Tutorial letter 101/3/2018

Linear Algebra

MAT3701

Semesters 1 & 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains important information about your module.

BAR CODE

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1 INTRODUCTION

Dear Student

Welcome to MAT3701 which is our third year module on Linear Algebra. We hope you will find it interesting and rewarding. This module is offered as a semester module. You will be well on your way to success if you start with your studies early and resolve to do the assignments properly.

1.1 Tutorial matter

Tutorial Letter 101 contains important information about the scheme of work, resources and assignments for this module. We urge you to read it carefully and to keep it at hand when working through the study material, preparing the assignments, preparing for the examination and addressing questions to your lecturers.

In this tutorial letter you will find the assignments as well as instructions on the preparation and submission of the assignments. This tutorial letter also provides information with regard to other resources and where to obtain them. Please study this information carefully.

Certain general and administrative information about this module has also been included. Please study this section of the tutorial letter carefully.

You must read all the tutorial letters you receive during the semester **immediately and carefully** as they always contain important and sometimes urgent information.

Some of this tutorial matter may not be available when you register. Tutorial matter that is not available when you register will be posted to you as soon as possible, but is also available on *myUnisa*.

We hope you will enjoy this module and wish you all the best!

2 PURPOSE OF AND OUTCOMES FOR THE MODULE

2.1 Purpose

The purpose of this module is to equip students with an integrated knowledge of the main concepts, theory and techniques of linear algebra over the real or complex field as it applies to inner product spaces, invariant subspaces, operators and their canonical forms. This will contribute to a knowledge base for further studies in mathematics and for application in other disciplines.

2.2 Outcomes

Outcome Demonstrate an understanding of and ability to work with:	**Assessment Criteria**
Theoretical concepts	• State definitions and results in own words. • Reproduce the proofs of a selection of theorems. • Do exercises of a theoretical nature that require deductive reasoning. • Verify whether definitions and results are satisfied in certain situations. • Construct counter examples.
Real and complex matrix theory	• Check and (if possible) diagonalise a matrix. • Calculate the spectral decomposition of a normal or self-adjoint matrix. • Use the adjoint of a matrix to solve least squares problems. • Use orthogonal matrices to eliminate the cross term in a conic section. • Use orthogonal matrices to describe rigid motions in the plane. • Test and (if possible) find the limit of a sequence of matrices. • Use Gerschgorin's Disk Theorem to locate the eigenvalues of a matrix. • Apply matrix theory to Markov chains. • Find the matrix representation and invariants of a bilinear form. • Compute condition numbers and use it for error estimates.
Real and complex vector spaces and inner product spaces	• Construct Lagrange polynomials and use it for interpolation. • Construct and use direct sum decompositions. • Construct invariant (including cyclic) subspaces. • Construct orthogonal bases, projections and complements. • Use inner products to find coordinate vectors and transition matrices.
Linear operators on real and complex vector spaces and inner product spaces	• Check and (if possible) find a diagonalising basis for a linear operator. • Find the adjoint of a linear operator. • Find the orthogonal projection on an inner product subspace. • Find the vector defining the action of a linear functional. • Find the spectral decomposition of a normal or self-adjoint operator. • Solve equations involving linear operators. • Find orthonormal eigenvector bases for unitary and normal operators.

3 LECTURERS AND CONTACT DETAILS

3.1 Lecturer(s)

The lecturer responsible for MAT3701 is:

Name	Office	Telephone	E-mail
*Prof JD Botha	Department of Mathematical Sciences GJ Gerwel Building C-Block 6-30 Florida Campus	011-670-9167 (RSA) 27-11-670-9167 (International)	bothajd@unisa.ac.za

* Prof Botha will be retiring at the end of 2017 but you will be informed of your new lecturer(s) early in 2018.

All queries that are not of a purely administrative nature but are about the content of this module should be directed to us. Email is the preferred form of communication to use. If you phone us please have your study material with you when you contact us. If you cannot get hold of us, leave a message with the Departmental Secretary. Please clearly state your name, reason for calling and how we can get back to you.

You are always welcome to come and discuss your work with us, but **please make an appointment before coming to see us**. Please come to these appointments well prepared with specific questions that indicate your own efforts to have understood the basic concepts involved.

