Tutorial letter 201/1/2018

Statistical Inference III

STA3702

Semester 1

Department of Statistics

SOLUTIONS TO ASSIGNMENT 01,
SOLUTIONS TO ASSIGNMENT 02,
TRIAL PAPER AND SOLUTIONS

Define tomorrow.

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SOLUTIONS TO ASSIGNMENT 01

Question 1 **[13]**

(a) (i) $E(X) = \theta p$ and $E(X^2) = Var(X) + [E(X)]^2 = \theta p(1-p) + \theta^2 p^2$. **(1+2=3)**

(b) $\bar{X}$ and $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ correspond to $E(X)$ and $E(X^2)$, respectively. **(1+1=2)**

(c) $\bar{X} = \tilde{\theta}\tilde{p}$ and $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \tilde{\theta}\tilde{p}(1-\tilde{p}) + \tilde{\theta}^2\tilde{p}^2$ where $\tilde{\theta}$ and $\tilde{p}$ are method of moments estimators of θ and p, respectively. **(1+1=2)**

(d) $\tilde{p} = \frac{\bar{X}}{\tilde{\theta}} \Longrightarrow \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \tilde{\theta}\frac{\bar{X}}{\tilde{\theta}}\left(1 - \frac{\bar{X}}{\tilde{\theta}}\right) + \tilde{\theta}^2\frac{\bar{X}^2}{\tilde{\theta}^2} = \bar{X}\left(1 - \frac{\bar{X}}{\tilde{\theta}}\right) + \bar{X}^2 = \bar{X} - \frac{\bar{X}^2}{\tilde{\theta}} + \bar{X}^2 \Longrightarrow$
$\frac{\bar{X}^2}{\tilde{\theta}} = \bar{X} - \frac{1}{n}\sum_{i=1}^{n} X_i^2 + \bar{X}^2 \Longrightarrow \tilde{\theta} = \dfrac{\bar{X}^2}{\bar{X} - \frac{1}{n}\sum_{i=1}^{n} X_i^2 + \bar{X}^2}$. **(6)**

Question 2 **[22]**

(a) **(4)**

(i) The likelihood function of μ and λ is

$$\begin{aligned}
L(\mu, \lambda|\mathbf{x}) &= \prod_{i=1}^{n} f(x_i|\mu, \lambda) \\
&= \prod_{i=1}^{n}\left(\frac{\lambda}{2\pi x_i^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2}\left(\frac{\sqrt{x_i}}{\mu} - \frac{1}{\sqrt{x_i}}\right)^2\right\} \\
&= (\lambda/2\pi)^{n/2}\left(\prod_{i=1}^{n} x_i\right)^{-3/2} \exp\left\{-\frac{\lambda}{2}\sum_{i=1}^{n}\left(\frac{\sqrt{x_i}}{\mu} - \frac{1}{\sqrt{x_i}}\right)^2\right\}
\end{aligned}$$

(2)

(ii) The log likelihood function of μ and λ is

$$\begin{aligned}
l(\mu, \lambda) &= \ln L(\mu, \lambda|\mathbf{x}) \\
&= -\frac{n}{2}\ln(2\pi) + \frac{n}{2}\ln\lambda - \frac{3}{2}\sum_{i=1}^{n}\ln x_i - \frac{\lambda}{2}\sum_{i=1}^{n}\left(\frac{\sqrt{x_i}}{\mu} - \frac{1}{\sqrt{x_i}}\right)^2
\end{aligned}$$

(2)

(b) **(16)**

Differentiating $l(\mu, \lambda)$ with respect to μ and equating the derivative to zero obtains:

$$0 = l'(\hat{\mu}, \lambda) = \frac{\lambda}{\hat{\mu}^2} \sum_{i=1}^{n} \left(\frac{\sqrt{X_i}}{\hat{\mu}} - \frac{1}{\sqrt{X_i}} \right) \sqrt{X_i} = \frac{\lambda}{\hat{\mu}^2} \sum_{i=1}^{n} \left(\frac{X_i}{\hat{\mu}} - 1 \right)$$

where $\hat{\mu}$ is the maximum likelihood estimator of μ. The solution for $\hat{\mu}$ in the equation is $\hat{\mu} = \bar{X}$.

