on the $y-$axis needs to be much smaller than the scale on the $x-$axis.) (4)
Solution	
2.4	**(2.4)** Determine the equation that defines the inverse function f^{-1}. (4)
Solution	$f(x) = 3^{x+3} - 18$ $\Rightarrow 3^{y+3} - 18 = x$ $\Rightarrow 3^{y+3} = x + 18$ $\Rightarrow y + 3 = \log_3\left(x+18\right)$ $\Rightarrow y = f^{-1}(x) = \log_3\left(x+18\right) - 3$
2.5	**(2.5)** Show that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ (6)
Solution	$\left(f \circ f^{-1}\right)(x) = x$ and $\left(f^{-1} \circ f\right)(x) = x$

	$\left(f \circ f^{-1}\right)(x)$ $= f\left[f^{-1}(x)\right]$ $= f\left[\log_3\left(x+18\right)-3\right]$ $= 3^{\left[\log_3\left(x+18\right)-3\right]+3} - 18$ $= 2^{\log_3(x+18)} - 18$ $= x+18-18 = x$ and $\left(f^{-1} \circ f\right)(x)$ $= f^{-1}\left[f(x)\right]$ $= f^{-1}\left[3^{x+3} - 18\right]$ $= \log_3\left[\left(3^{x+3} - 18\right)+18\right]-3$ $= \log_3\left[3^{x+3}\right]-3$ $= \left[x+3\right]-3 = x$

Q3

Question 3: 19 Marks



The sketch shows the graphs of the functions h and l defined by

$$y = h(x) = \log_3(x-p) + q \quad \text{and} \quad h = l(x) = mx + c.$$

The two graphs intersect at the points A and B.

Q3.1

(3.1) Use the graph to determine p and q, and hence write down the equation that defines h. (7)

Solution

$A(4,7)$ and $B(12,9)$ substitute in $h(x) = \log_3(x-p)+q$

$h(4) = \log_3(4-p)+q = 7$

$\Rightarrow \log_3(4-p) = 7-q \quad \Rightarrow 4-p = 3^{7-q} \quad \Rightarrow p = 4-3^{7-q}$

$h(12) = \log_3(12-p)+q = 9$

$\Rightarrow \log_3(12-p) = 9-q \quad \Rightarrow 12-p = 3^{9-q} \quad \Rightarrow p = 12-3^{9-q}$

$\therefore 4-3^{7-q} = 12-3^{9-q} \quad \Rightarrow -3^{7-q}+3^{9-q} = 12-4 \quad \Rightarrow 3^{7-q}-3^{9-q} = -8$

$\Rightarrow 3^{-q}\left(3^7-3^9\right) = -8 \quad \Rightarrow 3^{-q}(-17496) = -8\frac{1}{2} \quad \Rightarrow 3^{-q} = \frac{1}{2187} = 3^{-7}$

$\therefore q = 7 \qquad \therefore p = 4-3^{7-7} = 3 \qquad \therefore h(x) = \log_3(x-3)+7$

Q3.2

(3.2) Write down the sets that represent the domain and the range of the function h, and the equation of the asymptote of the graph of h. (3)

Solution

$Domain_h: \quad x \in (3,\infty) \qquad Range_h: \quad y \in \mathbb{R} \qquad asymptote: \quad x = 3$

Q3.3

(3.3) Describe the steps of the transformation process that you would apply to the graph of h to obtain the graph of $y = \log_3 x$. (2)

Solution

Step1: Shift the graph vertically 7 units down

Step2: Then shift the graph 3 units to the left.

Q3.4

(3.4) Use the graph to determine m and c, and hence write down the equation that defines l. (5)

Solution

$l(x) = mx + c$

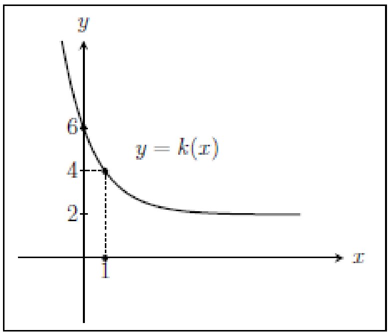
$l(4) = m(4)+c = 7 \Rightarrow \quad 4m+c = 7$

$l(12) = m(12)+c = 9 \Rightarrow \quad 12m+c = 9$

$\therefore m = \frac{1}{4} \qquad \therefore c = 6 \qquad \therefore l(x) = \frac{1}{4}x+6$

