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MODULE 4

MODULE 5

MODULE 6

LEARNING UNIT 4: FOURIER SERIES OVER ANY RANGE

MODULE 4
LAPLACE TRANSFORMS

LEARNING UNIT 1
PROPERTIES OF LAPLACE TRANSFORMS

OUTCOMES

At the end of this learning unit, you should be able to

- define a Laplace transform
- derive Laplace transforms of elementary functions using the properties of the Laplace transform
- use a standard list of Laplace transforms

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

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1.1 THE LAPLACE TRANSFORM

In recent years, the Laplace transform has been used increasingly in the solution of differential equations. While of great theoretical interest to mathematicians, the transform provides an easy and effective means of solving many problems occurring in the fields of science and engineering. Its usefulness lies in its ability to transform a differential equation into a purely algebraic problem. We shall begin by studying the theory of the Laplace transform and its inverse, and then go on to apply what we have learnt to the solution of differential equations.

1.1.1 Definition

Let $f(t)$ be a function of t defined for $t > 0$.

Then the Laplace transform of $f(t)$ is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt \qquad (1.1)$$

where, for our purposes, the parameter s is regarded as real (although in more advanced work it can be complex).

It is clear that $F(s)$ exists only if the improper integral (1.1) exists, that is, if the integral converges for some value of s. The values of s for which the transform does exist are those for which $e^{-st} f\left(t\right) \to 0$ if $t \to \infty$.

Consider the function $f(t) = e^{t^2}$.

The Laplace transform, if it exists, is given by $F(s) = \int_0^\infty e^{-st}.e^{t^2} dt = \int_0^\infty e^{t^2-st}\ dt$.

We can see that as $t \to \infty$, the integrand e^{t^2-st} also approaches infinity. Thus the integral does not exist, and so $f(t) = e^{t^2}$ possesses no Laplace transform.

We often denote the Laplace transform as an operator, and this is written as:

$$\mathcal{L}\{f(t)\} = F(s)$$

When referring to textbooks, you should take note of the notation used in the specific book. It is worth noting here that some textbooks use a capital letter for the function of *t*. In this case the corresponding small letter indicates the Laplace transform, that is:

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) \ dt$$

When writing by hand it is easy to confuse the letter *s* with the number 5. To avoid this confusion you may opt to use the letter *p* instead of *s* when writing by hand.

EXAMPLE 1

Use the definition to find the Laplace transform of $f(t) = 1$.

SOLUTION

By definition we have $F(s) = \mathcal{L}\{1\}$

$$= \int_0^\infty e^{-st}(1) \ dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= \left(\frac{0-1}{-s} \right) \qquad \text{(Remember } e^0 = 1\text{)}$$

$$= \frac{1}{s}$$

We stated above that the transform exists for values of *s* for which $e^{-st} f(t) \to 0$ as $t \to \infty$. Now if, in this example, $s < 0$, we would have $\lim_{t\to\infty} e^{-(-s)t} = \lim_{t\to\infty} e^{st}$ which is infinite and the transform will not exist.

But if $s > 0$, then $\lim_{t\to\infty} e^{-st} = 0$, so that the transform exists only for $s > 0$.

We therefore write the result as $\mathcal{L}\{1\} = \dfrac{1}{s}$, $s > 0$.

EXAMPLE 2

If $f(t) = e^{at}$, find $\mathcal{L}\{ f(t) \}$ by using the definition.

SOLUTION

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}.e^{at} \ dt$$

$$= \int_0^\infty e^{-(s-a)t} \ dt$$

$$\mathcal{L}\{e^{at}\} = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$
$$= \left(\frac{0-1}{-(s-a)}\right)$$
$$= \frac{1}{s-a}\ ,\ s > a$$

1.2 PROPERTIES OF THE LAPLACE TRANSFORM

Listed below are some of the more important properties of the Laplace transform with examples of each. These properties can often be used to derive the transforms of complicated functions from those of elementary ones.

1.2.1 Linearity property

If $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, then

$$\mathcal{L}\{a\ f_1(t) + b\ f_2(t)\}$$
$$= a\mathcal{L}\{f_1(t)\}\ + b\mathcal{L}\{f_2(t)\}$$
$$= a\,F_1(s)\qquad + b\,F_2(s)$$

where a and b are constants.

Because of this property we say that $\mathcal{L}$ is a linear operator. The above result can be extended to the sum of more than two functions.

EXAMPLE 3

Find $\mathcal{L}\{\cos at\}$ where a is any real constant.

SOLUTION

$\cos at$ can be expressed in exponential form as: $\cos at = \dfrac{e^{iat} + e^{-iat}}{2}$

Thus $\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\dfrac{e^{iat} + e^{-iat}}{2}\right\}$ and using the linear property of the transform

$$\mathcal{L}\{\cos at\} = \frac{1}{2}\mathcal{L}\left\{e^{iat}\right\} + \frac{1}{2}\mathcal{L}\left\{e^{-iat}\right\} \qquad (1.2)$$

Now $\mathcal{L}\left\{e^{iat}\right\} = \int_0^\infty e^{-st}.e^{iat}\ dt$ and $\mathcal{L}\left\{e^{-iat}\right\} = \int_0^\infty e^{-(s+ia)t}\ dt$

$$= \int_0^\infty e^{-(s-ia)t}\ dt$$

$$= \left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty$$

$$= \frac{1}{s-ia},\ \ s > 0$$

$$= \left[\frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^\infty$$

$$= \frac{1}{s+ia},\ \ s > 0$$

Thus equation (1.2) becomes:

$$\mathcal{L}\{\cos at\} = \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \frac{1}{2}\left(\frac{s+ia+s-ia}{s^2-i^2a^2}\right)$$

$$= \frac{s}{s^2+a^2}\ ,\ \ s > 0$$

We can, of course, apply the definition and evaluate $\mathcal{L}\{\cos at\} = \int_0^\infty \cos at.e^{-st}dt$ using integration by parts without using the linear property.

In the same way it can be shown that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2},\ \ s > 0$.

1.2.2 First translation or shifting property

> If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, where a is any real constant.

EXAMPLE 4

Evaluate $\mathcal{L}\{e^{-3t}\cos t\}$.

SOLUTION

From example 3 we know that $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$,

so that: $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$ (1.3)

Using the first translation property we have

$$\mathcal{L}\{e^{-3t}\cos t\} = \frac{s-(-3)}{(s-(-3))^2+1} = \frac{s+3}{(s+3)^2+1}$$

where the function of s in (1.3) has become the same function of $(s - a)$. Thus we have replaced the s with $s - a$, which in this case is $s - (-3) = s + 3$.

Thus $\mathcal{L}\{e^{-3t}\cos t\} = \dfrac{s+3}{s^2+6s+10}$.

1.2.3 Second translation or shifting property

> If $\mathcal{L}\{\, f(t)\,\} = F(s)$ and $g(t)$ is the function defined by $g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$
>
> then $\mathcal{L}\{g(t)\} = e^{-as}F(s)$, where a is any real constant.

EXAMPLE 5

Find $\mathcal{L}\{g(t)\}$ if $g(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right) & t > \dfrac{2\pi}{3} \\ 0 & t < \dfrac{2\pi}{3} \end{cases}$

SOLUTION

Using the result of example 3, $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1} = F(s)$ and the second shifting property

$$\mathcal{L}\{g(t)\} = e^{-as}F(s) = e^{-\frac{2\pi s}{3}}\left(\frac{s}{s^2+1}\right)$$

1.2.4 Change of scale property

> If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$ where a is any real constant.

EXAMPLE 6

If $\mathcal{L}\{f(t)\} = \dfrac{s^2-s+1}{(2s+1)^2(s-1)}$, find $\mathcal{L}\{f(2t)\}$.

SOLUTION

By the change of scale property $\mathcal{L}\{f(2t)\} = \frac{1}{2} F\left(\frac{s}{2}\right)$

We are given that $F(s) = \dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)}$

$$\text{Thus} \quad F\left(\frac{s}{2}\right) = \frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(\frac{2s}{2}+1\right)^2 \left(\frac{s}{2}-1\right)}$$

$$= \frac{\frac{1}{4}\left(s^2 - 2s + 4\right)}{\frac{1}{2}(s+1)^2 (s-2)}$$

$$= \frac{\left(s^2 - 2s + 4\right)}{2\left(s+1\right)^2 (s-2)}$$

$$\text{and} \quad \mathcal{L}\{f(2t)\} = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}$$

1.2.5 Multiplication by t^n

If $\mathcal{L}\{ f(t) \} = F(s)$, then: $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$

$$= (-1)^n F^{(n)}(s)$$

EXAMPLE 7

Find the Laplace transform of $t^2 e^{at}$.

SOLUTION

From example 2 we know that: $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

$$= F(s)$$

Thus $\mathcal{L}\{t^2 e^{at}\} = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{1}{s-a}\right)$

$$= -\frac{d}{ds}\left(\frac{1}{s-a}\right)^2$$

$$= \frac{2}{(s-a)^3}, \qquad s > a$$

In general $\mathcal{L}\{t^n e^{at}\} = (-1)^n \dfrac{d^n}{ds^n}\left(\dfrac{1}{s-a}\right)$ (1.4)

$$= \frac{n!}{(s-a)^{n+1}}, \qquad s > a$$

where $n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ and $0! = 1$ (by definition).

Putting $a = 0$ in (1.4) leads to:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad s > 0$$

1.2.6 Division by *t*

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$ provided that $\lim\limits_{t \to 0} \dfrac{f(t)}{t}$ exists.

EXAMPLE 8

Find $\mathcal{L}\left\{\dfrac{e^{-at} - e^{-bt}}{t}\right\}$.

SOLUTION

Let $f(t) = e^{-at} - e^{-bt}$. (1.5)

Before we can apply the rule for division by *t*, we must check that $\lim\limits_{t \to 0} \dfrac{f(t)}{t}$ exists.

Then $\lim\limits_{t - 0} \dfrac{f(t)}{t} = \lim\limits_{t - 0}\left(\dfrac{e^{-at} - e^{-bt}}{t}\right)$.

This has the indeterminate form $\frac{0}{0}$, so we have to use l'Hospital's rule to obtain:

$$
\begin{aligned}
\lim_{t-0} \frac{f(t)}{t} &= \lim_{t-0}\left[\frac{\frac{d}{dt}\left(e^{-at}-e^{-bt}\right)}{\frac{d}{dt}(t)}\right] \\
&= \lim_{t-0}\left(\frac{-ae^{-at}+be^{-bt}}{1}\right) \\
&= b-a
\end{aligned}
$$

Thus the limit exists and we can use the rule to find $\mathcal{L}\left\{\frac{f(t)}{t}\right\}$.

From equation (1.5) we have $\mathcal{L}\{f(t)\} = L\{e^{-at}-e^{-bt}\}$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

which, on using the result of example 2, becomes:

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{s+a} - \frac{1}{s+b} \\
&= F(s)
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty F(u)\,du \\
&= \int_s^\infty \left(\frac{1}{u+a} - \frac{1}{u+b}\right)\,du
\end{aligned}
$$

where the function of *s* has become the same function of *u*.

Thus

$$
\begin{aligned}
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \left[\ell n\ (u+a) - \ell n(u+b)\right]_s^\infty \\
&= \left[\ell n\left(\frac{u+a}{u+b}\right)\right]_s^\infty \\
&= 0 - \ell n\left(\frac{s+a}{s+b}\right) \\
&= \ell n\left(\frac{s+b}{s+a}\right)
\end{aligned}
$$

1.3 LAPLACE TRANSFORM OF DERIVATIVES

If $\mathcal{L}\{ f(t) \} = F(s)$ and $f(t)$ is continuous for $t \geq 0$, then $\mathcal{L}\left\{ f'(t) \right\} = sF(s) - f(0)$, where $f'(t) = \dfrac{d}{dt} f(t)$.

By definition (1.1) we have:

$$\mathcal{L}\left\{ f'(t) \right\} = \int_0^\infty e^{-st} f'(t) \; dt$$
$$= \lim_{B \to \infty} \int_0^B e^{-st} f'(t) \; dt$$

On integrating by parts this becomes:

$$\mathcal{L}\left\{ f'(t) \right\} = \lim_{B \to \infty} \left(\left[e^{-st} f(t) \right]_0^B + s\int_0^B e^{-st} f(t) \; dt \right)$$
$$= \lim_{B \to \infty} \left(e^{-sB} f(B) - f(0) + s\int_0^B e^{-st} f(t) \; dt \right)$$

Since we have assumed the existence of $F(s)$, it follows that $\int_0^\infty e^{-st} f(t) \; dt$

is convergent, and so $\lim_{t \to \infty} e^{-st} f(t) = \lim_{B \to \infty} e^{-sB} f(B) = 0$.

Thus $\mathcal{L}\left\{ f'(t) \right\} = -f(0) + s\int_0^\infty e^{-st} f(t) \; dt$ (1.6)

$$= sF(s) - f(0)$$

In a similar way it can be shown that $\mathcal{L}\left\{ f''(t) \right\} = s^2 F(s) - sf(0) - f'(0)$

and in general $\mathcal{L}\left\{ f^{(n)}(t) \right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0)$.

The above results are of particular importance in the solution of differential equations by means of Laplace transforms. We will return to these in learning unit 4.

EXAMPLE 9

If $f(t) = e^{at}$, find $\mathcal{L}\{ f'(t) \}$.

SOLUTION

Since $f(t) = e^{at}$, we have $f(0) = 1$ and, by example 2:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$
$$= F(s)$$

Thus $$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$
$$= \frac{s}{s-a} - 1$$
$$= \frac{a}{s-a}$$

1.4 LAPLACE TRANSFORM OF INTEGRALS

If $\mathcal{L}\{ f(t) \} = F(s)$, then $\mathcal{L}\left\{f(u)\, du\right\} = \dfrac{F(s)}{s}$.

Let $g(t) = \int_0^t f(u)\ du$

so that $g'(t) = f(t)$

and $G(s) = \mathcal{L}\{g(t)\}$
$$= \mathcal{L}\left\{\int_0^t f(u)\ du\right\}$$

Then, by equation (1.6) $F(s) = \mathcal{L}\{f(t)]$
$$= \mathcal{L}\{g'(t)]$$
$$= sG(s) - g(0)$$

But $$g(0) = \int_0^0 f(u)\ du$$
$$= 0$$

Thus $F(s) = s\,G(s)$

and $$\frac{F(s)}{s} = \mathcal{L}\left\{\int_0^t f(u)\ du\right\} \qquad (1.7)$$

EXAMPLE 10

Evaluate $\mathcal{L}\left\{\int_0^t \sin u\, du\right\}$.

SOLUTION

Let $f(u) = \sin u$

so that $f(t) = \sin t$.