(8)

Differentiating $l(\hat{\mu}, \lambda)$ with respect to λ and equating the derivative to zero obtains:

$$0 = l'(\hat{\mu}, \hat{\lambda}) = \frac{n}{2\hat{\lambda}} - \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\sqrt{X_i}}{\hat{\mu}} - \frac{1}{\sqrt{X_i}} \right)^2$$

where $\hat{\lambda}$ is the maximum likelihood estimator of λ. The solution for $\hat{\lambda}$ in the equation is $\hat{\lambda} = \dfrac{n}{\sum_{i=1}^{n} \left(\frac{\sqrt{X_i}}{\bar{X}} - \frac{1}{\sqrt{X_i}} \right)^2}$. **(8)**

(c) By the invariance property of maximum likelihood estimators, the maximum likelihood estimator of $\dfrac{1}{\lambda}$ is

$$\frac{1}{\hat{\lambda}} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\sqrt{X_i}}{\bar{X}} - \frac{1}{\sqrt{X_i}} \right)^2 .$$

(2)

Question 3 **[20]**

(a) **(6)**

(i) From given, $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Longrightarrow n - 1 = E\left(\dfrac{(n-1)S^2}{\sigma^2}\right) = \dfrac{(n-1)}{\sigma^2} E(S^2) \Longrightarrow E(S^2) = \sigma^2$. **(3)**

(ii) From given, $\dfrac{n\bar{S}^2}{\sigma^2} \sim \chi^2_{n-1} \Longrightarrow n - 1 = E\left(\dfrac{n\bar{S}^2}{\sigma^2}\right) = \dfrac{n}{\sigma^2} E\bar{(}S^2) \Longrightarrow E(\bar{S}^2) = \dfrac{n-1}{n}\sigma^2 \neq \sigma^2$. **(3)**

(b) **(8)**

(i) From given, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \Longrightarrow 2(n-1) = Var\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4}Var(S^2) \Longrightarrow Var(S^2) = \frac{2}{n-1}\sigma^4.$ **(4)**

(ii) From given, $\frac{n\bar{S}^2}{\sigma^2} \sim \chi^2_{n-1} \Longrightarrow 2(n-1) = Var\left(\frac{n\bar{S}^2}{\sigma^2}\right) = \frac{n^2}{\sigma^4}Var(\bar{S}^2) \Longrightarrow Var(\bar{S}^2) = \frac{2(n-1)}{n^2}\sigma^4.$ **(4)**

(c) **(6)**

That $E(S^2) = \sigma^2$ and $\lim_{n\to\infty} Var(S^2) = \lim_{n\to\infty} \frac{2\sigma^4}{n-1} = 0$ implies that S^2 is a consistent estimator of σ^2. (2)

That $\lim_{n\to\infty} E(\bar{S}^2) = \lim_{n\to\infty} \frac{n-1}{n}\sigma^2 = \sigma^2,$ (2)

and $\lim_{n\to\infty} Var(\bar{S}^2) = \lim_{n\to\infty} \frac{2(n-1)\sigma^4}{n^2} = 0$ implies that $\bar{S}^2$ is a consistent estimator of σ^2.

(2)

Question 4 **[35]**

(a) **(16)**

(i) The likelihood function of θ is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta} \left(\prod_{i=1}^{n} x_i\right)^{-1}$$

and the log likelihood function of θ is

$$l(\theta) = \ln L(\theta|\mathbf{x}) = n\ln\theta + \theta\sum_{i=1}^{n}\ln x_i - \ln\prod_{i=1}^{n} x_i.$$

Now

$$l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\ln x_i \text{ and } l''(\theta) = -\frac{n}{\theta^2} \Longrightarrow \mathbf{I}(\theta) = -l''(\theta) = \frac{n}{\theta^2}.$$

(8)

(ii) The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\hat{\theta}) = \frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \ln X_i.$$

The solution is $\hat{\theta} = \dfrac{-n}{\sum_{i=1}^{n} \ln X_i}$.

$$\mathbf{I}(\mathbf{x}) = -l''(\hat{\theta}) = \frac{n}{\hat{\theta}^2}.$$

(5)

(iii) The Cramer-Rao lower bound is $\left[\dfrac{\partial \theta}{\partial \theta}\right]^2 / \mathbf{I}(\theta) = \dfrac{\theta^2}{n}$. **(3)**

(b) **(7)**

From part $(a)(i)$ above:

$$\begin{aligned} L(\theta|\mathbf{x}) &= \left(\prod_{i=1}^{n} x_i\right)^{-1} \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta} \\ &= m_1(\mathbf{x}) \times m_2\left(\theta, \prod_{i=1}^{n} x_i\right) \end{aligned}$$

where $m_1(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{-1}$ and $m_2\left(\theta, \prod_{i=1}^{n} x_i\right) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}$. Hence $\prod_{i=1}^{n} X_i$ is a sufficient statistic for θ by the factorisation theorem. (3)