Q3.5	**(3.5)** **Use the graphs** of h and l (**not** the algebraic expressions for $h(x)$ and $l(x)$) to solve the inequality $l(x) - h(x) \leq 0$. (2)
Solution	$l(x)-h(x)\leq 0 \quad \Rightarrow l(x)\leq h(x)$ $\therefore x\in\left[4,12\right]$

Q4	**Question 4: 27 Marks** Suppose the functions f, g and h are defined by $$f(x) = 2^{4x} + 3 \cdot 2^{2x+1} \quad \text{and} \quad g(x) = \log_3 x + \log_3(x+6) \quad \text{and} \quad h(x) = e^{4x} - 10e^{2x}$$ respectively.
4.1	**(4.1)** Write down the sets D_f, D_g and D_h that represent the domains of f, g and h respectively. (3)
Solution	$D_f: \quad x \in \mathbb{R}$ $D_g: \quad x \in (0, \infty)$ $D_h: \quad x \in \mathbb{R}$
4.2	**(4.2)** Solve the equation $f(x) = 16$ for x. (8)
Solution	$2^{4x} + 3 \cdot 2^{2x+1} = 16 \Rightarrow 2^{4x} + 6 \cdot 2^{2x} - 16 = 0$ $\Rightarrow (2^{2x} + 8)(2^{2x} - 2) = 0$ $\Rightarrow 2^{2x} \neq= 8 \quad \text{or} \quad 2^{2x} = 2$ $\Rightarrow x = \frac{1}{2}$
4.3	**(4.3)** Solve the inequality $g(x) \geq 3$ for x. (8)
solution	$\log_3 x + \log_3(x+6) \geq 3$ $\Rightarrow \log_3(x^2 + 6x) \geq \log_3 27$ $\Rightarrow x^2 + 6x - 27 \geq 0$ $\Rightarrow (x+9)(x-3) \geq 0$ $\Rightarrow x \leq -9 \quad \text{or} \quad x \geq 3 \quad \text{but } D_f : x > 0$ $\therefore x \geq 3$
4.4	**(4.4)** Solve $h(x) = -9$ for x. Leave the answer in terms of ln, where necessary. (8)
	$e^{4x} - 10e^{2x} = -9 \Rightarrow e^{4x} - 10e^{2x} + 9 = 0$ $\Rightarrow (e^{2x} - 9)(e^{2x} - 1) = 0$ $\Rightarrow e^{2x} = 9 \quad \text{or} \quad e^{2x} = 1$ $\Rightarrow 2x = \ln 9 \quad \text{or} \quad 2x = 0$ $\Rightarrow x = \ln 3 \quad \text{or} \quad x = 0$

Q5	**Question 5: 10 Marks** 
Q5.1	**(5.1)** The sketch shows the graph of the function k defined by (6) $$y = k(x) = a^{-x+2} + b.$$ Use the graph to determine a and b, and hence write down the equation that defines k.
Solution	substitute points (1,4) and (0,6) into $k(x) = a^{-x+2} + b$ $k(0) = a^{-0+2} + b = 6 \quad \Rightarrow a^2 + b = 6$ $k(1) = a^{-1+2} + b = 4 \quad \Rightarrow a + b = 4$ $\therefore a^2 - a = 2 \Rightarrow (a-2)(a+1) = 0 \Rightarrow a = 2$ or $a \neq -1$ $\therefore b = 2 \quad \therefore k(x) = 2^{-x+2} + 2$
Q5.2	**(5.2)** What is the equation of the graph that is obtained by reflecting the graph of k around the line $y = x$? (4)
solution	reflecting in the line $y = x$ requires finding $k^{-1}(x)$ $\therefore 2^{-y+2} + 2 = x$ $\Rightarrow 2^{-y+2} = x - 2$ $\Rightarrow -y + 2 = \log_2(x-2)$ $\Rightarrow -y = \log_2(x-2) - 2$ $\Rightarrow y = k^{-1}(x) = -\log_2(x-2) + 2$