Then, from example 3: $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin\ t\}$

$$= \frac{1}{s^2+1}$$

$$= F(s)$$

By equation (1.7) we have $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$,

thus
$$\mathcal{L}\left\{\int_0^t \sin u\,du\right\} = \frac{1}{s(s^2+1)} \tag{1.8}$$

We can check this by evaluating the integral in (1.8):

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t \sin u\,du\right\} &= \mathcal{L}\left\{\left[-\cos\ u\right]_0^t\right\} \\
&= \mathcal{L}\{-\cos\ t+1\} \\
&= -\mathcal{L}\{\cos\ t\} + \mathcal{L}\{1\} \\
&= -\frac{s}{s^2+1} + \frac{1}{s} \\
&= \frac{-s^2+s^2+1}{s\left(s^2+1\right)} \\
&= \frac{1}{s\left(s^2+1\right)} \quad \text{which agrees with our previous result.}
\end{aligned}$$

We can now compile a standard list of Laplace transforms so that we need not find each transform from scratch using the definition and properties. This table will be supplied in the examination.

1.5 TABLE OF STANDARD LAPLACE TRANSFORMS

	$F(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{\,f(t)\,\}$	defined for
1.	1	$\frac{1}{s}$	$s > 0$
2.	t	$\frac{1}{s^2}$	$s > 0$
3.	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
4.	e^{at}	$\frac{1}{s-a}$	$s > a$
5.	$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
9.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
10.	$t \sin at$	$\frac{2as}{\left(s^2+a^2\right)^2}$	$s > 0$
11.	$t \cos at$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$	$s > 0$
12.	$t \sinh at$	$\frac{2as}{\left(s^2-a^2\right)^2}$	$s > \lvert a \rvert$
13.	t cosh at	$\frac{s^2+a^2}{\left(s^2-a^2\right)^2}$	$s > \lvert a \rvert$
14.	e^{at} sin bt	$\frac{b}{(s-a)^2+b^2}$	$s > a$
15.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

POST-TEST: MODULE 4 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 60 minutes

1. Evaluate $\mathcal{L}\{(t^2 + 1)^2\}$. (4)

2. Find the Laplace transforms of:

   (a) $e^{-2t}\cos 3t$ (5)

   (b) $2e^{3t}\sin 4t$ (5)

3. If $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, find $\mathcal{L}\{\sin 5t\}$ using the change of scale property. (5)

4. Find

   (a) $\mathcal{L}\{t^4 e^{4t}\}$ (2)

   (b) $\mathcal{L}\{t^2 \cos t\}$ (6)

5. If $f(t) = \sin 2t$, evaluate $\mathcal{L}\{f''(t)\}$. (8)

[35]

You should now be able to define a Laplace transform and use the definition and properties of Laplace transforms to find the transforms of elementary functions. You should also be able to use a standard list of Laplace transforms to determine the Laplace transform of a simple function.

We are now ready to move to the next learning unit and investigate inverse Laplace transforms.

MODULE 4
LAPLACE TRANSFORMS

LEARNING UNIT 2
INVERSE LAPLACE TRANSFORMS

OUTCOMES

At the end of this learning unit, you should be able to

- define the inverse Laplace transform
- use a standard list to determine the inverse Laplace transforms of simple functions
- determine inverse Laplace transforms by using partial fractions
- determine inverse Laplace transforms by completing the square

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

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2.1 THE INVERSE LAPLACE TRANSFORM

If $\mathcal{L}\left\{ f\left(t\right)\right\}$ = $F(s)$ = $\int_0^\infty e^{-st} f(t)\,dt$ is the Laplace transform of a function $f(t)$,

then $f(t) = \mathcal{L}^{-1}\left\{F\left(s\right)\right\}$= is called the inverse Laplace transform where $\mathcal{L}^{-1}$ is the inverse transform operator.

For example, since $\mathcal{L}\{e^{at}\}$ = $\dfrac{1}{s-a}$, the inverse transform is $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$.

The inverse transform $f(t)$ of a given $F(s)$ is unique in all the examples we shall deal with.

For real values of *s* we have to refer to a table of commonly occurring functions and their Laplace transforms to find function $f(t)$ whose transform is the given function $F(s)$. Therefore given a transform, we can write down the corresponding function in *t*, provided we can recognise it from a table of transforms. In practice, however, this is not always easy, since $F(s)$ is often a complicated function, which does not appear in the table in its given form. We then have to rewrite it as one or more "standard" functions as listed in the table. We use the same table as in learning unit 1 (page 13) to find the inverse Laplace transforms, this time reading from right to left. Remember that this table will be supplied in the examination. The table is given again on the next page.

Table of Laplace Transforms

	$F(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	
1.	1	$\frac{1}{s}$	$s > 0$
2.	t	$\frac{1}{s^2}$	$s > 0$
3.	t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
4.	e^{at}	$\frac{1}{s-a}$	$s > a$
5.	$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
9.	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
10.	$t \sin at$	$\frac{2as}{\left(s^2+a^2\right)^2}$	$s > 0$
11.	$t \cos at$	$\frac{s^2-a^2}{\left(s^2+a^2\right)^2}$	$s > 0$
12.	$t \sinh at$	$\frac{2as}{\left(s^2-a^2\right)^2}$	$s > \lvert a \rvert$
13.	$t \cosh$ at	$\frac{s^2+a^2}{\left(s^2-a^2\right)^2}$	$s > \lvert a \rvert$
14.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
15.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

2.2 PROPERTIES OF THE INVERSE LAPLACE TRANSFORM

For all the properties of the Laplace transform discussed in the previous unit, there are corresponding properties of the inverse transform. These are listed below with examples, with one useful addition, namely the convolution property.

2.2.1 Linearity property

If $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then

$$\begin{aligned}&\mathcal{L}^{-1}\{a F_1(s) + b F_2(s)\}\\ &= a\mathcal{L}^{-1}\{F_1(s)\} + b\mathcal{L}^{-1}\{F_2(s)\}\\ &= a\, f_1(t) + b\, f_2(t)\end{aligned}$$

where a and b are constants.

This result holds for any number of functions.

EXAMPLE 1

Find the inverse transform of $F(s) = \dfrac{5s+4}{s^3}$.

SOLUTION

There is nothing in the table which resembles $F(s)$ in the given form, so we rewrite it as:

$$F(s) = \frac{5}{s^2} + \frac{4}{s^3}$$

By the linearity property we then have:

$$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} \qquad (2.1)$$

Referring to the table, we see from no.2 that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$ and from no.3 that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = \dfrac{t^2}{2!}$.

Thus equation (2.1) becomes:

$$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3}\right\} = 5t + 2t^2$$

2.2.2 First translation or shifting property

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$.

EXAMPLE 2

Find $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^3}\right\}$.

SOLUTION

Let $F(s) = \dfrac{2}{s^3}$, then, by no. 3 of the table, $\mathcal{L}^{-1}\{F(s)\} = t^2$.

By the first translation property $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^3}\right\} = e^{3t}t^2$

where the function of s has become a function of $s - a$, which in this case is $(s - 3)$.

2.2.3 Second translation or shifting property

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

EXAMPLE 3

Find $\mathcal{L}^{-1}\left\{\dfrac{se^{-2s}}{s^2+16}\right\}$.

SOLUTION

Let $F(s) = \dfrac{s}{s^2+16}$,

then $\mathcal{L}^{-1}\{F(s)\} = \cos 4t$ (no. 6 of the table)

and $\mathcal{L}^{-1}\left\{e^{-as}\left(\dfrac{s}{s^2+16}\right)\right\} = \begin{cases} \cos 4(t-2) & t > 2 \\ 0 & t < 2 \end{cases}$

2.2.4 Change of scale property

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\{F(as)\} = \dfrac{1}{a} f\left(\dfrac{t}{a}\right)$.

EXAMPLE 4

Find $\mathcal{L}^{-1}\left\{\dfrac{2s}{4s^2+4}\right\}$.

SOLUTION

Let $F(s) = \dfrac{s}{s^2+4}$,

then $\mathcal{L}^{-1}\{F(s)\} = \cos 2t$ (no. 6 of the table)

and so
$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+4}\right\} = \mathcal{L}^{-1}\{F(2s)\} = \frac{1}{2}\cos\frac{2t}{2} = \frac{1}{2}\cos t$$

where we have used the change of scale property.

2.2.5 Multiplication by s^n

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$, then $\mathcal{L}^{-1}\{sF(s)\} = \dfrac{d}{dt} f(t) = f'(t)$.

This result follows from taking the inverse transform of equation (1.6) of learning unit 1, and it can be generalised to $\mathcal{L}^{-1}\{s^n F(s)\}$.

EXAMPLE 5

If $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \sin 2t$, find $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+4}\right\}$.

SOLUTION

Since $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\}$

$= \sin 2t$

$= f(t)$

and $f(0) = \sin 0 = 0$,

we have $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{s\left(\dfrac{2}{s^2+4}\right)\right\}$

$= \dfrac{d}{dt}\ (\sin 2t)$

$= 2\cos 2t$

2.2.6 Division by *s*

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(u)\,du$.

Thus division by *s* (or multiplication by $\frac{1}{s}$) has the effect of integrating $f(t)$ from 0 to *t*.

Taking the inverse transform of equation (1.7) in learning unit 1 leads to the above result, which can be generalised to $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s^n}\right\} = \int_0^t \ldots\ldots\ldots\ldots\ldots\ldots \int_0^t f(t)\,dt^n$.

EXAMPLE 6

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$.

SOLUTION

Let $F(s) = \dfrac{1}{s+1}$,

then, since $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}$

$= e^{-t}$

$= f(t)$

it follows that $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$

$$= \int_0^t e^{-u} du$$

where the function of *t* has become the same function of *u*, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \left[-e^{-u}\right]_0^t$$
$$= -e^{-t} + 1$$
$$= 1 - e^{-t}$$

2.2.7 The convolution property

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)\}G(s) = \int_0^t f(u)g(t-u)\, du = f * g$$

where $f * g$ is called the convolution of $f(t)$ and $g(t)$.

This property gives us a method of finding the inverse transform of a product.

EXAMPLE 7

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}$.

SOLUTION

If we let $F(s) = \frac{1}{s-2}$ and $G(s) = \frac{1}{s+3}$, then by no. 4 of the table:

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \quad \text{and} \quad \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$
$$= e^{2t} \qquad\qquad = e^{-3t}$$
$$= f(t) \qquad\qquad = g(t)$$

On using the convolution property we have $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u)\, du$

$$= \int_0^t e^{2u}.e^{-3(t-u)}\, du$$

where $f(t)$ has become the same function of *u* and $g(t)$ the same function of $(t - u)$.

Thus $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\} = \int_0^t e^{5u-3t}du$

$$= \frac{1}{5}\left[e^{5u-3t}\right]_0^t$$

$$= \frac{1}{5}\left(e^{2t} - e^{-3t}\right)$$

2.3 INVERSE LAPLACE TRANSFORMS OF DERIVATIVES

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then: $\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\}$ (2.2)

$$= (-1)^n t^n f(t)$$

This result follows directly from section 1.2.5 of learning unit 1 where it was stated that:

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

Taking the inverse transform of this gives equation (2.2). For n = 1 we have:

$$\mathcal{L}^{-1}\left\{F'(s)\right\} = -t\, f(t)$$

EXAMPLE 8

If $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$, find $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\}$.

SOLUTION

$$\frac{d}{ds}\left(\frac{1}{s+3}\right) = \frac{-1}{(s+3)^2}$$

and $\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{2}{(s+3)^3}$

Thus $\frac{1}{(s+3)^3} = \frac{1}{2}\,\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$

On taking the inverse transform of both sides we get

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s+3\right)^3}\right\} = \frac{1}{2}\ \mathcal{L}^{-1}\left\{\frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)\right\}$$
$$= \frac{1}{2}(-1)^2 t^2 e^{-3t}$$

where we have used equation (2.2) with $n = 2$.

Thus $\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^3}\right\} = \frac{1}{2}t^2 e^{-3t}$.

2.4 INVERSE LAPLACE TRANSFORMS OF INTEGRALS

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\left\{\int_s^\infty F(u)\ du\right\} = \frac{f(t)}{t}$ (2.3)

This follows from 1.2.6 of learning unit 1 which states that $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$.

Equation (2.3) is simply the inverse transform of this.

EXAMPLE 9

Find $\mathcal{L}^{-1}\left\{\int_0^\infty \frac{1}{u(u+1)}du\right\}$.

SOLUTION

Let $f(u) = \frac{1}{u(u+1)}$ so that $F(s) = \frac{1}{s(s+1)}$.

In example 6 we found that $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t} = f(t)$.

Thus $\mathcal{L}^{-1}\left\{\int_s^\infty F(u)\ du\right\} = \mathcal{L}^{-1}\left\{\int_s^\infty \frac{1}{u(u+1)}\,du\right\}$
$$= \frac{f(t)}{t}$$
$$= \frac{1-e^{-t}}{t}$$

EXAMPLE 10

Find the inverse Laplace transform of $F(s) = \dfrac{7s-6}{s^2-s-6}$.

SOLUTION

Since $s^2 - s - 6 = (s - 3)(s + 2)$, we can express $F(s)$ in partial fractions.

Let $\dfrac{7s-6}{(s-3)(s+2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2}$

We multiply both sides of the equation by $(s - 3)(s + 2)$ to get:

$$7s - 6 = A(s + 2) + B(s - 3)$$
$$= (A + B)s + (2A - 3B)$$

Equating coefficients of like terms gives the following:

s terms: $7 = A + B$

$A = 7 - B$

constant terms: $-6 = 2A - 3B$

$= 2(7 - B) - 3B$

$-20 = -5B$

$B = 4$

and $A = 3$

Thus $F(s) = \dfrac{7s-6}{s^2-s-6}$

$= \dfrac{3}{s-3} + \dfrac{4}{s+2}$

We can now use the linearity property. Each term is easily recognisable as the transform of an exponential function (see table, no. 4) and hence the inverse transform is:

$$\mathcal{L}^{-1}\left\{\frac{7s-5}{s^2-s-6}\right\} = 3e^{3t} + 4e^{-2t}$$

EXAMPLE 11

Find the inverse transform of $F(s) = \dfrac{2s+1}{(s+1)(s^2+1)}$

SOLUTION

Let $\dfrac{2s+1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$.

Since the second partial fraction has a denominator of the second degree, we assume the numerator to have the form $Bs + C$. On multiplying both sides of the equation by $(s + 1)(s^2 + 1)$ we obtain:

$$\begin{aligned} 2s+1 &= A(s^2+1) + (Bs+C)(s+1) \\ &= (A+B)s^2 + (B+C)s + (A+C) \end{aligned}$$

Equating coefficients of like terms gives the following:

s^2 terms: $A + B = 0$

$B = -A$

s terms: $2 = B + C$

$2 = -A + C$

constant terms: $1 = A + C$

Hence $C = \dfrac{3}{2}$, $A = -\dfrac{1}{2}$ and $B = \dfrac{1}{2}$

Thus
$$\begin{aligned} F(s) &= \frac{-1}{2(s+1)} + \frac{s+3}{2(s^2+1)} \\ &= -\frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{s}{s^2+1}\right) + \frac{3}{2}\left(\frac{1}{s^2+1}\right) \end{aligned}$$

and $\mathcal{L}^{-1}\{F(s)\} = -\dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\} + \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} + \dfrac{3}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

The inverse transforms of these partial fractions are found from the table and we obtain:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s+1}{(s+1)(s^2+1)}\right\} &= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{3}{2}\sin t \\ &= \frac{1}{2}(\cos t + 3\sin t - e^{-t}) \end{aligned}$$

EXAMPLE 12

Find the inverse transform of $F(s) = \dfrac{4s^2 - 4s + 2}{(s-1)^3}$.