Now let $Y_1, Y_2, Y_3, ..., Y_n$ be another random sample from the same distribution. Then

$$\frac{L(\theta|\mathbf{x})}{L(\theta|\mathbf{y})} = \left(\prod_{i=1}^{n} \frac{x_i}{y_i}\right)^{-1} \left(\frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} y_i}\right)^{\theta}$$

is independent of θ if $\prod_{i=1}^{n} x_i / \prod_{i=1}^{n} y_i = 1$ equivalently if $\prod_{i=1}^{n} y_i = \prod_{i=1}^{n} x_i$. This means that $\prod_{i=1}^{n} X_i$ is a minimal sufficient statistic for θ. (4)

(c) **(12)**

The maximum likelihood estimate of $\hat{\theta}$ is $\hat{\theta} = \dfrac{-10}{\sum_{i=1}^{10} \ln x_i} = \dfrac{10}{-14.6230} = 0.6839$. (4)

The standard error of $\hat{\theta}$ is $se(\hat{\theta}) = \sqrt{1/I(\mathbf{x})} = \sqrt{\hat{\theta}^2/10} = 0.2163$. (4)

The 95% confidence interval of θ is $.6839 \pm 1.96 \times .2163 = [0.2700; 1.1077]$. (4)

TOTAL: [90]

SOLUTIONS TO ASSIGNMENT 02

Question 1 **[30]**

(a) **(5)**

$$\begin{aligned} f(x|\theta) &= \frac{1}{\sqrt{2\pi\theta}} \exp\left\{-\frac{x^2}{2\theta}\right\} \\ &= \exp\{-\frac{x^2}{2\theta} - \ln\sqrt{2\pi} - \ln\sqrt{\theta}\} \\ &= \exp\{p(\theta)K(x) + S(x) + q(\theta)\} \end{aligned}$$

where $p(\theta) = \frac{1}{2\theta}$, $K(x) = -x^2$, $S(x) = -\ln\sqrt{2\pi}$ and $q(\theta) = -\ln\sqrt{\theta}$.

(b) **(3)**

$-\sum_{i=1}^{n} X_i^2$ is the complete sufficient statistic for θ - a property of the members of the exponential family of distributions.

(c) **(8)**

$$p'(\theta) = -\frac{1}{2\theta^2},\ p''(\theta) = \frac{1}{\theta^3},\ q'(\theta) = -\frac{1}{2\theta},\ q''(\theta) = \frac{1}{2\theta^2}.$$

(2)

$$-E\left[\sum_{i=1}^{n} X_i^2\right] = -n\frac{q'(\theta)}{p'(\theta)} = -n\frac{-1/(2\theta)}{-1/(2\theta^2)} = -n\theta.$$

(2)

$$\begin{aligned} Var\left[-\sum_{i=1}^{n} X_i^2\right] &= \frac{n}{[p'(\theta)]^3}\left\{p''(\theta)q'(\theta) - q''(\theta)p'(\theta)\right\} \\ &= \frac{n}{[-1/(2\theta^2]^3}\left\{(1/\theta)^3 \times (-1/(2\theta)) - (1/(2\theta^2)) \times (-1/(2\theta^2))\right\} \\ &= \frac{n}{[-1/(2\theta^2)]^3}\left\{-\frac{1}{2\theta^4} + \frac{1}{4\theta^4}\right\} = 2n\theta^2 \end{aligned}$$

(4)

(d) **(9)**

The likelihood function is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = (2\pi)^{-n/}\theta^{-n/2}\exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right\}$$

and the log likelihood function is

$$l(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2.$$

(4)

Now,

$$l'(\theta) = -\frac{n}{2\theta} + 12\theta^2\sum_{i=1}^{n} x_i^2 \text{ and } l''(\theta) = \frac{n}{2\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2.$$

(2)

The Fisher information about θ in the sample is

$$I(\theta) = -E\left[l''(\theta)\right] = -\frac{n}{2\theta^2} + \frac{1}{\theta^3}E\left(\sum_{i=1}^{n} X_i^2\right) = -\frac{n}{2\theta^2} + \frac{n\theta}{\theta^3} = \frac{n}{2\theta^2}.$$

(2)

Hence the required Cramer-Rao lower bound is $\frac{1}{I(\theta)} = \frac{2\theta^2}{n}$. (1)

(e) **(5)**

From part (c) above, $E\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] = \theta$. Therefore, $\frac{1}{n}\sum_{i=1}^{n} X_i^2$ is a minimum variance unbiased estimator of θ because it is an unbiased estimator of θ and a function of a complete sufficient statistic for θ. (3+2=5)

Question 2 [25]