Q6	**Question 6: 10 Marks** A roasted chicken is taken from an oven when its temperature has reached $95°C$ and is placed on a table in a room where the temperature is $20°C$. It starts to cool according to Newton's Law of Cooling. If the temperature of the chicken is $65°C$ after half an hour, what is the temperature after 45 minutes? Leave the answer in surd form.
Solution	$T(t) = T_s + D_0 \cdot e^{-kt}$ $T(30) = 65°C, \quad T_s = 20°C, \quad D_0 = (95-20)°C = 75°C$ $T(30) = 20 + 75 \cdot e^{-k(30)} = 65$ $e^{-k(30)} = \dfrac{65-20}{75} = \frac{3}{5}$ $k = \ln\frac{3}{5} \div -30 = -\dfrac{\ln\frac{3}{5}}{30}$ $\therefore T(t) = 20 + 75 \cdot e^{\frac{\ln\frac{3}{5}}{30}t}$ $\therefore T(45) = 20 + 75 \cdot e^{\frac{\ln\frac{3}{5}}{30}(45)}$ $= 20 + 75 \cdot e^{1.5\ln\frac{3}{5}} = 20 + 75 \cdot e^{\ln\frac{3}{5}^{1.5}}$ $= 20 + 75\left(\sqrt{\frac{3}{5}^3}\right) = 20 + 75\left(\frac{3}{25}\sqrt{15}\right) = \left(20 + 9\sqrt{15}\right)°C$

Q7	A culture starts with 680 bacteria. After just 30 minutes the count is 3 400.
Q7.1	**(7.1)** Find a formula for the number of bacteria after t hours. (5)
Solution	This is an example of exponential growth. So we use the equation of the form $n(t) = n_0 e^{rt}$ Where $n_0 = 680$ $n\left(\frac{1}{2}\right) = 3400$ If we now substitute in the equation we get: $n(\frac{1}{2}) = 680e^{r(\frac{1}{2})} = 3400$ $\Rightarrow e^{\frac{1}{2}r} = \frac{3400}{680} = 5$ We now need to solve for r using logarithms (in this case use ln) $e^{\frac{1}{2}r} = 5$ $\Rightarrow \ln\left(e^{\frac{1}{2}r}\right) = \ln 5$ $\Rightarrow \frac{1}{2}r = \ln 5$ $\Rightarrow r = 2\ln 5 = \ln 25$ This gives us the required equation: $n(t) = 680e^{(\ln 25)t} = 680e^{t\ln 25}$
Q7.2	**(7.2)** Use the formula to find the number of bacteria after 1 hour. (4)
Solution	$n(1) = 680e^{\ln 25} = 680 \times 25 = 17000$
Q7.3	**(7.3)** After how many minutes will the number of bacteria triple? Leave the answer in terms of ln but in simplified form. (5)
Solution	$680e^{t\ln 25} = 3 \times 680$ $\Rightarrow e^{t\ln 25} = 3$ $\Rightarrow t\ln 25 = \ln 3$ $\therefore t = \frac{\ln 3}{\ln 25}$ hours $\therefore t = \frac{60\ln 3}{\ln 5^2}$ minutes $= \frac{30\ln 3}{\ln 5}$ minutes

Q8	**Question 8: 14 Marks** The half-life of Radium−221 is 30 seconds. Suppose we have a 320 g sample.
8.1	**(8.1)** Find a formula for the mass remaining after t seconds. (4)
Solution	Find the equation in the form $Q(x) = a.e^{-kt}$ Intial mass (a) = 320g $Q(\mathrm{t}) = 320.e^{-kt}$ *Now* after 30 seconds the mass is 160g $160 = 320.e^{-k(30)}$ $e^{-30k} = 0.5$ $-30k = \ln 0.5$ $k = \frac{\ln 0.5}{-30} = 0.023104...$ $\therefore Q(\mathrm{t}) = 320 \cdot e^{-0.0231t}$
8.2	**(8.2)** Use the formula to determine how much of the sample remains after 210 seconds. (4)
Solution	$Q(\mathrm{t}) = 320 \cdot e^{-0.0231t}$ to find the mass: $Q(210) = 320 \cdot e^{-0.0231(210)} g \approx 2.503g$
8.3	**(8.3)** After how many minutes will only 10 g remain? (6)
Solution	$Q(\mathrm{t}) = 320 \cdot e^{-0.0231t}$ to find the time: $Q(t) = 320 \cdot e^{-0.0231(t)} = 10$ $e^{-0.0231(t)} = \frac{10}{320}$ $\therefore -0.0231(t) = \ln \frac{1}{32}$ $\therefore t = \ln \frac{1}{32} \div (-0.0231) \approx 150.032 \sec$

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