SOLUTION

Since $F(s)$ has the repeated factor $(s - 1)^3$ for its denominator, the corresponding partial fractions must have $s - 1$, $(s - 1)^2$ and $(s - 1)^3$ as denominators.

Let $\dfrac{4s^2 - 4s + 2}{(s-1)^3} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s-1)^3}$.

Multiplying both sides of the equation by $(s - 1)^3$ leads to:

$$\begin{aligned} 4s^2 - 4s + 2 &= A(s-1)^2 + B(s-1) + C \\ &= A(s^2 - 2s + 1) + B(s-1) + C \\ &= As^2 + (-2A + B)s + (A - B + C) \end{aligned}$$

We equate coefficients of like terms to get the following:

s^2 terms: $A = 4$

s terms: $-4 = -2A + B$

$-4 = -8 + B$

$B = 4$

constant terms: $2 = A - B + C$

$2 = 4 - 4 + C$

$C = 2$

Therefore $$F(s) = \frac{4}{s-1} + \frac{4}{(s-1)^2} + \frac{2}{(s-1)^3}$$

And $$\mathcal{L}^{-1}\{F(s)\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

We now use the table to find that:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{4s^2 - 4s + 2}{(s-1)^3}\right\} &= 4e^t + 4te^t + t^2e^t \\ &= e^t(4 + 4t + t^2) \\ &= e^t(t+2)^2 \end{aligned}$$

Remember that when resolving a function into partial fractions you must find the prime factors of the denominator.

For example, $\dfrac{1}{(s+2)(s^2-1)}$ is not in prime factors, since $(s^2 - 1) = (s - 1)(s + 1)$. When the denominator for $F(s)$ has no real factors, the use of the partial fraction method would involve the use of complex algebra. Instead, if we can, we write the denominator as the sum or difference of two squares.

For example: $$\begin{aligned} s^2+2s+5 &= (s^2+2s+1)+4 \\ &= (s+1)^2+4 \\ s^2-8s-9 &= (s^2-8s+16)-25 \\ &= (s-4)^2-25 \end{aligned}$$

The numerator of $F(s)$ is then expressed in terms of the same function of s (in the above examples $(s + 1)$ and $(s - 4)$). The first translation property now enables us to remove an exponential factor e^{-t} and e^{ut} above, and the transform of what remains can easily be found from the table.

EXAMPLE 13

Find $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s^2+4s+13}\right\}$.

SOLUTION

Since the denominator has no real factors we write it as

$$s^2 = 4s+13 = (s+2)^2+9$$

which is the sum of two squares. Next we express the numerator in terms of $s + 2$ as well, thus obtaining:

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{2s+1}{(s+2)^2+9}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2s+4-3}{(s+2)^2+9}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2(s+2)-3}{(s+2)^2+9}\right\} \\ &= \mathcal{L}^{-1}\left\{F(s+2)\right\} \end{aligned}$$

On using the first translation or shifting property this becomes

$$\mathcal{L}^{-1}\{F(s+2)\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+9}\right\}$$

$$= e^{-2t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right) \text{ by the linearity property.}$$

We now use the table to obtain:

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+4s+13}\right\} = e^{-2t}\left(2\cos 3t - \sin 3t\right)$$

EXAMPLE 14

Find $\mathcal{L}^{-1}\left\{\frac{1-5s}{s^2+2s-3}\right\}$.

SOLUTION

Using the method described above of expressing both numerator and denominator in terms of the same function $(s-a)$ we obtain

$$\mathcal{L}^{-1}\left\{\frac{1-5s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1-5s}{(s+1)^2-4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6-5(s+1)}{(s+1)^2-4}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{6-5s}{s^2-4}\right\}$$

where e^{-t} has been removed by virtue of the first translation property.

Use the linearity property to write

$$e^{-t}\mathcal{L}^{-1}\left\{\frac{6-5s}{s^2-4}\right\} = e^{-t}\left(3\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} - 5\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\}\right)$$

and from the table we find that

$$\mathcal{L}^{-1}\left\{\frac{1-5s}{s^2+2s-3}\right\} = e^{-t}(3\sinh 2t - 5\cosh 2t)\,.$$

Sometimes the removal of some power of s or $\frac{1}{s}$ will leave $F(s)$ in a form which can be more easily handled.

From learning unit 1 we use the facts that:

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then $\mathcal{L}^{-1}\{sF(s)\} = f'(t)$ and $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

EXAMPLE 15

Find $\mathcal{L}^{-1}\left\{\frac{s^2}{\left(s^2+4\right)^2}\right\}$.

SOLUTION

From the table we see that $\mathcal{L}^{-1}\left\{\frac{2as}{\left(s^2+a^2\right)^2}\right\} = t\sin at$

so that $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin at}{2a}$.

Let $F(s) = \frac{s}{(s^2+4)^2}$,

then
$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+4\right)^2}\right\}\\ &= \frac{t\sin 2t}{4}\\ &= f(t)\end{aligned}$$

It follows that:
$$\begin{aligned}\mathcal{L}^{-1}\{sF(s)\} &= \mathcal{L}^{-1}\left\{\frac{s^2}{\left(s^2+4\right)^2}\right\}\\ &= \frac{d}{dt}f(t)\\ &= \frac{d}{dt}\left(\frac{t\sin 2t}{4}\right)\\ &= \frac{1}{4}\left(\sin 2t + 2t\cos 2t\right)\end{aligned}$$

EXAMPLE 16

Find $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^5}\right\}$.

SOLUTION

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^5}\right\} = \frac{t^4e^{-t}}{4!}$$
$$= \frac{t^4e^{-t}}{24}$$

Let $F(s) = \dfrac{1}{(s+1)^5}$.

Then $\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^5}\right\}$

$$= \frac{t^4e^{-t}}{24}$$
$$= f(t)$$

Thus $\mathcal{L}^{-1}\left\{sF(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^5}\right\}$

$$= \frac{d}{dt}f(t)$$
$$= \frac{d}{dt}\left(\frac{t^4e^{-t}}{24}\right)$$
$$= \frac{e^{-t}}{24}\left(4t^3 - t^4\right)$$

EXAMPLE 17

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s+1)^2}\right\}$.

SOLUTION

Let $F(s) = \dfrac{1}{(s+1)^2}$,

then from the table we find that: $\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}$

$$= te^{-t}$$
$$= f(t)$$

Thus $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^2}\right\}$

$$= \int_0^t f(u) \; du$$

$$= \int_0^t ue^{-u} \; du$$

The integrand is the product of two functions of *u*, so we have to integrate by parts.

To refresh your memory of integration by parts: $\int f \frac{dg}{du} \; du = fg - \int g \frac{df}{du} \; du$

In our example we put $f = u$ and $\frac{dg}{du} = e^{-u}$.

Thus $\frac{df}{du} = 1$ and $g = -e^{-u}$.

Thus $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \left[-ue^{-u}\right]_0^t \left(-e^{-u}\right) \; du$

$$= -te^{-t} - \left[e^{-u}\right]_0^t$$

$$= -te^{-t} - e^{-t} + 1$$

$$= 1 - e^{-t}(1+t)$$

EXAMPLE 18

Use the convolution property of the inverse transform to find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$.

SOLUTION

Let $F(s) = \frac{1}{(s+2)^2}$ and $G(s) = \frac{1}{s-2}$

so that $\mathcal{L}^{-1}\{F(s)\} = te^{-2t} = f(t)$

and $\mathcal{L}^{-1}\{G(s)\} = e^{2t} = g(t)$.

Then, by the convolution property:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u) \; du$$

$$= \int_0^t u \; e^{-2u}e^{2(t-u)} \; du$$

$$= \int_0^t u \; e^{2t-4u} \; du$$

Again it is necessary to integrate by parts.

If we put $f = u$ and $\dfrac{dg}{du} = e^{2t-4u}$,

then $\dfrac{df}{du} = 1$ and $g = -\dfrac{1}{4}e^{2t-4u}$.

Thus $$\mathcal{L}^{-1}\{F(s)G(s)\} = \left[-\frac{1}{4}u\ \ e^{2t-4u}\right]_0^t - \int_0^t \left(-\frac{1}{4}\ \ e^{2t-4u}\right)du$$

$$= -\frac{1}{4}t\,e^{-2t} - \left[\frac{1}{16}e^{2t-4u}\right]_0^t$$

$$= -\frac{1}{4}t\,e^{-2t} - \frac{1}{16}\ \ e^{-2t} + \frac{1}{16}\ \ e^{2t}$$

Thus $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\} = \dfrac{1}{16}\left(e^{2t} - e^{-2t} - 4t\,e^{-2t}\right)$

You may remember that sinh at = $\dfrac{1}{2}(e^{at} - e^{-at})$, so that the above result can also be written as

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \frac{1}{8}(\sinh 2t - 2te^{-2t})$$

This example can also be done by using partial fractions.

Let $\dfrac{1}{(s+2)^2(s-2)} = \dfrac{A}{s+2} + \dfrac{B}{s+2} + \dfrac{C}{s-2}$.

Then $A = -\dfrac{1}{16}$, $B = -\dfrac{1}{4}$, $C = \dfrac{1}{16}$.

Thus $$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = -\frac{1}{16}\ \ \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{4}\ \ \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -\frac{1}{16}e^{-2t} - \frac{1}{4}t\ \ e^{-2t} + \frac{1}{16}e^{2t}$$

$$= \frac{1}{16}\left(e^{2t} - e^{-2t} - 4t\ \ e^{-2t}\right)$$

which agrees with our previous result.

2.5 POST-TEST: MODULE 4 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 108 minutes

1. Find the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+16}\right\}$ (2)

(b) $\mathcal{L}^{-1}\left\{\dfrac{2}{s^6}\right\}$ (2)

(c) $\mathcal{L}^{-1}\left\{\dfrac{4s-18}{9-s^2}\right\}$ (4)

(d) $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+9)}\right\}$ (4)

2. If $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-a^2}\right\}=\dfrac{\sinh\ at}{a}$, show that $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2-a^2)^2}\right\}=\dfrac{t\sinh at}{2a}$. (6)

3. Find:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6}{(s+3)^4}\right\}$ (4)

(b) $\mathcal{L}^{-1}\left\{\dfrac{s+4}{s^2+5s+6}\right\}$ (12)

(c) $\mathcal{L}^{-1}\left\{\dfrac{3s-2}{s^2-4s+8}\right\}$ (7)

(d) $\mathcal{L}^{-1}\left\{\dfrac{2}{s(s^2-1)}\right\}$ (7)

(e) $\mathcal{L}^{-1}\left\{\dfrac{4s(s^2-9)}{(s^2+9)^2}\right\}$ (8)

4. Use the convolution property of the inverse transform to show that:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}=e^t-t-1$$ (16)

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You should now be able to define the inverse Laplace transform and use a standard list to determine the inverse Laplace transforms of simple functions. You should also be able to determine inverse Laplace transforms by using partial fractions and by completing the square.

We are now ready to move to the next learning unit and study the Laplace transforms of two special functions, namely the Heaviside unit step function and the Dirac delta function.

MODULE 4

LAPLACE TRANSFORMS

LEARNING UNIT 3

SPECIAL FUNCTIONS

OUTCOMES

At the end of this learning unit, you should be able to

- define the Heaviside unit step function
- use a standard list to determine the Laplace transform of $H(t-c)$
- use a standard list to determine the Laplace transform of $H(t-c).f\left(t-c\right)$
- define the unit impulse or Dirac delta function
- use a standard list to determine the Laplace transform of $\delta\left(t-c\right)$
- determine the inverse Laplace transforms of $H(t-c)$ and $\delta\left(t-c\right)$

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

3.1 THE HEAVISIDE UNIT STEP FUNCTION

One of the main advantages of the Laplace transform in solving differential equations is the ability to handle discontinuous functions. These occur in problems where stimuli are suddenly applied, such as when a switch is opened or closed. They are also to be found in problems on discontinuous loading in beams where, for example, a load is uniformly distributed over one section of a beam, with no load on the remainder.

To deal with this type of problem where a "step discontinuity" is involved, we use the unit step function $U(t)$, also called Heaviside's unit function $H(t)$, which is defined as:

$$U(t) = H(t) = \begin{cases} 0 & \text{when } t < 0 \\ 1 & \text{when } t > 0 \end{cases} \qquad (3.1)$$

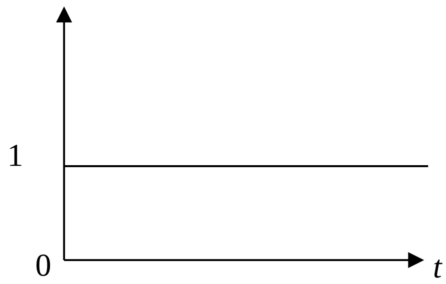

Figure 1

$U(t)$ has a unit step discontinuity at $t = 0$.

From the definition it follows that: $U(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Therefore $U(t - a)$ is a unit step function with step at $t = a$.

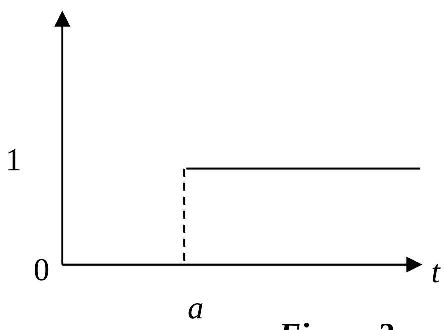

Figure 2

It is clear that $U(t - a)$ is the function obtained by shifting $U(t)$ a distance a along the positive t-axis.

3.2 LAPLACE TRANSFORM OF THE HEAVISIDE UNIT STEP FUNCTION

By definition, the Laplace transform of $U(t-a)$ is given by $\mathcal{L}\left\{U(t-a) = \int_0^\infty e^{-st} U(t-a)\,dt\right\}$.