You are also free to write to us about any of the difficulties you encounter with your work for this module. If these difficulties concern exercises which you are unable to solve, you must inculde your attempts so we can see where you are going wrong or what concepts you do not understand. Mail should be sent to one of us at:

Department of Mathematical Sciences
PO Box 392
UNISA
0003

PLEASE NOTE: Letters to lecturers may not be enclosed with or inserted into assignments.

3.2 Department

Fax number: 011 670 9171 (RSA)	+27 11 670 9171 (International)
Departmental Secretary: 011 670 9147 (RSA)	+27 11 670 9147 (International)

3.3 University (contact details)

If you need to contact the University about matters not related to the content of this module, please consult the publication *My studies @ Unisa* that you received with your study material. This brochure contains information on how to contact the University (e.g. to whom you can write for different queries, important telephone and fax numbers, addresses and details of the times certain facilities are open).

Always have your student number at hand when you contact the University.

4 MODULE-RELATED RESOURCES

4.1 Prescribed book

The prescribed textbook is

> **Stephen H. Friedberg, Arnold J. Insel and Lawrence E. Spencer:**
> *Linear Algebra*, 4^{th} (2013) edition, Pearson Education, ISBN 13: 978-1-292-02650-3
> (or any previous 2^{nd}, 3^{rd} or 4^{th} edition).

Please refer to the list of official booksellers and their addresses in the *my Studies @ Unisa* brochure. Prescribed books can be obtained from the University's official booksellers. If you have difficulty in locating your book(s) at these booksellers, please contact the Prescribed Book Section at tel: 012 429-4152 or e-mail vospresc@unisa.ac.za.

4.2 Recommended books

The following books may be consulted for additional reading:

(1) Petersen P: Linear Algebra, Springer, 2012, ISBN 978-1-4614-3612-6 (eBook)

(2) Robbiano L: Linear algebra for everyone, Springer, 2011, ISBN 978-88-470-1839-6 (eBook)

(3) Vasishtha AR, Sharma JN and Vasishtha AK, Linear Algebra: a course in finite-dimensional vector spaces, 45^{th} ed., Meerut: Krishna Prakashan Media, 2013, ISBN-13: 978-8182835757, ISBN-10: 8182835755

Textbook (3) (any edition) follows the presentation in the prescribed textbook the closest. Textbooks (1) and (2) are e-books (in pdf) and may be downloaded free of charge by registered students from the Unisa library. The procedure is as follows: go to the Unisa library catelogue (http://oasis.unisa.ac.za/). Then

- choose “author” instead of “title” from the drop-down menu for the first box
- enter “Petersen, Peter” in the second box
- choose “e-books” instead of “view the entire collection” from the drop-down menu for the third box

- click “submit”
- click on the link “View full text e-book at Access restricted to Unisa staff and students”

Follow the same procedure for textbook (2) by entering “Robbiano, Lorenzo” in the second box.

4.3 Electronic Reserves (e-Reserves)

There are NO e-Reserves for this module.

5 STUDENT SUPPORT SERVICES FOR THE MODULE

For information on the various student support systems and services available at Unisa (e.g. student counselling, tutorial classes, language support), please consult the publication *My studies @ Unisa* that you received with your study material.

5.1 Study groups

It is advisable to have contact with fellow students. One way to do this is to form study groups. The addresses of students in your area may be obtained from the following department:

Directorate: Student Administration and Registration
PO Box 392
UNISA
0003

5.2 myUnisa

If you have access to a computer that is linked to the internet, you can quickly access resources and information at the University. The *myUnisa* learning management system is Unisa’s online campus that will help you to communicate with your lecturers, with other students and with the administrative departments of Unisa all through the computer and the internet.

To go to the *myUnisa* website, start at the main Unisa website, www.unisa.ac.za, and then click on the “*myUnisa*” link below the orange tab labelled “Current students”. This should take you to the myUnisa website. You can also go there directly by typing my.unisa.ac.za in the address bar of your browser.

Please consult the publication *My studies @ Unisa* which you received with your study material for more information on *myUnisa*.

5.3 Group Discussions

There will be no group discussions for this module. Videos of a group discussion class held during 2014 are available on myUnisa under **Additional Resources**.

6 MODULE SPECIFIC STUDY PLAN

Study plan	Semester 1	Semester 2
Study units 1 - 9	9 March	17 August
Study units 10 - 17	13 April	21 September
Work through previous examination papers	20 April	27 September
Revision	until examination	until examination

7 MODULE PRACTICAL WORK AND WORK INTEGRATED LEARNING

There are no practicals for this module.