(a) **(5)**

The probability density function of Y is

$$\begin{aligned} g(y|\theta) &= \frac{\partial G(y|\theta)}{\partial y} = n[1 - F(y|\theta)]^{n-1} f(y|\theta) \\ &= \begin{cases} n\exp\{-(n-1)(y-\theta)\} \times \exp\{-(y-\theta)\} = n\exp\{-n(y-\theta)\} & \text{if } \theta < y < \infty, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) **(6)**

The moment generating function of Y is

$$\begin{aligned} M_y(t) &= \int_\theta^\infty n\exp\{ty\}\exp\{-n(y-\theta)\}dy \\ &= \int_\theta^\infty \frac{n}{n-t}(n-t)\exp\{-(n-t)(y-\theta)+t\theta\}dy \\ &= \frac{n}{n-t}\exp\{t\theta\}\int_\theta^\infty (n-t)\exp\{-(n-t)(y-\theta)\}dy = \frac{n}{n-t}\exp\{t\theta\} \end{aligned}$$

(4)

$$E[Y] = \left.\frac{\partial \ln M_y(t)}{\partial t}\right|_{t=0} = -\frac{1}{n} + \theta.$$

(2)

(c) **(5)**

From part (b) above, $E\left[Y + \frac{1}{n}\right] = \theta$. Therefore, $Y + \frac{1}{n}$ is a minimum variance unbiased estimator of θ because it is an unbiased estimator of θ and a function of a complete sufficient statistic for θ. (3+2=5)

(d) **(5)**

$$0.95 = P(\theta < Y < c) = \exp\{-n(c-\theta)\} \Longrightarrow -n(c-\theta) = \ln(0.95) \Longrightarrow c = \theta + \frac{0.05129}{n}.$$

(e) **(4)**

From part (d)

$$\begin{aligned} \theta < Y < \theta + \frac{0.05129}{n} &\Longrightarrow 0 < Y - \theta < \frac{0.05129}{n} \\ &\Longrightarrow -\frac{0.05129}{n} < \theta - Y < 0 \\ &\Longrightarrow Y - \frac{0.05129}{n} < \theta < Y \end{aligned}$$

is the 95% confidence interval for θ.

Question 3 **[15]**

(a) **(13)**

The likelihood function of θ is: $L(\theta|\mathbf{x}) = (2\pi\theta)^{-n/2} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{2} x_i^2\right\}$. (2)

Hence for any $\theta > 1$, $r(1, \theta|\mathbf{x}) = \frac{L(1|\mathbf{x})}{L(\theta|\mathbf{x})} = \theta^{n/2} \exp\left\{-\left(\frac{\theta-1}{2\theta}\right)\sum_{i=1}^{n} x_i^2\right\}$. (3)

According to the Neyman-Pearson theorem, the best test of the hypotheses rejects H_0 if

$$\begin{aligned} r(1,\theta|\mathbf{x}) &= \theta^{n/2} \exp\left\{-\left(\frac{\theta-1}{2\theta}\right)\sum_{i=1}^{n} x_i^2\right\} \leq k \in (0,1] \\ &\equiv \text{ if } \exp\left\{-\left(\frac{\theta-1}{2\theta}\right)\sum_{i=1}^{n} x_i^2\right\} \leq \theta^{-n/2}k = k^* \\ &\equiv \text{ if } -\left(\frac{\theta-1}{2\theta}\right)\sum_{i=1}^{n} x_i^2 \leq \ln k^* \\ &\equiv \text{ if } \sum_{i=1}^{n} x_i^2 \geq -\left(\frac{2\theta}{\theta-1}\right)\ln k^* = c \end{aligned}$$

where c solves the probability equation $0.05 = P\left(\frac{\sum_{i=1}^{n} X_i^2}{\theta} \geq c|\theta = 1\right)$. (8)

(b) $c = 24.996$. **(2)**

Question 4 **[25]**

(a) **(13)**

The likelihood function of θ is: $L(\theta|\mathbf{x}) = \frac{e^{-10\theta}\theta^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x_i!}$. (2)

Hence, $r(0.1, 0.5|\mathbf{x}) = \frac{L(0.1|\mathbf{x})}{L(0.5|\mathbf{x})} = e^{(-1+5)}\left(\frac{0.1}{0.5}\right)^{\sum_{i=1}^{10} x_i} = e^4(0.2)^{\sum_{i=1}^{10} x_i}$. (3)

According to the Neyman-Pearson theorem, the best test of the hypotheses rejects H_0 if

$$\begin{aligned} r(0.1, 0.5|\mathbf{x}) &= e^4(0.2)^{\sum_{i=1}^{10} x_i} \leq k \in (0, 1] \\ &\equiv \text{ if } (0.2)^{\sum_{i=1}^{10} x_i} \leq e^{-4}k = k^* \\ &\equiv \text{ if } \sum_{i=1}^{10} x_i \geq \ln k^* / \ln 0.2 = c \end{aligned}$$

where c solves the probability equation $\alpha = P\left(\sum_{i=1}^{10} X_i \geq c|\theta = 0.1\right)$. (8)