Since $U(t-a)$ has the value 0 when $t < a$ and 1 when $t > a$, it follows that:

$$\begin{aligned}\int_0^\infty e^{-st} U(t-a)\ dt &= \int_a^\infty e^{-st}(1)dt + \int_0^a e^{-st}(0)dt \\ &= \int_a^\infty e^{-st}\ dt \\ &= \left[\frac{e^{-st}}{-s}\right]_a^\infty \\ &= \left(0 - \frac{e^{-sa}}{-s}\right)\end{aligned}$$

Thus $\mathcal{L}\{U(t-a)\} = \dfrac{e^{-as}}{s}$ (3.2)

and $\mathcal{L}^{-1}\left\{\dfrac{e^{-as}}{s}\right\} = U(t-a)$ (3.3)

Putting $a = 0$ in equations (3.2) and (3.3) gives:

$$\mathcal{L}\{U(t)\} = \frac{1}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = U(t)$$

EXAMPLE 1

Write the function $f(t) = \begin{cases} e^{-t} & t < 3 \\ 0 & t > 3 \end{cases}$ in terms of the unit step function and find $\mathcal{L}\{f(t)\}$.

SOLUTION

In its given form $f(t)$ does not represent a unit step function. If we subtract e^{-t} from both sides of the equation we get:

$$f(t) - e^{-t} = \begin{cases} e^{-t} - e^{-t} & t < 3 \\ 0 - e^{-t} & t > 3 \end{cases}$$

$$= \begin{cases} 0 & t < 3 \\ -e^{-t} & t > 3 \end{cases}$$

$$= -e^{-t} \begin{cases} 0 & t < 3 \\ 1 & t > 3 \end{cases}$$

This is a unit step function, with step discontinuity of magnitude – e^{-t} at $t = 3$.

Thus $\quad f(t) - e^{-t} = -e^{-t}U(t-3)$

$$f(t) = e^{-t} - e^{-t}U(t-3)$$

We now take Laplace transforms of both sides to obtain:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{e^{-t} - e^{-t}U(t-3)\right\} \\
&= \mathcal{L}\left\{e^{-t}\right\} - \mathcal{L}\left\{e^{-t}U(t-3)\right\} \\
&= \frac{1}{s+1} - \int_0^\infty e^{-st}e^{-t}U(t-3)\ dt \\
&= \frac{1}{s+1} - \int_3^\infty e^{-st}e^{-t}(1)\,dt - \int_0^3 e^{-st}(0)\,dt \\
&= \frac{1}{s+1} - \int_3^\infty e^{-(s+1)t}dt \\
&= \frac{1}{s+1} - \left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_3^\infty \\
&= \frac{1}{s+1} - \frac{e^{-3(s+1)}}{(s+1)} \\
&= \frac{1 - e^{-3(s+1)}}{s+1}
\end{aligned}$$

EXAMPLE 2

Express the following function in terms of unit step functions. Hence find the Laplace transform of $f(t)$:

$$f(t) = \begin{cases} 0 & t < \mathrm{a} \\ 2 & a < t < b \\ 0 & t > \mathrm{b} \end{cases}$$

SOLUTION

$f(t)$ may be represented graphically as follows:

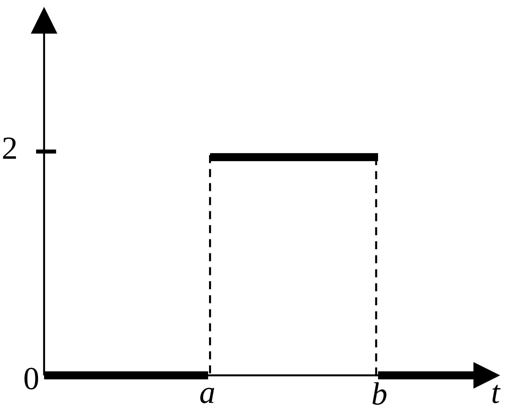

Figure 3

[This function is called a pulse of duration ($b - a$), magnitude 2 and strength $2(b - a)$. The strength of a pulse is the product of its magnitude and duration, that is the area under the graph.]

We would like to express $f(t)$ in terms of the unit step function. We begin by introducing step functions at each point where there is a kink or jump in the graph. In this case at $t = a$ and $t = b$. At each jump we will add in the new behaviour and subtract the old behaviour.

$$\begin{aligned} f(t) &= (2-0)U(t-a) + (0-2)U(t-b) \\ &= 2U(t-a) - 2U(t-b) \qquad (3.4) \\ &= 2\left[U(t-a) - U(t-b)\right] \end{aligned}$$

[In general, a pulse is defined by $f(t) = k\left[U(t-a) - U(t-b)\right]$ where ($b - a$) is the duration and k the magnitude.]

We now take Laplace transforms of both sides of equation (3.4), obtaining:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{2[U(t-a) - U(t-b)]\} \\ &= 2\mathcal{L}\{U(t-a)\} - 2\mathcal{L}\{U(t-b)\} \\ &= \frac{2e^{-as}}{s} - \frac{2e^{-bs}}{s} \\ &= 2\left(\frac{e^{-as} - e^{-bs}}{s}\right), \ s > 0 \end{aligned}$$

$U(t - a)$ can be written in another form using the second translation or shifting property of the inverse transform which states that if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = \begin{cases} 0 & t < a \\ f\left(t-a\right) & t > a \end{cases}$

We also have $f(t-a)U(t-a) = f(t-a)\begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

$$= \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

from which it follows that $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)U(t-a)$

and $e^{-as}F(s) = \mathcal{L}\left\{f(t-a)U(t-a)\right\}$.

EXAMPLE 3

Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$.

SOLUTION

Let $F(s) = \dfrac{1}{s^2}$ so that: $\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$

$$= t$$
$$= f(t)$$

In this example $a = 2$, so that $f(t-a) = f\left(t-2\right)$ and we obtain

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = (t-2)U(t-2).$$

3.3 THE UNIT IMPULSE FUNCTION

A pulse of very large magnitude and infinitesimally short duration (but of finite strength) is called an impulse. For example, in mechanics we use the term when speaking of a large force that acts for a very short time, but produces a finite change in momentum. It is also used in beam problems where a concentrated load acts at a point, thus giving a very great load intensity.

Let $f_x(t) = \begin{cases} \dfrac{1}{\alpha} & 0 \le t \le \alpha \\ 0 & t > \alpha \end{cases}$

represent a pulse of duration α from $t = 0$ to $t = \alpha$ and of magnitude $\frac{1}{\alpha}$.

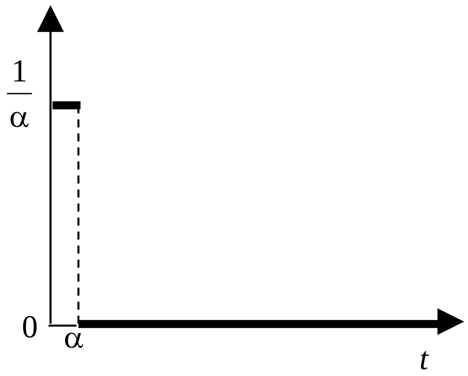

Figure 4

As $\alpha \to 0$, the height of the rectangle increases indefinitely, while the width decreases in such a way that the strength of the pulse, i.e. the area, is always equal to $\frac{1}{\alpha} \times \alpha = 1$. The limit of $f_\alpha(t)$ as $\alpha \to 0$ is denoted by the Dirac delta $\delta(t)$ and we call this limiting function the unit impulse or Dirac delta function. It represents an impulse of unit strength at $t = 0$.

$\delta(t)$ has the following properties:

(a) $\int_0^\infty \delta(t) \; dt = 1$

(b) $\int_0^\infty \delta(t) g(t) \; dt = g(0)$

(c) $\int_0^\infty \delta(t-a) g(t) \, dt = g(a)$ where $g(t)$ is any continuous function

Since $f_\alpha(t)$ is a pulse of magnitude $\frac{1}{\alpha}$ and duration α, it can be expressed in terms of unit step functions (see example 2) as $f_\alpha(t) = \frac{1}{\alpha}\left[U(t) - U(t-\alpha\right]$.

Hence $\delta(t) = \lim_{\alpha \to 0} \; f_\alpha(t) = \lim_{\alpha \to 0} \; \frac{1}{\alpha} \; \left(\left[U(t) - U(t-\alpha\right]\right)$.

A unit impulse at $t = a$ is represented by $\delta(t - a)$ where

$$\delta(t-a) = \lim_{\alpha \to 0} f_\alpha(t-a)$$

And $$f_\alpha(t-a) = \begin{cases} \dfrac{1}{\alpha} & a \le t \le a+\alpha \\ 0 & t > a+\alpha \end{cases}$$

3.4 LAPLACE TRANSFORM OF THE UNIT IMPULSE FUNCTION

In order to evaluate the Laplace transform of $\delta(t-a)$, we first find the Laplace transform of $f_\alpha(t-a)$ and then take the limit of this as $\alpha \to 0$, that is:

$$\mathcal{L}\{\delta(t-a)\} = \lim_{\alpha \to 0} \mathcal{L}\{f_\alpha(t-a)\}$$

By definition:

$$\begin{aligned}
\mathcal{L}\{f_{\alpha(t-a)}\} &= \int_0^\infty e^{-st} f_\alpha(t-a)\ dt \\
&= \int_{a+\alpha}^\infty e^{-st}(0)\, dt + \int_a^{a+\alpha} e^{-st} \tfrac{1}{\alpha}\ dt \\
&= \int_a^{a+\alpha} \frac{e^{-st}}{\alpha}\, dt \\
&= \left[\frac{e^{-st}}{-s\alpha}\right]_a^{a+\alpha} \\
&= \frac{e^{-s(a+\alpha)} - e^{-s\alpha}}{-s\alpha} \\
&= \frac{e^{-as}\left(e^{-s\alpha} - 1\right)}{-s\alpha}
\end{aligned}$$

$$\mathcal{L}\{f_\alpha(t-a)\} = \frac{e^{-as}(1-e^{-s\alpha})}{s\alpha}$$

and $\mathcal{L}\{\delta(t-a)\} = \lim_{\alpha \to 0} \ \mathcal{L}\{f_\alpha(t-a)\}$

$$= e^{-as} \ \lim_{\alpha \to 0}\left(\frac{1-e^{-s\alpha}}{s\alpha}\right)$$

But $\lim_{\alpha - 0}\left(\dfrac{1-e^{-s\alpha}}{s\alpha}\right) = \dfrac{0}{0}$, which is an indeterminate form, so we use l'Hospital's rule, to obtain:

$$\lim_{\alpha\to 0}\left(\frac{1-e^{-s\alpha}}{s\alpha}\right)=\lim_{\alpha\to 0}\left(\frac{\frac{d}{d\alpha}(1-e^{-s\alpha})}{\frac{d}{d\alpha}(s\alpha)}\right)$$

$$=\lim_{\alpha\to 0}\left(\frac{s\,e^{-s\alpha}}{s}\right)$$

$$=\lim_{\alpha\to 0} e^{-s\alpha}$$

$$=1$$

Then $\mathcal{L}\{\delta(t-a)\}=\lim_{\alpha\to 0}\ \mathcal{L}\{f_\alpha(t-a)\}$

$$=e^{-as}$$

Setting a = 0 in this equation, we find that:

$$\mathcal{L}\{\delta(t)\}=\lim_{\alpha\to 0}\ \mathcal{L}\{f_\alpha(t)\}$$

$$=1$$

The inverse transforms are $\delta(t-a)=\mathcal{L}^{-1}\{e^{-as}\}$

and $\delta(t)=\mathcal{L}^{-1}\{1\}$.

EXAMPLE 4

Show $f_\alpha(t-3)$ graphically and find $\mathcal{L}\{\delta(t-3)\}$.

SOLUTION

The function $f_\alpha(t-3)$ is defined as:

$$f_\alpha(t-3)=\begin{cases}\dfrac{1}{\alpha} & 3\le t\le 3+\alpha\\ 0 & t>3+\alpha\end{cases}$$

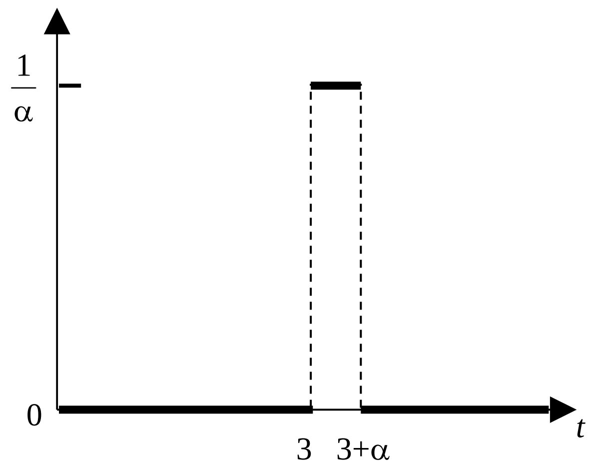

Figure 5

Now $\delta(t-3) = \lim\limits_{\alpha \to 0} f_\alpha(t-3)$

and since $\mathcal{L}\{\delta(t-a)\} = e^{-as}$,

it follows that $\mathcal{L}\{\delta(t-3)\} = e^{-3s}$.

EXAMPLE 5

Find $\mathcal{L}\{(t-1)\delta(t-1)\}$.

SOLUTION

$$\mathcal{L}\{(t-1)\delta(t-1)\} = \lim_{\alpha \to 0} \mathcal{L}\{(t-1)\, f_\alpha(t-1)\}$$

$$\text{where } f_\alpha(t-1) = \begin{cases} \dfrac{1}{\alpha} & 1 \le t \le 1+\alpha \\ 0 & t > 1+\alpha \end{cases}$$

$$\text{Then } \mathcal{L}\{(t-1)\, f_\alpha(t-1)\} = \int_0^\infty e^{-st}(t-1)\, f_\alpha(t-1)\, dt$$

$$= \int_1^{1+\alpha} \frac{e^{-st}}{\alpha}(t-1)\, dt$$

Since the integrand is the product of two functions of *t*, we have to integrate by parts.

Let $f = (t-1)$ and $\dfrac{dg}{dt} = \dfrac{e^{-st}}{\alpha}$.

Then $\dfrac{df}{dt} = 1$ and $g = \dfrac{e^{-st}}{-s\alpha}$.