8 ASSESSMENT

8.1 Assessment plan

In each semester there are two assignments for MAT3701. The questions for the assignments for both semesters are given at the end of this tutorial letter. Both assignments count towards your semester mark. Please make sure that you **answer the questions for the semester for which you are registered**. If you answer questions from the wrong semester, your solutions will not be marked and you will get zero marks for the assignments.

It is not necessary to write out the questions with your answers. However, you must explain carefully the reasoning for your answer. Labelled diagrams should be included when necessary.

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the examination. It is therefore good practice to work through a complete set of four assignments for a given year. For this reason the assignments and worked solutions of previous years are made available under **Additional Resources** on myUnisa – see also the letter **MAT3701 Exam Preparation** under **Additional Resources**.

Worked solutions to all the questions for an assignment will be made available on myUnisa shortly after the due date. When marking the assignments, constructive comments will be made on your work, which will then be returned to you. The assignments and the comments on these assignments constitute an important part of your learning and should help you to be better prepared for the next assignment and the examination.

If you need help with problems you must include your attempt at a solution so that we may see where you are going wrong. You need to specify your problem clearly. For example, it is no good saying you don't understand Chapter 4; all we can then do is to repeat what is in the study guide or reference books.

To be admitted to the examination you need to submit the first assignment by its due date.

Your semester mark for MAT3701 counts 20% and your examination mark 80% of your final mark. Both assignments carry the same weight.

Your final mark will therefore be calculated according to the formula

$$\text{Final mark } = 0.8P_E + 0.1P_1 + 0.1P_2$$

where P_E, P_1 and P_2 denote your percentage in the examination, Assignment 01 and Assignment 02 respectively.

8.2 General assignment numbers

The assignments are numbered as 01 and 02 for each semester.

8.2.1 Unique assignment numbers

Please note that each assignment has its own unique assignment number which has to be written on the cover of your assignment upon submission.

8.2.2 Due dates of assignments

The due dates for the submission of the assignments in 2018 are:

Assignment number	Semester 1	Semester 2
1	9 March	17 August
2	13 April	21 September

8.3 Submission of assignments

You may submit written assignments and assignments completed on mark-reading sheets either by post or electronically via *myUnisa*. Assignments may **not** be submitted by fax or e-mail.

For detailed information on assignments, please refer to the *myStudies @ Unisa* brochure, which you received with your study package.

To submit an assignment via myUnisa:

- Go to myUnisa.
- Log in with your student number and password.
- Select the module.
- Click on assignments in the menu on the left-hand side of the screen.
- Click on the assignment number you wish to submit.
- Follow the instructions.

PLEASE NOTE: Although students may work together when preparing assignments, each student must write and submit his or her own individual assignment. In other words, you must submit your own calculations in your own words. It is unacceptable for students to submit identical assignments on the basis that they worked together. That is copying (a form of plagiarism) and none of these assignments will be marked. Furthermore, you may be penalised or subjected to disciplinary proceedings by the University.

8.4 Assignments

The assignment questions for Semester 1 are contained in Addendum A.

Assignment 1, pages 13–15

Assignment 2, pages 16–18

The assignment questions for Semester 2 are contained in Addendum B.

Assignment 1, pages 19–21

Assignment 2, pages 22–24

9 EXAMINATIONS

9.1 Examination admission

To be admitted to the examination you must submit the compulsory assignment, i.e. Assignment 01, by the due date (9 March 2018 for Semester 1 and 17 August 2018 for Semester 2).

9.2 Examination period

This module is offered in a semester period of fifteen weeks. This means that if you are registered for the first semester, you will write the examination in May/June 2018 and the supplementary examination in October/November 2018. If you are registered for the second semester, you will write the examination in October/November 2018 and the supplementary examination in May/June 2019.

During the semester, the Examination Section will provide you with information regarding the examination in general, examination venues, examination dates and examination times.

9.3 Examination paper

The textbook and Study Guide form the basis for this course. The relevant topics that you need to study are listed in the Study Guide. The examination will be a single written paper of two hours' duration.

Refer to the *myStudies @ Unisa* brochure for general examination guidelines and examination preparation guidelines.

You are not allowed to use a calculator in the examination. Previous examination papers with solutions will be made available on myUnisa under **Additional Resources**.