(b) **(12)**

(i) $\alpha = P\left(\sum_{i=1}^{10} X_i \geq 3|\theta = 0.1\right) = 1 - P\left(\sum_{i=1}^{10} X_i \leq 2|\theta = 0.1\right)$
$= 1 - (0.3679 + 0.3679 + 0.1839) = 0.0803.$ **(6)**

(ii) The probability of the type II error is $\beta = P\left(\sum_{i=1}^{10} X_i \leq 2|\theta = 0.5\right)$
$= 0.006747 + 0.03369 + 0.08422 = 0.12465.$ (5)

Hence the power of the test is $1 - \beta = 0.8753.$ (1)

TOTAL: [95]

1 Trial Examination Paper

INSTRUCTIONS

1. Answer all questions.

2. Show intermediate steps.

Abbreviations	
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
pdf	probability density function
cdf	cumulative distribution function
$MVUE$	Minimum Variance Unbiased Estimator
MME	Method of Moments Estimator
$CRLB$	Cramer Rao Lower Bound
pmf	probability mass function
mgf	moment generating function
re	relative efficiency

Question 1 **[33]**

A distribution belongs to the regular 1-parameter exponential family if among other regularity conditions its pdf or pmf has the form:

$$f(y|\theta) = a(\theta)g(y)\exp\{b(\theta)r(y)\}, \ \ y \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty)$$

where $g(y) > 0$, $r(y)$ is a function of y which does not depend on θ, and $a(\theta) > 0$ and $b(\theta) \neq 0$ are real valued functions of θ. Furthermore, for this distribution (assuming the first and second derivatives of $a(\theta)$ and $b(\theta)$ exist)

$$E[r(Y)] = -\frac{a'(\theta)}{a(\theta)b'(\theta)}.$$

Suppose that $Y_1, Y_2, ..., Y_n$ is a random sample from this distribution.

(a) Show that $f(y|\theta)$ can also be expressed as:

$$f(y|\theta) = g(y)\exp\{b(\theta)r(y) - q(\theta)\},\ y \in \chi \subset (-\infty, \infty) \text{ and } \theta \in \Theta \subset (-\infty, \infty).$$

(2)

(b) Prove that $\sum_{i=1}^{n} r(Y_i)$ is a minimal sufficient statistic for θ. **(5)**

(c) Justify that $\sum_{i=1}^{n} r(Y_i)$ is also a complete sufficient statistic for θ. **(1)**

(d) Write down a function of the complete sufficient statistic for θ that is a minimum variance unbiased estimator ($MVUE$) of $E[r(Y)]$. **Justify your answer.** **(3)**

(e) Suppose that $X_1, X_2, ..., X_n$ is random sample from a distribution with pdf :

$$f(x|\theta) = \begin{cases} \dfrac{\theta}{(1+x)^{1+\theta}} & \text{if } x > 0 \text{ and } \theta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Show that $f(x|\theta)$ belongs to the regular 1-parameter exponential family by showing that the pdf can be expressed as:

$$f(x|\theta) = a(\theta)g(x)\exp\{b(\theta)r(x)\},\ x > 0 \text{ and } \theta > 0.$$

(5)

(ii) What is the mean of the complete sufficient statistic for θ? **Justify your answer.** **(6)**

(iv) What is the maximum likelihood estimator (MLE) of θ? **Justify your answer.** **(6)**

(v) What is the method of moments estimator (MME) of θ? **Justify your answer.** **(2)**

(vi) What is the minimum variance unbiased estimator ($MVUE$) of $\frac{1}{\theta}$? **Justify your answer.** **(3)**

Question 2 **[30]**

Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pdf :

$$f(x|\theta) = \begin{cases} \dfrac{1}{\theta} & \text{if } 0 < x \leq \theta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. Let $Y_n = \max\{X_1, X_2, ..., X_n\}$. The probability density function of Y_n is

$$g(y|\theta) = \begin{cases} ny^{n-1}\theta^{-n} & \text{if } 0 < y \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that the ten independent observations,

$$23,\ 23,\ 24,\ 23,\ 23,\ 23,\ 24,\ 22,\ 24,\ 25,$$

are from a population with probability density function $f(x|\theta)$.