Thus: $\mathcal{L}\left\{(t-1)\left(f_\alpha(t-1)\right)\right\} = \left[(t-1)\dfrac{e^{-st}}{-s\alpha}\right]_1^{1+\alpha} - \displaystyle\int_1^{1+\alpha} \dfrac{e^{-st}}{-s\alpha}\,dt$

$$= -\frac{\alpha e^{-s(1+\alpha)}}{s\alpha} - \left[\frac{e^{-st}}{s^2\alpha}\right]_1^{1+\alpha}$$

$$= -\frac{e^{-s(1+\alpha)}}{s} - \frac{e^{-s(1+\alpha)} - e^{-s}}{s^2\alpha}$$

In the limit as $\alpha \to 0$, this becomes:

$$\mathcal{L}\left\{(t-1)\delta(t-1)\right\} = \lim_{\alpha\to 0}\ \mathcal{L}\left\{(t-1)\,f_\alpha(t-1)\right\}$$

$$= \lim_{\alpha\to 0}\left(-\frac{e^{-s(1+\alpha)}}{s} - \frac{e^{-s(1+\alpha)} - e^{-s}}{s^2\alpha}\right)$$

$$= -\frac{e^{-s}}{s} - \lim_{\alpha\to 0}\left(\frac{e^{-s(1+\alpha)} - e^{-s}}{s^2\alpha}\right)$$

The term in brackets has the indeterminate form $\dfrac{0}{0}$ so we use l'Hospital's rule to obtain:

$$\lim_{\alpha\to 0}\left(\frac{-e^{-s(1+\alpha)} - e^{-s}}{s^2\alpha}\right) = \lim_{\alpha\to 0}\left[\frac{\dfrac{d}{dx}\left(-e^{-s(1+\alpha)} - e^{-s}\right)}{\dfrac{d}{d\alpha}\left(s^2\alpha\right)}\right]$$

$$= \lim_{\alpha\to 0}\left(\frac{se^{-s(1+\alpha)}}{s^2}\right)$$

$$= \frac{e^{-s}}{s}$$

Thus $\mathcal{L}\left\{(t-1)\delta(t-1)\right\} = -\dfrac{e^{-s}}{s} + \dfrac{e^{-s}}{s}$

$$= 0$$

Remember that to find $\mathcal{L}\{\delta(t-a)\}$, we must first evaluate $\mathcal{L}\{f_\alpha(t-a)\}$ and then take the limit as $\alpha \to 0$, that is

$$\mathcal{L}\left\{\delta(t-a)\right\} = \lim_{\alpha\to 0} L\left\{f_\alpha(t-a)\right\}$$

3.5 TABLE OF LAPLACE TRANSFORMS OF SPECIAL FUNCTIONS

The following table is a summary of the results of this unit and will be added to the table of Laplace transforms that will be given with your examination paper:

$f(t)$	$\mathcal{L}\{f(t)\}$
$U(t)$	$\frac{1}{s}$
$U(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}

3.6 POST-TEST: MODULE 4 (LEARNING UNIT 3)

(*Solutions on myUnisa under additional resources*)

Time: 40 minutes

1. Write the function $f(t) = \begin{cases} 1 & t < 2 \\ 0 & t > 2 \end{cases}$

   in terms of the unit step function and hence find $\mathcal{L}\{ f(t) \}$. (6)

2. Find the Laplace transform of $f(t) = \begin{cases} e^{2t} & t < 1 \\ 0 & t > 1 \end{cases}$ (9)

3. Express $f(t) = \begin{cases} 0 & t < 3 \\ 6 & 3 < t < 5 \\ 0 & t > 5 \end{cases}$

   in terms of the unit step function and find $\mathcal{L}\{ f(t) \}$. (5)

4. Write the function $f(t) = \begin{cases} \dfrac{3}{\alpha} & 2 \leq t \leq 2+\alpha \\ 0 & t > \alpha \end{cases}$

   in terms of the unit impulse function and find its Laplace transform. (4)

[24]

You should now be able to define the Heaviside unit step function and the unit impulse function, and find their Laplace transforms and inverse Laplace transforms using a table of standard transforms.

We are now ready to start using Laplace transforms to solve ordinary differential equations as will be explained in the next learning unit.

MODULE 4

LAPLACE TRANSFORMS

LEARNING UNIT 4

SOLVING DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

OUTCOMES

At the end of this learning unit, you should be able to

- understand the procedure to solve differential equations using Laplace transforms
- solve differential equations using Laplace transforms

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

4.1 SOLUTION OF DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Laplace transforms can also be used to solve differential equations. We will use the following results from learning unit 1:

If	$\mathcal{L}\{f(t)\} = F(s)$,	
then	$\mathcal{L}\{f\,'(t)\} = sF(s) - f(0)$	(4.1)
and	$\mathcal{L}\{f\,''(t)\} = s^2F(s) - s\,f(0) - f\,'(0)$.	(4.2)

Consider the second-order linear differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(t)$$

or

$$ay''(t) + by'(t) + cy(t) = f(t) \qquad (4.3)$$

where a, b and c are constants. Suppose we are asked to solve this equation subject to the initial or boundary conditions $y(0) = A$ and $y'(0) = B$, where A and B are given constants.

On taking Laplace transforms of both sides of equation (4.3) we obtain:

$$a\mathcal{L}\{y''(t)\} + b\mathcal{L}\{y'(t)\} + c\mathcal{L}\{y(t)\} = \mathcal{L}\{f(t)\} \qquad (4.4)$$

Let $\mathcal{L}\{y(t)\} = Y(s)$. (4.5)

By equations (4.1) and (4.2) it follows that

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) \qquad (4.6)$$

and $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$ (4.7)

Equation (4.4) thus becomes

$$a_1\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = F(s)$$

and substitution of the initial conditions gives:

$$a\left(s^2Y(s) - sA - B\right) + b\left(sY(s) - A\right) + cY(s) = F(s)$$

This is an algebraic equation from which we can determine Y(s), that is:

$$Y(s) = \frac{F(s) + a\left(As + B\right) + bA}{as^2 + bs + c}$$

The required solution is then obtained by taking the inverse Laplace transform to find:

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

The above method can be extended to differential equations of higher order.

For first-order differential equations *a* = 0 and so:

$$Y(s) = \frac{F(s) + bA}{bs + c}$$

EXAMPLE 1

Solve the differential equation $\frac{d^2y}{dt^2} + y = t$ given the initial conditions

$y(0) = 0$ and $y'(0) = 1$.

These conditions are also sometimes written as $y = 0$ and $\frac{dy}{dt} = 1$ when $t = 0$.

SOLUTION

We take Laplace transforms of both sides of the given equation to obtain:

$$\mathcal{L}\{y''(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

On using equations (4.5) and (4.7) this becomes

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2} \qquad (4.8)$$

where $\mathcal{L}\{t\}$ are found from the table of transforms.

Substitution of the initial values gives:

$$s^2Y(s) - 1 + Y(s) = \frac{1}{s^2}$$

$$\left(s^2 + 1\right)Y(s) = \frac{1}{s^2} + 1$$

$$= \frac{1 + s^2}{s^2}$$

$$Y(s) = \frac{1 + s^2}{s^2(1 + s^2)}$$

$$= \frac{1}{s^2}$$

Thus the original differential equation has been transformed into the algebraic equation:

$$Y(s) = \frac{1}{s^2}$$

We now take the inverse transform of each side, that is:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

Using the table shows that $y(t) = t$.

EXAMPLE 2

Solve the equation $y'(t) + 3y(t) = t$ given that $y(0) = 1$.

SOLUTION

On taking Laplace transforms of both sides we obtain:

$$\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

We now use equations (4.5) and (4.6) to get

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s^2}$$

Substitution of the initial values gives:

$$\begin{aligned} sY(s) - 1 + 3Y(s) &= \frac{1}{s^2} \\ (s+3)Y(s) &= \frac{1}{s^2} + 1 \\ &= \frac{1+s^2}{s^2} \\ Y(s) &= \frac{1+s^2}{s^2(s+3)} \end{aligned}$$

Once again we have an algebraic equation giving $Y(s)$ in terms of s and the required solution is obtained by taking the reverse transform of each side. In order to do this we express $Y(s)$ in partial fractions.

Let $\frac{1+s^2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$.

We multiply both sides of the equation by $s^2(s + 3)$ so that:

$$1+s^2 = As(s+3)+B(s+3)+Cs^2$$
$$= As^2+3As+Bs+3B+Cs^2$$
$$= (A+C)s^2+(3A+B)s+3B$$

Equating coefficients of like terms yields the following:

constant terms: 1 = 3B

B = $\frac{1}{3}$

s-terms: 0 = 3A + B

A = $-\frac{1}{9}$

s^2-terms: 1 = A + C

C = $\frac{10}{9}$

Thus $$Y(s) = \frac{1+s^2}{s^2(s+3)}$$
$$= -\frac{1}{9s}+\frac{1}{3s^2}+\frac{10}{9(s+3)}$$
$$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}+\frac{10}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

Referring to the table we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}=e^{-t}$$

so that $y(t)=\frac{10}{9}e^{-3t}+\frac{t}{3}-\frac{1}{9}$.

EXAMPLE 3

Solve the equation $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+3y=e^{-t}\sin t$ given $y(0)=0$ and $y'(0)=1$.

SOLUTION

We take Laplace transforms of both sides of the given equation to obtain:

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + 3\mathcal{L}\{y(t)\} = \mathcal{L}\left\{e^{-t}\sin t\right\}$$

Reference to the table shows that $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

so that by the first translation or shifting property:

$$\begin{aligned}\mathcal{L}\left\{e^{-t}\sin t\right\} &= \frac{1}{(s+1)^2+1}\\ &= \frac{1}{s^2+2s+2}\end{aligned}$$

Thus $s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 3Y(s) = \dfrac{1}{s^2+2s+2}$

Using the given initial conditions we have:

$$\begin{aligned}s^2Y(s) - 1 + 2sY(s) + 3Y(s) &= \frac{1}{s^2+2s+2}\\ \left(s^2+2s+3\right)Y(s) &= \frac{1}{s^2+2s+2} + 1\\ &= \frac{1+s^2+2s+2}{s^2+2s+2}\\ &= \frac{s^2+2s+3}{s^2+2s+2}\\ Y(s) &= \frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+3\right)}\\ &= \frac{1}{s^2+2s+2}\\ &= \frac{1}{\left(s+1\right)^2+1}\end{aligned}$$

We now take the inverse transform of each side to obtain

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\end{aligned}$$

by the first translation property.

Using the table we find that the solution is:

$$y(t) = e^{-t}\sin t$$

EXAMPLE 4

Solve the equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x}$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 0$.

SOLUTION

In this example the independent variable is x, so we use x instead of t in defining the Laplace transform, that is, we put $Y(s) = \int_0^\infty e^{-sx} f(x)\, dx$.

Taking the Laplace transform of both sides of the given equation we have:

$$\mathcal{L}\{y''(x)\} + 2\mathcal{L}\{y'(x)\} + \mathcal{L}\{y(x)\} = \mathcal{L}\{e^{-x}\}$$

Thus $s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \dfrac{1}{s+1}$

and substituting the initial values:

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s+1}$$

$$\left(s^2 + 2s + 1\right)Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)\left(s^2 + 2s + 1\right)}$$

$$= \frac{1}{(s+1)^3}$$

We now take the inverse transform of each side to get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

The table shows that $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^3}\right\} = \dfrac{x^2e^{-x}}{2}$

and the required solution is $y(x) = \dfrac{x^2e^{-x}}{2}$

EXAMPLE 5

Solve the equation $y''(t) - 9y(t) = 6\sinh 3t$ given $y(0) = 0$ and $y'(0) = 4$.

SOLUTION

We take Laplace transforms of both sides of the equation obtaining

$$\mathcal{L}\{y''(t)\} - 9\mathcal{L}\{y(t)\} = 6\mathcal{L}\{\sinh 3t\}$$

from which we get $\quad s^2Y(s) - sy(0) - y'(0) - 9Y(s) = 6\left(\dfrac{3}{s^2-9}\right)$

Substitution of the initial values gives:

$$\begin{aligned} s^2Y(s) - 4 - 9Y(s) &= 6\left(\frac{3}{s^2-9}\right) \\ \left(s^2-9\right)Y(s) &= \frac{18}{s^2-9} + 4 \\ &= \frac{18 + 4s^2 - 36}{s^2-9} \\ Y(s) &= \frac{4s^2 - 18}{\left(s^2-9\right)^2} \end{aligned}$$

In order to take inverse transforms of this we shall have to rewrite the equation in a form which appears in the table. Reference to the table shows that

$$\mathcal{L}^{-1}\left\{\frac{s^2+9}{\left(s^2-9\right)^2}\right\} = t\cosh 3t$$

so we write the equation as:

$$\begin{aligned} Y(s) &= \frac{s^2 + 9 + 3s^2 - 27}{\left(s^2-9\right)^2} \\ &= \frac{s^2+9}{\left(s^2-9\right)^2} + \frac{3\left(s^2-9\right)}{\left(s^2-9\right)^2} \end{aligned}$$

Thus $\quad \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{s^2+9}{\left(s^2-9\right)^2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{3}{s^2-9}\right\}$

and $\quad y(t) = t\cosh 3t + \sinh 3t$.

EXAMPLE 6

Solve the equation $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = t$ given that $y(0) = 0$ and $y'(0) = 0$.

SOLUTION

On taking Laplace transforms of each side we get

$$\mathcal{L}\{y''(t)\} - \mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\} = \mathcal{L}\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) = \frac{1}{s^2}$$

and we insert the given initial conditions to obtain:

$$\left(s^2 - s - 2\right)Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2\left(s^2 - s - 2\right)}$$

Since the denominator of $Y(s)$ can be factorised, we resolve it into partial fractions to enable us to find the inverse transform.

Let $\frac{1}{s^2\left(s+1\right)\left(s-2\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-2}$.

On multiplying both sides of the equation by $s^2(s+1)(s-2)$

we obtain: $1 = As\left(s^2 - s - 2\right) + B\left(s^2 - s - 2\right) + Cs^2(s-2) + Ds^2(s+1)$

$$= As^3 - As^2 - 2As + Bs^2 - Bs - 2B + Cs^3 - sCs^2 + Ds^3 + Ds^2$$

$$= \left(A + C + D\right)s^3 + \left(-A + B - 2C + D\right)s^2 + (-2A - B)s - 2b$$

Equating coefficients of like terms gives the following:

s^3-terms: $A + C + D = 0$

s^2-terms: $-A + B - 2C + D = 0$

s-terms: $-2A - B = 0$

constant terms: $-2B = 1$

Hence $B = -\frac{1}{2}, \quad A = \frac{1}{4}, \quad C = -\frac{1}{3}, \quad D = \frac{1}{12}$.

Thus $Y(s) = \frac{1}{s^2(s+1)(s-2)}$

$$= \frac{1}{4s} - \frac{1}{2s^2} - \frac{1}{3(s+1)} + \frac{1}{12(s-2)}$$

and $\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{12}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$

Reference to the table gives the solution:

$$y(t) = \frac{1}{4} - \frac{1}{2}t - \frac{1}{3}e^{-t} + \frac{1}{12}e^{2t}$$

EXAMPLE 7

Solve the equation $y''(t) + 4y(t) = \delta(t)$ given $y(0) = 0$ and $y'(0) = 1$.

SOLUTION

We have $\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t)\}$.

Thus $s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 1$

and on substituting the initial conditions this becomes:

$$s^2Y(s) - 1 + 4Y(s) = 1$$

so that $(s^2 + 4)Y(s) = 2$

$$Y(s) = \frac{2}{(s^2+4)}$$

Then $\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

and $y(t) = \sin 2t,\ t > 0$.

EXAMPLE 8

Solve the equation $y''(x) - 6y'(x) + 9y(x) = 0$ given that $y(0) = 0$ and $y(1) = 1$.