If you are not successful in the May/June or October/November examination (i.e. if you have less than 50%) you may write the supplementary examination in October/November or May/June respectively, **provided** that you obtained at least 40% for the previous examination. Supplementary examination dates will be provided by the Examination Section.

10 OTHER ASSESSMENT METHODS

There are no other assessment methods for this module.

11 FREQUENTLY ASKED QUESTIONS

The *my Studies @ Unisa* brochure contains an A-Z guide of the most relevant study information. Please refer to this brochure for any other questions.

12 CONCLUSION

We trust that you will have a successful academic year. You are important to us and we are willing and available to assist you with problems related to the content of this course.

Our best wishes

Your MAT3701 lecturers.

ADDENDUM A: ASSIGNMENTS FOR SEMESTER 1

A.1 Assignment 01

ONLY FOR SEMESTER 1 STUDENTS
ASSIGNMENT 01
Based on Study Units 1 - 9
FIXED CLOSING DATE: 9 MARCH 2018
UNIQUE NUMBER: 751222

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the examination. It is therefore good practice to work through a complete set of four assignments for a given year. For this reason the assignments and worked solutions of previous years are made available under **Additional Resources** on myUnisa – see also the letter **MAT3701 Exam Preparation** under **Additional Resources**.

Worked solutions to all the questions for this assignment will be made available on myUnisa shortly after the due date. Your answers to the assignment questions should be fully motivated.

QUESTION 1

Consider the vector space $V = C^2$ with scalar multiplication over the *real* numbers $R.$ Let

$$U = \{(z_1, z_2) \in V : z_2 = (1+i)z_1\}$$

(a) Show that U is a subspace of $V.$

(b) Find a basis for $U.$

(c) Determine whether $V = U \oplus W$ where W is the subspace defined by

$$W = \{(z_1, z_2) \in V : z_1 = z_2\}$$

QUESTION 2

Given that

$$B = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 14 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 14 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 14 \end{bmatrix} \right\}$$

is a generating set for C^3, find a basis for C^3 from among the vectors of $B.$

QUESTION 3

Let $f_{-2},\ f_{-1}$ and f_0 denote the Lagrange polynomials associated with $-2, -1$ and 0 respectively. Let $\gamma = \{f_{-2}, f_{-1}, f_0\}$ and $\beta = \{1, x, x^2\}$.

(a) Find f_{-2}, f_{-1}, f_0 and express each one in standard polynomial form, i.e. $a + bx + cx^2$ where a, b and c are real numbers.

(b) **Use the Lagrange interpolation formula** to express $1, x$ and x^2 as linear combinations of f_{-2}, f_{-1} and f_0.

(c) Without any further computations, explain why γ is a basis for $P_2(R)$.

(d) Write down the change of coordinate matrix P which changes $\beta-$coordinates to $\gamma-$coordinates.

(e) Without any further computations, write down P^{-1}.

QUESTION 4

Let $T : C^3 \to C^3$ be a linear operator such that $T^3 = 0$ and dim$(N(T)) = 2$. Show there exists a basis β for C^3 such that $[T]_\beta = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}$ where $a, b \in C$.

QUESTION 5

Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and let $T : M_{2\times 2}(C) \to M_{2\times 2}(C)$ be defined by $T(X) = AX$.

(a) Show that T is a linear operator over C.

(b) Find a basis for $R(T)$.

(c) Find a basis for $N(T)$.

(d) Determine whether or not $M_{2\times 2}(C) = R(T) \oplus N(T)$.

(e) Briefly explain whether or not T is diagonalizable.

QUESTION 6

Let $T : M_{2\times 2}(R) \to M_{2\times 2}(R)$ denote the projection on $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a - d = 0 \right\}$ along $U = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$.

(a) Find the matrix representation of T with respect to $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) Find the formula for $T \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ expressed as a matrix in terms of a, b, c, d.

QUESTION 7

Let $T : P_2(C) \to P_2(C)$ be the linear operator defined by

$$T\left(a + bx + cx^2\right) = a + (a + b + c)\,x + (a + 2c)\,x^2.$$

(a) Show that T satisfies the test for diagonalizability.

(b) Find a basis τ for $P_2(C)$ consisting of eigenvectors of T and write down $[T]_\tau$.

(c) Show that $T - I$ is a projection.

