Tutorial Letter 104/1/2017

REAL ANALYSIS

MAT3711

Semester 1

Department of Mathematical Sciences

IMPORTANT INFORMATION:

Preparation for the May/June 2017 exam.

BAR CODE

Learn without limits.

UNISA university of south africa

Preparation for the May/June 2017 exam

Dear Students

In order to assist you in the preparation for the upcoming exam, I enclose some hints and suggestions on this matter. I also include a copy of the October/November 2012 exam as well as selected hints, tips and solutions which may be useful to you when studying.

You should study all the proofs as you learn this subject. Studying and mastering proofs will train you in the art of constructing proofs of your own. Besides, the beauty of Mathematics lies not only in the applications, but also in the ingenuity of some of the proofs. This is important for you to do as Real Analysis is typical of the definition-theorem-proof kind of mathematics. Furthermore, going through proofs is part and parcel of learning the subject. If you have worked carefully through the Study Guide, you should do very well in the exam.

You should know all definitions, all exercises, all statements of theorems, corollaries, lemmas, etc. In particular, you should pay attention to the proofs of the following theorems:

Learning Unit 2:
02.4, 02.5, 02.6, 02.1, 03.2, 03.3, 03.4, 03.7, 03.8, 03.10, 03.11, 03.12, 03.17, 04.4, 04.5

Learning Unit 3:
01.4, 01.5, 01.6, 01.7, 01.10, 02.2, 02.4, 02.5, 02.6, 02.9, 03.3, 03.5

Learning Unit 4:
01.2, 01.3, 01.4, 01.5, 01.6, 02.2, 02.7, 02.8, 02.9, 02.10, 02.12, 02.14

Learning Unit 5:
01.2, 01.3, 01.5, 01.7

Learning Unit 6:
01.2, 01.3, 01.5, 02.2, 03.1, 03.2, 03.3, 03.4, 5.4.1, 04.2, 04.3, 04.4, 05.2

Learning Unit 7:
01.2, 01.3, 01.4, 01.5, 02.3, 02.1, 02.5, 02.6

Learning Unit 8:
01.3, 02.2, 02.4, 03.3, 03.4, 03.9, 03.10, 03.11

Learning Unit 9:
01.4, 01.5, 01.6, 02.1, 02.2, 03.1

Please note that an exclusive focus on these theorems alone should not be regarded as sufficient preparation for the exam.

The past papers and assignments may be useful to you when studying.

I hope the above information helps you to prepare for the exam. Good luck in your studies!

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Selected hints, tips and solutions to the October/November 2012 exam

Question 1

(a) Definitions

(b) (i) Observe that S contains only rational numbers and, for instance, $1 \in S$. For any $r > 0$, $B(1, r) = (1 - r, 1 + r)$, which contains irrational numbers because between any two rational numbers there is an irrational number. So $B(1, r) \nsubseteq S$, and hence S is not open.

(ii) Consider the sequence $\{x_n\}$ given by $x_n = 2 - \frac{1}{n}$. This is a sequence in S such that $x_n \to 2$. So $2 \in \overline{S}$. But $2 \notin S$ and hence $S \neq \overline{S}$. Therefore S is not closed.

(c) Theorem 1.3.30

Question 2

(a) Definitions

(b) Theorem 2.2.2

(c) Let X be $\mathbb{R}\backslash\{0\}$ with the usual metric. Put $x_n = \frac{1}{n}$ for $n = 1, 2, \ldots$. Then $\{x_n\}$ is a Cauchy sequence. Indeed, given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ so that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. Then for $m, n \geq n_0$ we have $d(x_m, x_n) = |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon$. The sequence however does not converge in X.

Question 3

(a) Definitions

(b) Theorem 4.1.7

(c) (i) It is known that $\mathbb{R}$ is complete and that closed subsets of a complete space are complete. Now $[1, \infty)$ is a closed subset of $\mathbb{R}$. Therefore (X, d) is complete.

(ii) Let $x, y \in X$. Then

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{\lambda + x}{1 + x} - \frac{\lambda + y}{1 + y}\right| \\
&= \left|\frac{(\lambda + x)(1 + y) - (\lambda + y)(1 + x)}{(1 + x)(1 + y)}\right| \\
&= \left|\frac{\lambda(y - x) + (x - y)}{(1 + x)(1 + y)}\right| \\
&\leq \frac{1}{4}|x - y|\,|\lambda - 1| \quad \text{since } 1 + x \geq 2 \text{ and } 1 + y \geq 2 \\
&= \frac{\lambda - 1}{4}|x - y|
\end{aligned}$$

since $\lambda > 1$. But now $\lambda < 2$ implies $\lambda - 1 < 1$, so

$$|f(x) - f(y)| < \frac{1}{4}|x - y|.$$

Therefore f is a contraction on X.

(iii) We know this from Banach's Fixed Point Theorem.

(iv) $f(x) = x \Leftrightarrow \lambda + x = x + x^2 \Leftrightarrow x^2 = \lambda \Leftrightarrow x = \pm\sqrt{\lambda}$. Since x must be in $[1, \infty)$, it follows that the fixed point of f is $\sqrt{\lambda}$.

Question 4

(a) (i) Definition

(ii) For all $v \in V$ where V is such that $T : V \to W$.

(iii) Theorem 7.3.10 (a) and (b)

(b) (i) First, let us note that $T(x, y) = \langle (x, y), (1, 2) \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product. Thus,

$$\begin{aligned} ||T(x,y)|| &= |\langle (x,y), (1,2) \rangle| \\ &\leq ||(x,y)||\ ||(1,2)|| \\ &= \sqrt{5}||(x,y)||. \end{aligned}$$

Therefore $||T|| \leq \sqrt{5}$ and we may conclude that T is a bounded linear operator.

(ii) Example 7.3.14

Question 5

(a) Definition

(b) Let $P = \{x_0, \ldots x_n\}$ be a partition of $[0, 1]$ such that $\frac{1}{2} \in P$. Let $x_j = \frac{1}{2}$. Then,

$$M_k(f) = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, j-1 \\ 2 & \text{if } k = j, \ldots n. \end{cases}$$

Also

$$m_k(f) = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, j \\ 2 & \text{if } k = j+1, \ldots n \end{cases}$$

and

$$\Delta\alpha_k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, j \\ 1 & \text{if } k = j+1 \\ 0 & \text{if } k = j+2, \ldots n \end{cases}.$$

Then $U(P, f, \alpha) = 2$ and $L(P, f, \alpha) = 2$. Thus

$$\int_0^1 f \, d\alpha = 2.$$