SOLUTION

Taking Laplace transforms of both sides we have:

$$\mathcal{L}\{y''(x)\} - 6\mathcal{L}\{y'(x)\} + 9\mathcal{L}\{y(x)\} = 0$$

Thus $s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) + 9Y(s) = 0$

Since the value of $y'(0)$ is not given, we write $y'(0) = C$ and substitute for $y(0)$ and $y'(0)$ in the above equation to get:

$$s^2Y(s) - C - 6sY(s) + 9Y(s) = 0$$
$$(s^2 - 6s + 9)Y(s) = C$$
$$Y(s) = \frac{C}{s^2 - 6s + 9}$$
$$= \frac{C}{(s-3)^2}$$

Take the inverse transform of each side obtaining:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{C}{(s-3)^2}\right\}$$

Thus $\qquad y(x) = cxe^{3x}$

To find the value of C we substitute the given initial condition $y(1) = 1$ in this equation to get:

$$1 = Ce^3$$
$$C = \frac{1}{e^3} = e^{-3}$$

The solution is therefore: $\qquad y(x) = e^{-3} \times e^{3x}$

$$= xe^{3(x-1)}$$

In all the examples so far worked out, initial or boundary conditions were supplied. These enable us to find particular solutions of the given differential equations. However, you will remember from earlier units that frequently initial conditions are not given, and we then have to find the general solution of the equation. This solution contains arbitrary constants, equal in number to the order of the equation.

EXAMPLE 9

Find the general solution of $\frac{d^2y}{dt^2} + 16y = 0$.

SOLUTION

On taking Laplace transforms we get:

$$\mathcal{L}\{y''(t)\} + 16\mathcal{L}\{y(t)\} = 0$$

Thus $\qquad s^2Y(s) - sy(0) - y'(0) + 16Y(s) = 0$

Let $y(0) = a$ and $y'(0) = b$

so that $\left(s^2+16\right)Y(s)=as+b$

$$Y(s)=\frac{as+b}{s^2+16}$$

We therefore have $\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\left\{\frac{as+b}{s^2+16}\right\}$

$$=a\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}+\frac{b}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\}$$

$$=a\cos 4t+\frac{b}{4}\sin 4t$$

If we put $a = c_1$ and $\frac{b}{4}=c_2$, then the general solution of the given equation is:

$$y(t)=c_1\cos 4t+c_2\sin 4t$$

4.2 POST-TEST: MODULE 4 (LEARNING UNIT 4)

(*Solutions on myUnisa under additional resources*)

Time: 75 minutes

Use Laplace transforms to solve the following differential equations:

1. $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+4y=0$ given that $y(0) = 0$ and $y'(0) = 2$ (10)

2. $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=t^2e^{-2t}$ given that $y(0) = 0$ and $y'(0) = 0$ (5)

3. $y''(t)-y'(t)-2y(t)=4$ given that $y(0) = 1$ and $y'(0) = 6$ (12)

4. $y''(t)+y(t)=t$ given that $y(0) = 2$ and $y'(0) = 1$ (8)

5. $y'(x)-y(x)=1$ given that $y(0) = 0$ (8)

6. $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+10y=\left(25t^2+16t+2\right)e^{3t}$ given that $y(0) = 0$ and $y'(0) = 0$ (8)

[51]

You should now be able to solve differential equations using Laplace transforms.

In the next learning unit, we shall study examples of how to use Laplace transforms to solve a system of simultaneous equations.

MODULE 4

LAPLACE TRANSFORMS

LEARNING UNIT 5

SOLVING SIMULTANEOUS DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

OUTCOMES

At the end of this learning unit, you should be able to

- understand the procedure to solve differential equations using Laplace transforms
- solve differential equations using Laplace transforms

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

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5.1 SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

We can also solve a system of simultaneous equations using Laplace transforms.

You will not be able to master the outcomes for this learning unit without referring to your prescribed book.

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

5.2 POST-TEST: MODULE 4 (LEARNING UNIT 5)

(*Solutions on myUnisa under additional resources*)

Time: 60 minutes

Use Laplace transforms to solve the following simultaneous differential equations:

1. $$\frac{dx}{dt} - \frac{dy}{dt} + y = -e^{t}$$ (10)

$$x + \frac{dy}{dt} - y = e^{2t}$$

2. $$\frac{dx}{dt} - x + \frac{dy}{dt} + 3y = e^{-t} - 1$$ (12)

$$\frac{dx}{dt} + 2x + \frac{dy}{dt} + y = e^{2t} + t$$

3. $$\frac{dx}{dt} + 2x - 3y = t$$ (11)

$$\frac{dy}{dx} - 3x + 2y = e^{2t}$$

[33]

You should now be able to solve simultaneous differential equations using Laplace transforms.

In the next learning unit, we shall study examples of how to use Laplace transforms in problems relating to electrical circuits, vibrations and beams.

MODULE 4
LAPLACE TRANSFORMS

LEARNING UNIT 6
PRACTICAL APPLICATIONS

OUTCOMES

At the end of this learning unit, you should be able to solve problems using Laplace transforms relating to

- electric circuits
- vibrating systems
- beams

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

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6.1 PRACTICAL APPLICATIONS OF THE LAPLACE TRANSFORM

In Module 1 we studied some ways in which differential equations are used in engineering. We discussed the solution of problems on electric circuits, vibrating systems and the deflection of beams. We shall now see how the same type of problem can be solved using Laplace transforms.

6.1.1 Application to electric circuits

The simple electric circuit shown below has an inductance L, resistance R and a capacitance C, connected in series with an electromotive force E.

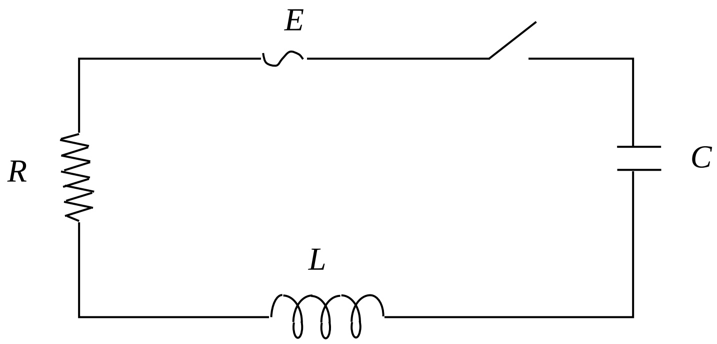


Figure 1

Let Q be the charge which flows to the capacitor plates when the switch is closed. Then, by Kirchhoff's second law:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E$$

By solving this equation, we can find the charge Q and also the current I, which is the rate of flow of charge, that is:

$$I = \frac{dQ}{dt}$$

EXAMPLE 1

An inductor of 1 H, a resistor of 6 Ω and a capacitor of 0.1 F are connected in series with an electromotive force of 20sin2t volts. At t = 0 the charge on the capacitor and the current in the circuit are zero. Find the charge and current at any time t > 0.

SOLUTION

We are required to solve the equation

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 10Q = 20\sin 2t$$

given the initial values

$$Q(0) = 0 \text{ and } Q'(0) = 0.$$

Taking Laplace transforms on both sides of the equation we obtain:

$$\mathcal{L}\{Q''(t)\} + 6\mathcal{L}\{Q'(t)\} + 10\mathcal{L}\{Q(t)\} = 20\mathcal{L}\{\sin 2t\}$$

Let $\mathcal{L}\{Q(t)\} = q(s)$.

Then $\mathcal{L}\{Q'(t)\} = 2q(s) - Q(0)$

and $\mathcal{L}\{Q''(t)\} = s^2q(s) - sQ(0) - Q'(0)$.

We therefore have

$$s^2q(s) - sQ(0) - Q'(0) + 6sq(s) - 6Q(0) + 10Q(s) = 20\left(\frac{2}{s^2+4}\right)$$

and substituting the given initial values leads to

$$(s^2 + 6s + 10)q(s) = \frac{40}{s^2+4}$$

so that $q(s) = \dfrac{40}{(s^2+4)(s^2+6s+10)}$.

In order to find the inverse transform, we resolve $q(s)$ into partial fractions.

Let $$\frac{40}{(s^2+4)(s^2+6s+10)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+6s+10}$$

Multiplication of both sides by $(s^2 + 4)(s^2 + 6s + 10)$ gives:

$$\begin{aligned}
40 &= (As+B)(s^2+6s+10) + (Cs+D)(s^2+4) \\
&= As^3 + 6As^2 + 10As + Bs^2 + 6Bs + 10B + Cs^3 + 4Cs + Ds^2 + 4D \\
&= (A+C)s^3 + (6A+B+D)s^2 + (10A+6B+4C)s + (10B+4D)
\end{aligned}$$

We now equate coefficients of like terms to get the following:

s^3-terms: $A + C = 0$

s^2-terms: $6A + B + D = 0$

s-terms: $10A + 6B + 4C = 0$

constant terms: $10B + 4D = 40$

Hence: $A = -\dfrac{4}{3},\ B = \dfrac{4}{3},\ C = \dfrac{4}{3}$ and $D = \dfrac{20}{3}$

Thus:
$$q(s) = \frac{40}{(s^2+4)(s^2+6s+10)}$$
$$= \frac{-4s+4}{3(s^2+4)} + \frac{4s+20}{3(s^2+6s+10)}$$
$$= -\frac{4}{3}\left(\frac{s}{s^2+4}\right) + \frac{2}{3}\left(\frac{2}{s^2+4}\right) + \frac{4}{3}\left(\frac{(s+3)+2}{(s+3)^2+1}\right)$$

and
$$\mathcal{L}^{-1}\{q(s)\} = -\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{(s+3)+2}{(s+3)^2+1}\right\}$$
$$= -\frac{4}{3}\cos 2t + \frac{2}{3}\sin 2t + \frac{4}{3}e^{-3t}\left(\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right)$$

The solution is therefore:
$$Q(t) = -\frac{4}{3}\cos 2t + \frac{2}{3}\sin 2t + \frac{4}{3}e^{-3t}\cos t + \frac{8}{3}e^{-3t}\sin t$$

and
$$I(t) = \frac{dQ}{dt}$$
$$= \frac{8}{3}\sin 2t + \frac{4}{3}\cos 2t - \frac{4}{3}e^{-3t}\cos t - \frac{28}{3}e^{-3t}\sin t$$

For large values of t, those terms of Q and I that contain e^{-3t} are negligible and are called the transient part of the solution. The remaining terms are called the steady-state part of the solution. The transient terms of I, that is the last two terms
$$-\frac{4}{3}e^{-3t}\cos t - \frac{28}{3}e^{-3t}\sin t,$$
vanish as $t \to \infty$ and the steady-state solution is:
$$I(t) = \frac{8}{3}\sin 2t + \frac{4}{3}\cos 2t$$

EXAMPLE 2

An inductor of 1 H, a resistor of 4 Ω and a capacitor of 0.05 F are connected in series with an electromagnetic force of 100 V. At t = 0 the charge on the capacitor and the current in the circuit are zero. Find the charge and the current at any time t > 0.

SOLUTION

Let Q and I be the instantaneous charge and current, respectively, at time t. By Kirchhoff's second law we have

$$\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + \frac{Q}{0.05} = 100$$

or

$$\frac{d^2Q}{dt^2} + 4\frac{dQ}{dt} + 20Q = 100$$

with initial conditions

$$Q(0) = 0 \quad \text{and} \quad Q'(0) = 0 .$$

Taking Laplace transforms, we find:

$$\mathcal{L}\{Q''(t)\} + 4\mathcal{L}\{Q'(t)\} + 20\mathcal{L}\{Q(t)\} = 100\mathcal{L}\{1\}$$

Thus $s^2\, q(s) - s\,Q(0) - Q'(0) + 4s\,q(s) - 4Q(0) + 20\,q(s) = \dfrac{100}{s}$

which, on inserting the initial conditions, becomes

$$(s^2 + 4s + 20)q(s) = \frac{100}{s}$$

so that

$$q(s) = \frac{100}{s(s^2 + 4s + 20)}$$

Let $\dfrac{100}{s(s^2 + 4s + 20)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 4s + 20}$.

Thus

$$\begin{aligned} 100 &= A(s^2 + 4s + 20) + (Bs + C)s \\ &= (A + B)s^2 + (4A + C)s + 20A \end{aligned}$$

Equating coefficients of like terms gives the following:

constant terms: $100 = 20A$

$A = 5$

s-terms: $0 = 4A + C$

$C = -20$

s^2-terms: $0 = A + B$

$B = -5$

Thus $$q(s) = \frac{100}{s(s^2+4s+20)}$$

$$= \frac{5}{s} - \frac{5s+20}{s^2+4s+20}$$

$$= \frac{5}{s} - \frac{5(s+2)+10}{(s+2)^2+16}$$

and $$\mathcal{L}^{-1}\{Q(s)\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - e^{-2t}\mathcal{L}^{-1}\left\{\frac{5s+10}{s^2+16}\right\}$$
$$= 5 - e^{-2t}\left(5\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\}\right)$$

Thus $$\mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+36}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{s^2+36}\right\}$$

Thus $$Q(t) = 5 - 5e^{-2t}\cos 4t - \frac{5}{2}e^{-2t}\sin 4t$$

and $$I(t) = 10e^{-2t}\cos 4t + 20e^{-2t}\sin 4t + 5e^{-2t}\sin 4t - 10e^{-2t}\cos 4t$$
$$= 25e^{-2t}\sin 4t$$

6.1.2 Application to vibrating systems

Suppose a mass m hangs in equilibrium from the lower end of a vertical spring. If it is given a small vertical displacement and then released, the mass will undergo vertical vibrations.

Let $y(t)$ be the instantaneous displacement of m at time t.

Then the equation of motion of the system is

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + k\,y = f(t)$$

where $c\,\dfrac{dy}{dt}$ is the damping force proportional to the velocity of *m*, *ky* is the restoring force and $f(t)$ is an external, variable force acting vertically on *m*.

Sometimes no damping and/or external forces are present, in which case

$$c\frac{dy}{dt}=0 \quad \text{and/or} \quad f(t)=0\,.$$

EXAMPLE 3

The equation of motion of a mass performing free vibrations is $\dfrac{d^2y}{dt^2}+36y=0$

if $y=-\dfrac{1}{4}$ and $\dfrac{dy}{dt}=-2$ when $t=0$.

Determine the amplitude and period of the motion.

SOLUTION

The initial values are $y(0)=-\dfrac{1}{4}$ and $y'(0)=-2$.

Note that $c\dfrac{dy}{dt}=0$ and $f\left(t\right)=0,$ there are no damping and external forces acting on *m*, so the vibrations are said to be free.