QUESTION 8

Let $T : M_{3\times3}(C) \to M_{3\times3}(C)$ be the linear operator over C defined by $T(X) = AX$ where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and let W be the T–cyclic subspace of $M_{3\times3}(C)$ generated by $J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) Find the T–cyclic basis for W.

(b) Find the characteristic polynomial of T_W.

(c) Determine whether T_W is one-to-one.

(d) Determine whether T_W is onto.

(e) For each eigenvalue of T_W, find a corresponding eigenvector in W expressed as a linear combination of the T-cyclic basis for W.

A.2 Assignment 02

ONLY FOR SEMESTER 1 STUDENTS
ASSIGNMENT 02
Based on Study Units 10 - 17
FIXED CLOSING DATE: 13 APRIL 2018
UNIQUE NUMBER: 782661

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the examination. It is therefore good practice to work through a complete set of four assignments for a given year. For this reason the assignments and worked solutions of previous years are made available under **Additional Resources** on myUnisa – see also the letter **MAT3701 Exam Preparation** under **Additional Resources**.

Worked solutions to all the questions for this assignment will be made available on myUnisa shortly after the due date. Your answers to the assignment questions should be fully motivated.

QUESTION 1

Let $T : C^4 \to C^4$ be the linear operator such that

$$[T]_\beta = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & i \\ 0 & i & i & 0 \end{bmatrix}$$

where β is the standard basis for C^4 and let

$$W = \text{ span } \{(1,0,0,0)\,,\ (0,1,-1,0)\}\,.$$

(a) Show that W is $T-$ invariant.

(b) Find a basis for $W^\perp$.

(c) Show that $W^\perp$ is T^*- invariant.

QUESTION 2

Find the minimal solution of the system

$$\begin{aligned} 5x \quad\quad +2z &= 19 \\ 2x \quad +y \quad\quad &= 5 \end{aligned}$$

with respect to the ordinary inner product in R^3.

QUESTION 3

(This exercise illustrates *Friedberg:* Theorem 6.15, p. 371.)

Let $A = \begin{bmatrix} 1-i & i & 0 \\ i & 1-i & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $w_1 = \begin{bmatrix} 2-i \\ -2+i \\ 0 \end{bmatrix}$ and let $w_2 = \begin{bmatrix} i \\ i \\ 1-i \end{bmatrix}$.

(a) Show that A is normal.

(b) Show that $\|Aw_1\| = \|A^* w_1\|$.

(c) Show that w_1 and w_2 are eigenvectors of A and find their respective eigenvalues λ_1 and λ_2.

(d) Show that $A^* w_1 = \bar{\lambda}_1 w_1$.

(e) Show that w_1 and w_2 are orthogonal.

QUESTION 4

Let V be an inner product space and let $y, z \in V$ be non-zero vectors. Define $T : V \to V$ by $T(x) = x - \langle x, y \rangle z$ for all $x \in V$.

(a) Show that T is linear.

(b) Find a formula for T^* similar to the one given for T.

QUESTION 5

Eliminate the xy-term in

$$3x^2 + 2xy + 3y^2 = 1 \qquad \text{... } (*)$$

by a rotation of the axes, i.e. find a rotation matrix $P \in M_{2\times 2}(R)$ such that ($*$), expressed in terms of x', y' defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix},$$

contains no cross term. Express x, y in terms of x', y' and state the (counterclockwise) angle of rotation.

QUESTION 6

Consider the inner product space $P_2(C)$ over C with $\langle \cdot, \cdot \rangle$ defined by

$$\langle g, h \rangle = g(-1)\,\overline{h(-1)} + g(0)\overline{h(0)} + g(1)\,\overline{h(1)}$$

Let $T : P_2(C) \to P_2(C)$ be the linear operator defined by $T(f(x)) = f(-x)$.

(a) Show that T is linear.

(b) Show that $T^2 = I$.

(c) Show that $\langle T(f(x)), g(x)\rangle = \langle f(x), T(g(x))\rangle$ for all $f, g \in P_2(C)$.

(d) Show that T is unitary.

(e) Find the eigenvalues and bases for the associated eigenspaces of T.

QUESTION 7

Find the spectral decomposition of

$$A = \begin{bmatrix} 1+i & 1 & 0 \\ 1 & 1+i & 0 \\ 0 & 0 & 1+2i \end{bmatrix}$$

Write each orthogonal projection matrix as a single matrix.

QUESTION 8

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$.