(a) What **are** the method of moments **estimator** and **estimate** of θ? **(4)**

(b) What **are** the maximum likelihood **estimator** and **estimate** of θ? **(8)**

(c) Find the mean and the variance, in terms of θ and n, of the maximum likelihood **estimator** of θ. Is the estimator consistent? **(10)**

(d) What is an estimate of the standard error of the maximum likelihood estimate of θ? **(3)**

(e) Find an approximate 95% confidence interval for θ. **(5)**

Question 3 **[18]**

Let $X_1, X_2, ..., X_n$ be a random sample from the distribution with pdf :

$$f(x|\theta_1, \theta_2) = \begin{cases} \dfrac{\theta_1}{\theta_2^{\theta_1}} x^{\theta_1 - 1} & \text{if } 0 < x \leq \theta_2 \text{ and } \theta_1 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the likelihood function and the log-likelihood function of θ_1 and θ_2. **(6)**

(b) Show that the respective maximum likelihood estimators ($MLE's$) of θ_1 and θ_2 are

$$\hat{\theta}_1 = \frac{n}{n \ln X_{(n)} - \sum_{i=1}^{n} \ln X_i} \text{ and } \hat{\theta}_2 = X_{(n)} = \max\{X_1, X_2, ..., X_n\},$$

respectively. **Hint: First find the MLE of θ_2 and then that of θ_1.** **(12)**

Question 4 **[19]**

A random variable X has probability density function

$$f(x|\theta) = \frac{\exp\{x-\theta\}}{(1+\exp\{x-\theta\})^2}, \quad -\infty < x < \infty \text{ and } -\infty < \theta < \infty,$$

and **cumulative** distribution function

$$F(x|\theta) = 1 - \frac{1}{1+\exp\{x-\theta\}}, \quad -\infty < x < \infty \text{ and } -\infty < \theta < \infty.$$

It is desired to test the null hypothesis $H_0: \ \theta = 0$ against the alternative $H_1: \ \theta = 1$ using one random observation X.

(a) Calculate the level of significance and the power of the test which rejects H_0 if $X > 1$. **(12)**

(b) Consider the test which rejects H_0 if $X_1 \geq c$. Find c for which the level of significance of the test is 0.1. **(7)**

TOTAL [100]

2 Trial Examination Solutions

Question 1 **[Total marks=30]**

(a)

$$\begin{aligned} f(y|\theta) &= a(\theta)g(y)\exp\{b(\theta)R(y)\} \\ &= g\,(y)\exp\{b(\theta)r\,(y) - -\ln[a(\theta)]\} \\ &= g\,(y)\exp\{b(\theta)r\,(y) - q\,(\theta)\} \end{aligned}$$

where $q(\theta) = -\ln[a(\theta)]$. **(2)**

(b) The likelihood function of θ is

$$\begin{aligned} L\,(\theta|y) &= \prod_{i=1}^{n} f\,(y_i|\theta) = \left(\prod_{i=1}^{n} g\,(y_i)\right)\exp\left\{b\,(\theta)\sum_{i=1}^{n} r\,(y_i) - nq\,(\theta)\right\} \\ &= m_1\,(y) \times m_2\left(\theta, \sum_{i=1}^{n} r\,(y_i)\right) \end{aligned}$$

where $m_1\,(\mathbf{y}) = \prod_{i=1}^{n} g\,(y_i)$ and $m_2\left(\theta, \sum_{i=1}^{n} r\,(y_i)\right) = \exp\left\{b\,(\theta)\sum_{i=1}^{n} r\,(y_i) - nq\,(\theta)\right\}$. This means $\sum_{i=1}^{n} r\,(Y_i)$ is sufficient statistic for θ by the factorisation theorem.

Now let $X_1, X_2, ..., X_n$ be another random sample from the same distribution. Then

$$\frac{L\,(\theta|y)}{L\,(\theta|x)} = \prod_{i=1}^{n} \frac{g\,(y_i)}{g\,(x_i)} \exp\left\{b\,(\theta)\sum_{i=1}^{n} (r\,(y_i) - r\,(x_i))\right\}$$

is independent of θ if $\sum_{i=1}^{n} (r\,(y_i) - r\,(x_i)) = 0$ equivalently if $\sum_{i=1}^{n} r\,(y_i) = \sum_{i=1}^{n} r\,(x_i)$. This means that $\sum_{i=1}^{n} r\,(Y_i)$ is a minimal sufficient statistic for θ. **(5)**

(c) It is because $f(y|\theta)$ belongs to the 1–parameter exponential family. **(1)**

(d) $\frac{1}{n}\sum_{i=1}^{n} r(Y_i)$ because it is a function of the complete sufficient statistic for θ and $E\left[\frac{1}{n}\sum_{i=1}^{n} r(Y_i)\right] = E[r(Y)]$. **(3)**