Taking Laplace transforms of the equation we have:

$$\mathcal{L}\{y''(t)\}+36\mathcal{L}\{y(t)\}=0$$

Thus $\quad s^2Y(s)-sy(0)-y'(0)+36Y(s)=0$

Substitution of the initial conditions gives

$$(s^2+36)Y(s)=-\frac{s}{4}-2$$

so that

$$\begin{aligned} Y(s) &= \frac{-s}{4(s^2+36)}-\frac{2}{(s^2+36)} \\ &= -\frac{1}{4}\frac{s}{(s^2+6^2)}-\frac{6}{3(s^2+6^2)} \end{aligned}$$

We want a 6 as numerator in the second term so that we can easily use the table of transforms.

Thus $\mathcal{L}^{-1}\left\{Y(s)\right\} = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+36}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{s^2+36}\right\}$

and $y(t) = -\frac{1}{4}\cos 6t - \frac{1}{3}\sin 6t$

The period of the motion is $T = \frac{2\pi}{6} = \frac{\pi}{3}$

and the amplitude is $a = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{5}{12}$.

EXAMPLE 4

A system vibrates according to the equation $\frac{d^2y}{dt^2} + 4y = 3\sin t$.

If $y = 0$ at $t = 0$ and $\frac{dy}{dt} = 1$ at $t = \frac{\pi}{2}$, find the displacement of the mass at any time t.

SOLUTION

In this case we have an external force $f(t)$ = 3sint acting on m, but no damping force. The vibrations are said to be forced.

Take Laplace transforms on both sides of the equation:

$$\mathcal{L}\left\{y''(t)\right\} + 4\mathcal{L}\left\{y(t)\right\} = 3\mathcal{L}\left\{\sin\ t\right\}$$

Thus $s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{3}{s^2+1}$

We are given $y(0) = 0$, but since we do not know the value of $y'(0)$, we put $y'(0) = c_1$.

Substituting these values into the equation gives:

$$\left(s^2+4\right)Y(s) = \frac{3}{s^2+1} + c_1$$

and
$$Y(s) = \frac{3 + c_1s^2 + c_1}{\left(s^2+1\right)\left(s^2+4\right)}$$

This must be expressed in partial fractions so that we can find the inverse transform.

Let $\frac{c_1s^2 + c_1 + 3}{\left(s^2+1\right)\left(s^2+4\right)} = \frac{As+B}{s^2+1} + \frac{Ds+E}{s^2+4}$.

Then $$c_1s^2 + c_1 + 3 = (As + B)(s^2 + 4) + (Ds + E)(s^2 + 1)$$
$$= As^3 + 4As + Bs^2 + 4B + Ds^3 + Ds + Es^2 + E$$
$$= (A + D)s^3 + (B + E)s^2 + (4A + D)s + (4B + E)$$

We equate coefficients of like terms to get

s^3-terms: $0 = A + D$

s^2-terms: $c_1 = B + E$

s-terms: $0 = 4A + D$

constant terms: $c_1 + 3 = 4B + E$

from which it follows that

$$A = D = 0,\ B = 1 \text{ and } E = c_1 - 1.$$

Thus $$Y(s) = \frac{c_1s^2 + c_1 + 3}{(s^2 + 1)(s^2 + 4)}$$
$$= \frac{1}{s^2 + 1} + \frac{c_1 - 1}{s^2 + 4}$$

and $$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} + \frac{c_1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

so that $$y(t) = \sin t + \frac{c_1}{2}\sin 2t - \frac{1}{2}\sin 2t\ .$$

To find the value of c_1 we make use of the initial condition

$y'\left(\frac{\pi}{2}\right) = 1$, that is $y'(t) = 1$ when $t = \frac{\pi}{2}$.

Differentiate the equation of $y(t)$ to get

$$y'(t) = \cos t + c_1 \cos 2t - \cos 2t$$

and then substitute the given values of $y'(t)$ and t to get:

$$1 = \cos\frac{\pi}{2} + c_1 \cos \pi - \cos \pi$$

Since $\cos\frac{\pi}{2} = 0$ and $\cos \pi = -1$, this becomes:

$$1 = c_1 + 1$$
$$c_1 = 0$$

Substituting this value for c_1 we obtain:

$$y(t) = \sin t - \frac{1}{2}\sin 2t$$

6.1.3 Application to beams

Consider a transversely loaded beam that is co-incident with the *x*-axis. Let the length of the beam be *b* and its load per unit length *k*. The axis of the beam has a transverse deflection *y*(*x*) at any point *x* which satisfies the differential equation

$$\frac{d^4y}{dx^4} = \frac{k}{E\,\mathrm{I}} \qquad 0 < x < \mathrm{b}$$

where *E* is the modulus of elasticity for the beam and I is the moment of inertia of a small cross-section at *x*.

This transverse deflection is often called the curve of deflection of the beam and is the shape into which the axis of the beam is bent by gravity and other forces acting upon it.

Solution of the equation subject to given initial conditions will yield the equation of the curve of deflection, from which we can find the deflection *y*(*x*) at any point. The initial conditions associated with the equation depend on the manner in which the beam is supported. Most commonly we find

(a) clamped, built-in or fixed

(b) simply supported or hinged

(c) free end

EXAMPLE 5

A simply supported beam of length *b* carries a uniform load *k* per unit length. Find the deflection at any point, given the differential equation

$$\frac{d^4y}{dx^4} = \frac{k}{E\,\mathrm{I}} \qquad 0 < x < b$$

and initial conditions $y(0) = 0$, $y''(0) = 0$, $y(b) = 0$ and $y''(b) = 0$.

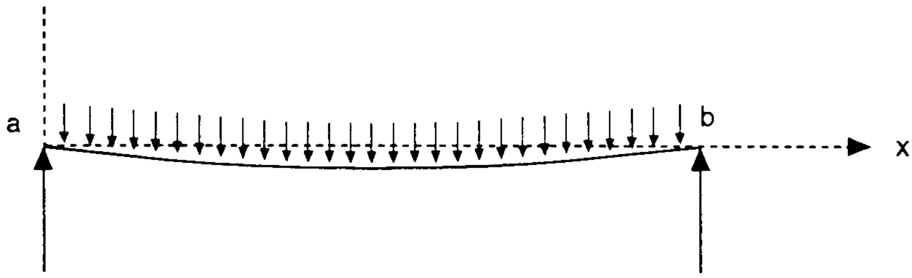


Figure 2

SOLUTION

Taking Laplace transforms of both sides of the equation gives:

$$\mathcal{L}\left\{y^{4}(x)\right\} = \frac{k}{EI}\mathcal{L}\{1\}$$

Thus: $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \dfrac{k}{EI}\dfrac{1}{s}$

Since we do not know the values of $y'(0)$ and $y'''(0)$, we put $y'(0) = c_1$ and $y'''(0) = c_2$.

We substitute these and the given initial values in the equation to get:

$$s^4Y(s) - s^2c_1 - c_2 = \frac{k}{EI}\frac{1}{s}$$

Thus $$s^4Y(s) = \frac{k}{EI}\frac{1}{s} + s^2c_1 + c_2$$

and $$Y(s) = \frac{k}{EI}\frac{1}{s} + \frac{c_1}{s^2} + \frac{c_2}{s^4}$$

Taking the inverse transforms we get:

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{K}{EI}\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} + c_1\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

so that $$y(x) = \frac{k}{EI}\left(\frac{x^4}{4!}\right) + c_1x + c_2\left(\frac{x^3}{3!}\right)$$

$$= \frac{kx^4}{24EI} + c_1x + \frac{c_2x^3}{6}$$

To evaluate the constants c_1 and c_2, we differentiate this equation twice.

Thus $\quad y'(x) = \dfrac{kx^3}{6E\,\text{I}} + c_1 + \dfrac{c_2 x^2}{2}$

$$y''(x) = \frac{kx^2}{2E\,\text{I}} + c_2 x$$

We make use of the initial conditions to get:

$$0 = \frac{kb^2}{2E\,\text{I}} + c_2 b$$

$$c_2 = -\frac{kb}{3E\,\text{I}}$$

Substituting this and the initial values we get:

$$0 = \frac{kb^4}{24E\,\text{I}} + c_1 b - \frac{kb^4}{12E\,\text{I}}$$

$$c_1 = \frac{kb^3}{24E\,\text{I}}$$

Thus $\quad y(x) = \dfrac{kx^4}{24E\,\text{I}} + \dfrac{kb^3 x}{24E\,\text{I}} - \dfrac{kbx^3}{12E\,\text{I}}$

$$= \frac{k}{24E\,\text{I}}\left(x^4 - 2bx^3 + b^3 x\right)$$

EXAMPLE 6

A cantilever beam is fixed at one end $x = 0$ and free at the other end $x = b$.

It carries a load per unit length given by:

$$k(x) = \begin{cases} k_o & 0 < x < \dfrac{b}{3} \\ 0 & \dfrac{b}{3} < x < b \end{cases}$$

If the differential equation and initial conditions are

$$\frac{d^4 y}{dx^4} = \frac{k(x)}{E\,\text{I}} \text{ given that } y(0) = 0,\ \ y'(0) = 0,\ \ y''(b) = 0 \ \text{ and } \ y'''(b) = 0,$$

find the deflection at any point of the beam.

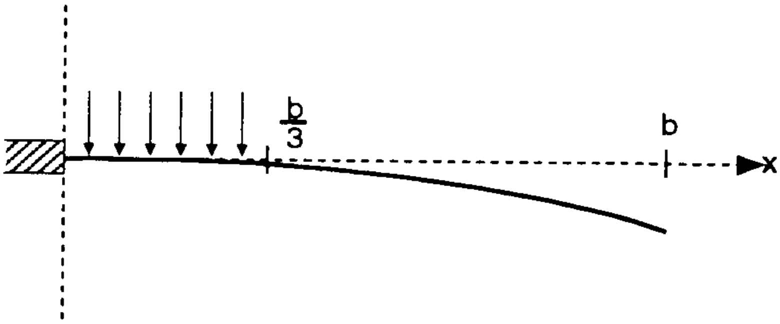


Figure 3

SOLUTION

Since the beam has a load uniformly distributed over one-third of its length and no load on the remainder, there is a step discontinuity of magnitude k_o at $x = \frac{b}{3}$.

$k(x)$ can therefore be written in terms of unit step functions as:

$$k(x) = k_o U(x) - k_o U\left(x - \frac{b}{3}\right)$$

Thus $\quad \dfrac{d^4 y}{dx^4} = \dfrac{k_o}{E\mathrm{I}}\left[U(x) - U\left(x - \frac{b}{3}\right)\right]$

Taking Laplace transforms we get: $\mathcal{L}\{y''''(x)\} = \dfrac{k_o}{E\mathrm{I}}\mathcal{L}\{U(x)\} - \dfrac{k_o}{E\mathrm{I}}\mathcal{L}\left\{U\left(x - \frac{b}{3}\right)\right\}$

Thus: $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \dfrac{k_o}{E\mathrm{I}}\dfrac{1}{s} - \dfrac{k_o e^{-\frac{bs}{3}}}{E\mathrm{I}}\dfrac{1}{s}$

Let $y''(0) = c_1$ and $y'''(0) = c_2$ and substitute these and the initial conditions to obtain:

$$s^4Y(s) - sc_1 - c_2 = \frac{k_o}{E\mathrm{I}}\frac{1}{s} - \frac{k_o e^{-\frac{bs}{3}}}{E\mathrm{I}}\frac{1}{s}$$

Thus
$$s^4Y(s) = sc_1 + c_2 + \frac{k_o}{E\mathrm{I}s} - \frac{k_o e^{-\frac{bs}{3}}}{E\mathrm{I}s}$$

and
$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{k_o}{E\mathrm{I}s^5} - \frac{k_o e^{-\frac{bs}{3}}}{E\mathrm{I}s^5}$$

Thus $\mathcal{L}^{-1}\{Y(s)\} = c_1\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + \frac{k_o}{E\mathrm{I}}\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} - \frac{k_o}{E\mathrm{I}}\mathcal{L}^{-1}\left\{\frac{e^{-\frac{bs}{3}}}{s^5}\right\}$

and $$y(x) = \frac{c_1x^2}{2} + \frac{c_2x^3}{6} + \frac{k_ox^4}{24E\,\mathrm{I}} - \frac{k_o}{24E\,\mathrm{I}}\left(x-\frac{b}{3}\right)^4 U\left(x-\frac{b}{3}\right) \ldots\ldots\ldots\ldots\ldots(6.1)$$

where the inverse transform of $\dfrac{e^{-\frac{bs}{3}}}{s^5}$ was obtained using the following:

If $\mathcal{L}^{-1}\{F(s)\} = f(x)$
then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(x-a)U(x-a)$

In this example $e^{-as} = e^{-\frac{bs}{3}}$.

Thus $a = \dfrac{b}{3}$

and $F(s) = \dfrac{1}{s^5}$

so that $f(x) = \dfrac{x^4}{4!}$

and $f(x-a) = \dfrac{1}{24}\left(x-\dfrac{b}{3}\right)^4$.

Thus $\mathcal{L}^{-1}\left\{\dfrac{e^{-\frac{bs}{3}}}{s^5}\right\} = \dfrac{1}{24}\left(x-\dfrac{b}{3}\right)^4 U\left(x-\dfrac{b}{3}\right)$

Equation (6.1) is equivalent to:

$$y(x) = \begin{cases} \dfrac{c_1x^2}{2} + \dfrac{c_2x^3}{6} + \dfrac{k_ox^4}{24\ E\ \mathrm{I}} & 0 < x < \dfrac{b}{3} \\ \dfrac{c_1x^2}{2} + \dfrac{c_2x^3}{6} + \dfrac{k_ox^4}{24E\,\mathrm{I}} - \dfrac{k_o}{24E\,\mathrm{I}}\left(x-\dfrac{b}{3}\right)^4 & x > \dfrac{b}{3} \end{cases}$$

Since we are given that $y''(x) = 0$ and $y'''(x) = 0$ when $x = b$, we differentiate $y(x)$ three times using the value for $x > \dfrac{b}{3}$.

$$y'(x) = c_1 x + \frac{c_2 x^2}{2} + \frac{k_o x^3}{6EI} - \frac{k_o}{6EI}\left(x - \frac{b}{3}\right)^3$$

$$y''(x) = c_1 + c_2 x + \frac{k_o x^2}{2EI} - \frac{k_o}{2EI}\left(x - \frac{b}{3}\right)^2 \quad ....(6.2)$$

$$y'''(x) = c_2 + \frac{k_o x}{EI} - \frac{k_o}{EI}\left(x - \frac{b}{3}\right)$$

On substituting $y'''(x) = 0$ we get:

$$0 = c_2 + \frac{k_o}{EI}\left(b - b + \frac{b}{3}\right)$$

Thus $$c_2 = -\frac{k_o b}{3EI}.$$

We now substitute this value and $y''(b) = 0$ in equation (6.2) to obtain:

$$0 = c_1 - \frac{k_o b^2}{3EI} + \frac{k_o b^2}{2EI} - \frac{k_o}{2EI}\left(\frac{2b}{3}\right)^2$$

$$c_1 = \frac{k_o}{EI}\left(\frac{6b^2 - 9b^2 + 4b^2}{18}\right)$$

$$= \frac{k_o b^2}{18EI}$$

Thus the required equation of the curve of deflection of the given beam is:

$$y(x) = \begin{cases} \dfrac{k_o}{EI}\left[\dfrac{b^2 x^2}{36} - \dfrac{bx^3}{18} + \dfrac{x^4}{24}\right] & 0 < x < \dfrac{b}{3} \\ \dfrac{k_o}{EI}\left[\dfrac{b^2 x^2}{36} - \dfrac{bx^3}{18} + \dfrac{x^4}{24} - \dfrac{1}{24}\left(x - \dfrac{b}{3}\right)^4\right] & x > \dfrac{b}{3} \end{cases}$$

EXAMPLE 7

Show that a concentrated load W acting at a point $x = a$ on a beam can be expressed in terms of the unit impulse or Dirac delta function as $W\delta(x-a)$.