(a) Find $\|A\|$, $\|A^{-1}\|$ and cond(A) where $\|\cdot\|$ denotes the Euclidean norm.

(b) Suppose x and $\widetilde{x}$ are vectors such that $Ax = b, \|b\| = 1$ and $\|b - A\widetilde{x}\| \leq 0.001$. Use (a) to determine upper bounds for $\|\widetilde{x} - A^{-1}b\|$ (the absolute error) and $\|\widetilde{x} - A^{-1}b\| / \|A^{-1}b\|$ (the relative error).

ADDENDUM B: ASSIGNMENTS FOR SEMESTER 2

B.1 Assignment 01

ONLY FOR SEMESTER 2 STUDENTS
ASSIGNMENT 01
Based on Study Units 1 - 9
FIXED CLOSING DATE: 17 AUGUST 2018
UNIQUE NUMBER: 868142

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the examination. It is therefore good practice to work through a complete set of four assignments for a given year. For this reason the assignments and worked solutions of previous years are made available under **Additional Resources** on myUnisa – see also the letter **MAT3701 Exam Preparation** under **Additional Resources**.

Worked solutions to all the questions for this assignment will be made available on myUnisa shortly after the due date. Your answers to the assignment questions should be fully motivated.

QUESTION 1

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, and let

$$W_1 = \{X \in M_{2\times 2}(C) : AX = XB\}$$

and

$$W_2 = \{X \in M_{2\times 2}(C) : AX = iX\}.$$

(a) Show that W_1 is a subspace of $M_{2\times 2}(C)$.

(b) Given that W_2 is a subspace of $M_{2\times 2}(C)$, find a basis for $W_1 \cap W_2$.

(c) Explain whether or not $M_{2\times 2}(C) = W_1 \oplus W_2$.

QUESTION 2

Let f_a, f_b, f_c be the Lagrange polinomials associated with the distinct real numbers a, b, c respectively. Define $T : P_2(R) \to P_2(R)$ by $T(g) = g - g(a) f_a$.

(a) Show that T is a linear operator.

(b) Explain whether or not T is a projection.

(c) Find $[T]_\beta$ where $\beta = \{f_a, f_b, f_c\}$.

QUESTION 3

Let $T : V \to V$ be a linear operator on a vector space V over F.

(a) Show that $T^2 = I \Leftrightarrow R(T + I) \subseteq N(T - I)$.

(b) Suppose that $V = R^4$ and $F = R$. Find the formula for T such that $T^2 = I,\ R(T + I) =$ span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $N(T + I) = R(T + I)^{\perp}$.

QUESTION 4

Let $T : M_{2\times 2}(C) \to M_{2\times 2}(C)$ be the linear operator defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b - c \\ -b + c & d \end{bmatrix}$$

(a) Show that T satisfies the test for diagonalizability.

(b) Find a basis τ for $M_{2\times 2}(C)$ consisting of eigenvectors of T and write down $[T]_\tau$.

(c) Show that T satisfies the equation $(T - I)^3 = T - I$.

QUESTION 5

Let

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix} \in M_{2\times 2}(R)$$

Find A^n where n is an arbitrary positive integer and express it as a single matrix.

QUESTION 6

Let $T : C^3 \to C^3$ be a nonzero linear operator such that $T^2 = -T$.

(a) Show that $\lambda = -1$ is an eigenvalue of T.

(b) Show that $R(T) = E_{-1}(T)$, the eigenspace of T associated with $\lambda = -1$.

(c) Explain whether or not $C^3 = R(T) \oplus N(T)$.

(d) Briefly explain whether or not T is diagonalizable.

QUESTION 7

Let $T : C^3 \to C^3$ be the linear operator defined by

$$T(z_1, z_2, z_3) = (z_1, 2z_2 + iz_3,\ 2iz_2 - z_3).$$

(a) Show that T is a projection.

(b) Find a basis for the space onto which T projects.

(c) Find a basis for the space along which T projects.

QUESTION 8

Let

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

(a) Show that A is a regular transition matrix.

(b) Find $\lim\limits_{m \to \infty} A^m$.

(c) Describe the Gerschgorin discs in which the eigenvalues of A lie.