(e) (i)

$$\begin{aligned} f(x|\theta) &= \frac{\theta}{(1+x)^{1+\theta}} \\ &= \theta(1+x)^{-1}(1+x)^{-\theta} \\ &= \theta(1+x)^{-1}\exp\{-\theta\ln(1+x)\} \\ &= a(\theta)\,g(x)\exp\{b(\theta)\,r(x)\} \end{aligned}$$

where $a(\theta) = \theta;\ g(x) = (1+x)^{-1};\ b(\theta) = \theta;$ and $r(x) = -\ln(1+x)$. **(5)**

(ii) The complete sufficient statistic is $\sum_{i=1}^{n} r(X_i) = -\sum_{i=1}^{n}\ln(1+X_i)$ since $f(x|\theta)$ belongs to the 1–parameter exponential family. $E[r(X)] = -\frac{a'(\theta)}{a(\theta)\,b'(\theta)} = \frac{1}{\theta\times 1} = \frac{1}{\theta} \Rightarrow E\left[-\sum_{i=1}^{n}\ln(1+X_i)\right] = -\frac{n}{\theta}$. **(6)**

(iii) The likelihood function is

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n \prod_{i=1}^{n}(1+x_i)^{-1}\prod_{i=1}^{n}(1+x_i)^{-\theta}$$

and the log-likelihood function is

$$l(\theta) = n\ln\theta + \ln\left[\prod_{i=1}^{n}(1+x_i)^{-1}\right] - \theta\sum_{i=1}^{n}\ln(1+x_i).$$

The MLE of θ is $\hat{\theta}$ which solves the equation

$$0 = l'(\theta) = \frac{n}{\hat{\theta}} - \sum_{i=1}^{n} \ln(1 + x_i).$$

The solution is $\hat{\theta} = \dfrac{n}{\sum_{i=1}^{n} \ln(1 + X_i)}$. **(6)**

(iv) $E[\ln(1+X)] = \dfrac{1}{\theta} \Longrightarrow \theta = \dfrac{1}{E[\ln(1+X)]} \Longrightarrow$ the MME of θ is $\widetilde{\theta} = \dfrac{n}{\sum_{i=1}^{n} \ln(1 + X_i)}$. **(2)**

(v) The minimum variance unbiased estimator $(MVUE)$ of $\dfrac{1}{\theta}$ is $\dfrac{1}{n}\sum_{i=1}^{n} \ln(1 + X_i)$ because the estimator is a function of a complete sufficient statistic for θ and $E\left[\dfrac{1}{n}\sum_{i=1}^{n} \ln(1 + X_i)\right] = \dfrac{1}{\theta}$. **(3)**

Question 2 **[Total marks=30]**

(a) $E(X) = \int_0^\theta x/\theta dx \left[\dfrac{x^2}{2\theta}\right]_0^\theta = \dfrac{\theta}{2} \Rightarrow \theta = 2E[X]$. This means the method of moments estimate (MME) of θ is $\widetilde{\theta} = 2\bar{x} = 2 \times 23.2 = 46.4$. **(4)**

(b) The likelihood function of θ

$$L(\theta|\mathbf{x}) = \begin{cases} \theta^{-n} & \text{if } \theta \geq x_{(n)} = \max\{x_1, x_2, ..., x_n\}. \\ 0 & \text{otherwise,} \end{cases}$$

decreases as θ increase. Thus the function attains its maximum at the smallest value of θ which is $x_{(n)}$. This means the maximum likelihood estimate of θ is $x_{(10)} = 25$. **(8)**

(c) $E\left[Y_n\right] = \int_0^\theta yny^{n-1}\theta^{-n}dy = \left[\frac{n}{n+1}y^{n+1}\theta^{-n}\right]_0^\theta = \frac{n}{n+1}\theta.$ **(2)**

$$E\left[Y_n^2\right] = \int_0^\theta y^2ny^{n-1}\theta^{-n}dy = \left[\tfrac{n}{n+2}y^{n+2}\theta^{-n}\right]_0^\theta = \frac{n}{n+2}\theta^2 \Rightarrow$$

$$Var\left(Y_n\right) = E\left[Y_n^2\right] - \left(E\left[Y_n\right]\right)^2 = \left(\frac{n}{n+2} - \tfrac{n^2}{(n+1)^2}\right)\theta^2 = \frac{n}{(n+2)\left(n+1\right)^2}\theta^2.$$ **(4)**

That the bias of the estimate is $\theta - E\left[Y_n\right] = \frac{1}{n+1}\theta \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} Var\left(Y_n\right) = 0$ means Y_n is a consistent estimator of θ. **(4)**