B.2 Assignment 02

ONLY FOR SEMESTER 2 STUDENTS **ASSIGNMENT 02** **Based on Study Units 10 - 17** **FIXED CLOSING DATE: 21 SEPTEMBER 2018** **UNIQUE NUMBER: 711122**

Please note that we will only mark a selection of the questions. It is therefore in your own best interest to do all the questions. The fact that a question is not marked does not mean that it is less important than one that is marked. We try to cover the whole syllabus over the two semesters (4 assignments) and to use these assignments to help you prepare for the examination. It is therefore good practice to work through a complete set of four assignments for a given year. For this reason the assignments and worked solutions of previous years are made available under **Additional Resources** on myUnisa – see also the letter **MAT3701 Exam Preparation** under **Additional Resources**.

Worked solutions to all the questions for this assignment will be made available on myUnisa shortly after the due date. Your answers to the assignment questions should be fully motivated.

QUESTION 1

Let $<,>$ denote the standard inner product on C^n, i.e. $< x, y >= y^*x$ where x and y are column vectors in C^n. Let $A \in M_{n\times n}(C)$ be a non-singular matrix. Show that $<,>_1$: $C^n \times C^n \to C$ defined by $< x, y >_1 = < Ax, Ay >$ is an inner product on C^n over C.

QUESTION 2

Given that the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 6 \\ 3x_1 + x_2 &= 0 \end{aligned}$$

is inconsistent, find a least squares approximate solution in R^2 (see SG: Section 12.6, p 120).

QUESTION 3

Let $T : C^3 \to C^3$ be the linear operator defined by

$$T(z_1, z_2, z_3) = (z_2 + z_3, iz_1 + z_2 + z_3,\ z_2 + z_3)$$

(a) Find the formula for $T^*(z_1, z_2, z_3)$.

(b) Find a basis for $N(T)$.

(c) Find a basis for $R(T^*)$.

(d) Show that $R(T^*) = N(T)^{\perp}$.

QUESTION 4

Let V be a finite-dimensional inner product space and let $y \in V$ be a unit vector. Define $T : V \to V$ by $T(x) = x - \langle x, y\rangle y$ for all $x \in V$.

(a) Show that T is linear.

(b) Show that T is a projection.

(c) Show that T is a self-adjoint.

(d) Briefly explain why T is an orthogonal projection.

(e) Describe the subspace onto which T projects.

QUESTION 5

Let T be a linear operator on a finite-dimensional inner product space V and let W be a T- invariant subspace of V. Show that:

(a) $W^{\perp}$ is T^*- invariant.

(b) If T is invertible, then W is T^{-1} - invariant.

(c) If T is unitary, then $W^{\perp}$ is T- invariant.

QUESTION 6

Let $f : R^2 \to R^2$ be a rigid motion and $v_0 \in R^2$ a fixed vector. Define $(f^{-1} + v_0)(v) = f^{-1}(v) + v_0$ for all $v \in R^2$.

(a) Show that $f^{-1} + v_0$ is a rigid motion.

(b) Suppose f is defined by

$$f(a,b) = (1+b, 3+a)$$

Express f in the form $f = g \circ T$ where g is a translation and T an orthogonal operator on R^2. Find the vector v_0 by which g translates. If T is a rotation, find the angle θ through which it rotates. If T is a reflection about a line L through the origin, find the equation of L and the angle α it makes with the positive x-axis.

QUESTION 7

Let $V = P_2(R)$ denote the inner product space over R with inner product defined by

$$\langle g, h \rangle = g(-1)\,h(-1) + g(0)h(0) + g(1)\,h(1)$$

Let $T : V \to V$ be the orthogonal projection on

$$W = \text{span}\,(S) \text{ where } S = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\,x \right\}$$

(a) Show that S is orthonormal.

(b) Find the formula for $T(ax^2 + bx + c)$ expressed as a single polynomial in terms of $a,\ b,\ c.$

(c) Use T to express x^2 as $x^2 = f + g$ where $f \in W$ and $g \in W^{\perp}$.

(d) Find the polynomial in W closest to x^2.

QUESTION 8

It is given that $A \in M_{3\times 3}(C)$ is a normal matrix with eigenvalues i and $-i$ and corresponding eigenspaces

$$E_i = \text{span}\left\{ \frac{1}{3}(-2,1,2), \frac{1}{3}(2,2,1) \right\}$$

and

$$E_{-i} = \text{span}\left\{ \frac{1}{3}(1,-2,2) \right\}$$

(a) Find the spectral decomposition of A.

(b) Find A.