(d) The estimate is $\sqrt{Var\left(Y_n\right)|_{n=10,\ \theta=x_{(10)}=25}} = \sqrt{\frac{10}{12 \times 11^2} \times 25^2\widetilde{\theta}} = 2.0747.$ **(3)**

(e) The 95% confidence interval is $25 \pm 1.960 \times 2.0747 = [20.9334;\ 29.0664]$. **(5)**

Question 3 **[Total marks=18]**

(a) The likelihood function of θ_1 and θ_2 is

$$L\left(\theta_1, \theta_2|\mathbf{x}\right) = \prod_{i=1}^n f\left(x_i|\theta_1, \theta_2\right) = \begin{cases} \left(\prod_{i=1}^n x_i\right)^{-1} \frac{\theta_1^n}{\theta_2^{n\theta_1}}\left(\prod_{i=1}^n x_i\right)^{\theta_1} & \text{if } \theta_1 > 0,\ \theta_2 \geq \text{ all the } x_i's, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$L\left(\theta_1, \theta_2|\mathbf{x}\right) = \prod_{i=1}^n f\left(x_i|\theta_1, \theta_2\right) = \begin{cases} \left(\prod_{i=1}^n x_i\right)^{-1} \frac{\theta_1^n}{\theta_2^{n\theta_1}}\left(\prod_{i=1}^n x_i\right)^{\theta_1} & \text{if } \theta_1 > 0,\ \theta_2 \geq x_{(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

where $x_{(n)} = \max\{x_1, x_2, x_3, ..., x_n\}$. **(3)**

The log likelihood function is

$$\begin{aligned} l(\theta_1, \theta_2) &= \ln L(\theta_1, \theta_2) \\ &= \begin{cases} \ln\left(\prod_{i=1}^{n} x_i\right)^{-1} + n \ln\theta_1 - n\theta_1 \ln\theta_2 + \theta_1 \ln\left(\prod_{i=1}^{n} x_i\right) & \text{if } \theta_1 > 0,\ \theta_2 \geq x_{(n)}, \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

(3)

(b) For any $\theta_1 > 0,\ L(\theta_1, \theta_2|\mathbf{x})$ decreases as θ_2 increases. Thus the function attains its maximum at the smallest value of θ_2 which is $x_{(n)}$. This means the maximum likelihood estimator of θ_2 is $X_{(n)}$. **(6)**

Consider the log likelihood

$$l\left(\theta_1, x_{(n)}\right) = \ln\left(\prod_{i=1}^{n}\right)^{-1} + n\ln\theta_1 - n\theta_1 \ln x_{(n)} + \theta_1 \ln\left(\prod_{i=1}^{n} x_i\right)$$

whose derivative with respect to θ_1 is

$$l'\left(\theta_1 x_{(n)}\right) = \frac{n}{\theta_1} - n\ln x_{(n)} + \ln\left(\prod_{i=1}^{n} x_i\right) = \frac{n}{\theta_1} - n\ln x_{(n)} + \sum_{i=1}^{n} \ln x_i.$$

The MLE of θ_1 is $\hat{\theta}_1$ which solves the equation

$$0 = l'\left(\hat{\theta}_1, X_{(n)}\right) = \frac{n}{\hat{\theta}_1} - n\ln X_{(n)} + \ln\left(\prod_{i=1}^{n} X_i\right) = \frac{n}{\hat{\theta}_1} - n\ln X_{(n)} + \sum_{i=1}^{n} \ln X_i.$$

The solution is

$$\hat{\theta}_1 = \frac{n}{n\ln X_{(n)} - \sum_{i=1}^{n} \ln X_i} = \frac{n}{n\ln X_{(n)} - \ln\left(\prod_{i=1}^{n} X_i\right)}.$$ (6)

Question 4 **[Total marks=19]**

(a) $\alpha = P(X > 1|\theta = 0) = 1 - P(X < 1|\theta = 0) = 1 - F(1|0) = \dfrac{1}{1 + \exp(1)} = 0.2689.$ **(5)**

$\beta = P(X < 1|\theta = 1) = F(1|1) = 1 - \dfrac{1}{1 + \exp(0)} = \dfrac{1}{2}.$ **(5)**

Hence the power $= 1 - \beta = \dfrac{1}{2}.$ **(2)**

(b) $0.1 = P(X \geq c|\theta = 0) = 1 - F(c|0) = \dfrac{1}{1 + \exp(c)} \Rightarrow$

$10 = 1 + \exp(c) \Rightarrow 9 = \exp(c) \Rightarrow c = \ln 9 = 2.1972.$ **(7)**

TOTAL [100]