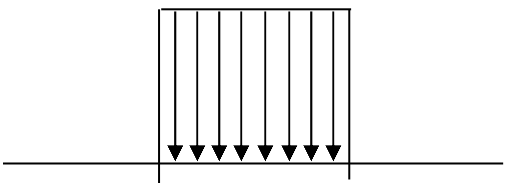


Figure 4

SOLUTION

Let k_o represent a uniform loading per unit length over a portion of the beam from a to α. Then the total loading for this portion is $k_o(a + \alpha - a) = k_o\alpha$ which must be equal to W.

Thus: $k_o\alpha = W$

$$k_o = \frac{W}{\alpha} \qquad \text{for} \quad a < x < a+\alpha$$

Since the loading is equal to zero, we can write:

$$k(x) = \begin{cases} k_o & a < x < a+\alpha \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{W}{\alpha} & a < x < a+\alpha \\ 0 & \text{otherwise} \end{cases}$$

This represents a pulse of duration α from $x = a$ to $x = a + \alpha$ and magnitude $\frac{W}{\alpha}$.

The strength of the pulse is, of course, $\frac{W}{\alpha} \times \alpha = W$.

In the limit as α tends to zero, the portion from $x = a$ to $x = a + \alpha$ reduces to the point $x = a$. An impulse of strength W at $x = a$ is represented by

$$\lim_{\alpha \to 0} k(x) = W\delta(x-a) .$$

EXAMPLE 8

A beam of length b with its end built in has a concentrated load W acting at its centre. Find the resulting deflection. The differential equation is

$$\frac{d^4y}{dx^4} = \frac{W}{E\mathrm{I}}\delta\left(x - \frac{b}{2}\right)$$

with initial conditions $y(0) = 0$, $y'(0) = 0$, $y(b) = 0$ and $y'(b) = 0$.

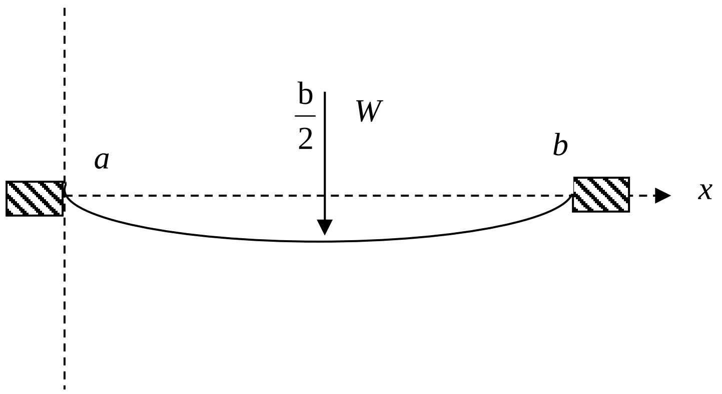


Figure 5

SOLUTION

Take Laplace transforms of both sides of the given equation to get:

$$\mathcal{L}\{y''''(x)\} = \frac{W}{E\mathrm{I}}\mathcal{L}\left\{\delta\left(x - \frac{b}{2}\right)\right\}$$

Thus $\quad s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{W}{E\,\mathrm{I}}e^{-\frac{bs}{2}}$.

Using the initial conditions and writing $y''(0) = c_1$ and $y'''(0) = c_2$, we have:

$$s^4Y(s) - sc_1 - c_2 = \frac{W}{E\,\mathrm{I}}e^{-\frac{bs}{2}}$$

$$s^4Y(s) = sc_1 - c_2 + \frac{W}{E\,\mathrm{I}}e^{-\frac{bs}{2}}$$

$$Y(s) = \frac{c_1}{s_3} + \frac{c_2}{s^4} + \frac{We^{-\frac{bs}{2}}}{E\,\mathrm{I}\,s^4}$$

Thus $$\mathcal{L}^{-1}\{Y(s)\} = c_1\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + c_2\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} + \frac{W}{E\,\mathrm{I}}\mathcal{L}^{-1}\left\{\frac{e^{-\frac{bs}{2}}}{s^4}\right\}$$

and $$y(x)=\frac{c_1x^2}{2}+\frac{c_2x^3}{6}+\frac{W}{6\ E\mathrm{I}}\left(x-\frac{b}{2}\right)^3 U\left(x-\frac{b}{2}\right)$$

or $$y(x)=\frac{c_1x^2}{2}+\frac{c_2x^3}{6}+\frac{W}{6\ E\mathrm{I}}\left(x-\frac{b}{2}\right)^3 H\left(x-\frac{b}{2}\right)$$

We can also write *y*(*x*) as:

$$y(x)=\begin{cases}\dfrac{c_1x^2}{2}+\dfrac{c_2x^3}{6} & 0 < x < \dfrac{b}{2}\\[2mm] \dfrac{c_1x^2}{2}+\dfrac{c_2x^3}{6}+\dfrac{W}{6\ E\mathrm{I}}\left(x-\dfrac{b}{2}\right)^3 & x > \dfrac{b}{2}\end{cases}$$

In order to use the initial conditions, we differentiate *y*(*x*) using the value for $x > \frac{b}{2}$.

$$y'(x)=c_1x+\frac{c_2x^2}{2}+\frac{3W}{6\ E\mathrm{I}}\left(x-\frac{b}{2}\right)^2$$

We now substitute $y\left(b\right)=0$ in the equation for *y*(*x*) and $y'(x)$, respectively, obtaining

$$0=\frac{c_1b^2}{2}+\frac{c_2b^3}{6}+\frac{W}{6\ E\mathrm{I}}\left(\frac{b}{2}\right)^3$$

and $y'(b)=0$ in the equation for $y'(x)$, obtaining

$$0=c_1b+\frac{c_2b^2}{2}+\frac{W}{2\ E\mathrm{I}}\left(\frac{b}{2}\right)^2.$$

Solving these simultaneous equations we find:

$$c_1=\frac{Wb}{8E\mathrm{I}}$$

$$c_2=-\frac{W}{2E\mathrm{I}}$$

The required equation of the curve of deflection is therefore:

$$y(x)=\begin{cases}\dfrac{W}{E\mathrm{I}}\left[\dfrac{bx^2}{16}-\dfrac{x^3}{12}\right] & 0 < x < \dfrac{b}{2}\\[2mm] \dfrac{W}{E\mathrm{I}}\left[\dfrac{bx^2}{16}-\dfrac{x^3}{12}+\dfrac{1}{6}\left(x-\dfrac{b}{2}\right)^3\right] & x > \dfrac{b}{2}\end{cases}$$

Examples 6 and 8 above may seem rather long and involved, but they have been included to demonstrate the use of unit step and impulse functions in beam problems.

6.2 POST-TEST: MODULE 4 (LEARNING UNIT 6)

(*Solutions on myUnisa under additional resources*)

Time: 70 minutes

1. An inductor of 1 H, a resistor of 2 Ω and a capacitor of 0.5 F are connected in series with an electromagnetic force of $\cos t$ volts. At $t = 0$, the charge on the capacitor and the current in the circuit are zero. Find the current at any time $t > 0$ and discuss the solution as $t \to \infty$. (25)

2. A system vibrates according to the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$$

where y is the displacement at time t. If the initial conditions are $y(0) = 0$ and $y'(0) = 3$, find y in terms of t. (9)

3. A cantilever beam, clamped at $x = 0$ and free at $x = b$, carries a uniform load k per unit length. Given the differential equation

$$\frac{d^4y}{dx^4} = \frac{k}{E\mathrm{I}} \qquad 0 < x < b$$

and initial conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(b) = 0 \text{ and } y'''(b) = 0,$$

show that the equation of the curve of deflection of the beam is:

$$y(x) = \frac{k}{24E\mathrm{I}}\left(x^4 - 4bx^3 + 6b^2x^2\right)$$ (16)

[50]

Using Laplace transforms, you should now be able to solve problems relating to electric circuits, vibrating systems and beams.

This concludes our study of Laplace transforms. In the next module we shall continue our studies on linear algebra and learn how to solve a system of equations using Gaussian elimination.

MODULE 5
LINEAR ALGEBRA

LEARNING UNIT 1
GAUSS ELIMINATION

OUTCOMES

At the end of this learning unit you should be able to

- understand the process of Gaussian elimination
- solve a system of simultaneous equations using Gaussian elimination

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS — PAGE

1.1 INTRODUCTION

Linear algebra involves the systematic solving of linear algebraic or differential equations that arise during mathematical modelling of an electrical, mechanical or other engineering system where two or more components are interacting with each other. In the first-level mathematics engineering course you used elimination, substitution and Cramer's rule to solve simultaneous equations. In the second level you used matrix multiplication and the inverse of the coefficient matrix to solve simultaneous equations. We are now adding another method to solve a system of simultaneous equations called Gaussian elimination.

Recall that a system of m linear equations in n unknowns may have

(1) no solution, in which case it is called an inconsistent system

(2) exactly one solution, called a unique solution

(3) an infinite number of solutions

In the latter two cases, the system is said to be consistent.

Before continuing:

- Refer to myUnisa to refresh your memory on determinants, Cramer's rule and using the inverse of a matrix to solve simultaneous equations.
- See Tutorial Letter 101 to find the pages you must study from your prescribed book on Gaussian elimination.

1.2 GAUSSIAN ELIMINATION

In each step of the process, we may perform one of the following operations to keep an equivalent set of equations:

1. Interchange two equations.
2. Multiply or divide a row by a non-zero constant.
3. Add (or subtract) a constant multiple of one row to (or from) another.

To keep track of our solution we number the equations.

Our object is to find a solution for one of the unknowns and then substitute backwards in the other equations to find the solution to all the unknowns.

EXAMPLE 1

Use Gaussian elimination to solve the following system of linear equations:

$$\begin{aligned} x-2y+3z&=9\\ -x+3y&=-4\\ 2x-5y+5z&=17 \end{aligned}$$

SOLUTION

$x-2y+3z=9$ Equation 1

$-x+3y \quad =-4$ Equation 2

$2x-5y+5z=17$ Equation 3

$x-2y+3z=9$ Equation 1

$+y+3z=5$ Equation 4 = Equation 2 + Equation 1

$-y-z=-1$ Equation 5 = Equation 3 – 2 times equation 1

$x-2y+3z=9$ Equation 1

$y+3z=5$ Equation 4

$2z=4$ Equation 6 = Equation 5 + Equation 4

$x-2y+3z=9$ Equation 1

$y+3z=5$ Equation 4

$z=2$ Equation 7 = Equation 6 divided by 2

We can now substitute $z = 2$ in equation 4 to find $y = –1$

and then substitute z and y in equation 1 to find $x = 1$.

Instead of writing out the full equations, we can use matrix notation. We will now redo example 1 using matrix notation.

Writing our equations in matrix form gives:

$$\begin{bmatrix} 1 & -2 & 3\\ -1 & 3 & 0\\ 2 & -5 & 5 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 9\\ -4\\ 17 \end{bmatrix}$$

We then form what is called the **augmented** matrix by entering the right-hand side into the left-hand matrix and numbering our rows as follows:

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Since each row of the augmented matrix corresponds to an equation, the three operations listed earlier correspond to the following three row operations on the augmented matrix:

1. Interchange two rows.
2. Multiplyor divide a row by a non-zero constant.
3. Add (or subtract) a constant multiple of one row to (or from) another row.

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 3 & 5 & 5 \\ 2 & -5 & 5 & 17 \end{array}\right] \qquad \text{Row 4 = Row 2 + Row 1}$$

$$\text{Row 1}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 3 & 5 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \qquad \text{Row 5 = Row 3 } - \text{ } 2\left(\text{Row 1}\right)$$

$$\text{Row 1}\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 3 & 5 & 5 \\ 0 & 0 & 2 & 4 \end{array}\right] \qquad \text{Row 6 = Row 3 } - \text{ } 2\left(\text{Row 1}\right)$$

We now have $2z = 4$,

thus $z = 2$

and substituting backwards give $y = -1$ and $x = 1$.

EXAMPLE 2

A simple pulley system gives the equations

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 &= 0 \\ -3\ddot{x}_1 + T &= 3g \\ -2\ddot{x}_2 + T &= 2g \end{aligned}$$

where $\ddot{x}_1, \ddot{x}_2$ represent acceleration, T is the tension in the rope and g is the acceleration due to gravity. Determine $\ddot{x}_1, \ddot{x}_2$ and T using Gaussian elimination. (To clarify the notation used, derivatives to time, t, may be represented by a dot, thus $\ddot{x}_1 = \frac{d^2 x_1}{dt^2}$ and $\ddot{x}_2 = \frac{d^2 x_1}{dt^2}$.)

SOLUTION

In matrix form: $\begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ 3g \\ 2g \end{pmatrix}$

In augmented matrix form and indicating which row operations have been performed:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 3g \\ 0 & -2 & 1 & 2g \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3g \\ 0 & -2 & 1 & 2g \end{array}\right) \quad R_2 + 3R_1 = \mathrm{R}_4$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 3g \\ 0 & 0 & \frac{5}{3} & 4g \end{array}\right) \quad R_3 + \tfrac{2}{3} R_4$$

From the last row $\frac{5}{3}T = 4g$

Thus $T = \frac{12}{5} g$

Substitute T in the second row $3\ddot{x}_2 + \frac{12}{5} g = 3g$

$\ddot{x}_2 = \frac{1}{5} g$

From the first row $\ddot{x}_1 + \ddot{x}_2 = 0$

Thus $\ddot{x}_1 = -\frac{1}{5} g$

1.3 POST-TEST: MODULE 5 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 60 minutes

1. Use Gaussian elimination to solve the following sets of simultaneous equations:

(a) (10)

$$\begin{aligned} 2x + y - z &= 2 \\ x + 3y + 2z &= 1 \\ x + y + z &= 2 \end{aligned}$$

(b) (10)

$$\begin{aligned} 10x + y - 5z &= 18 \\ -20x + 3y + 20z &= 14 \\ 5x + 3y + 5z &= 9 \end{aligned}$$

3. By applying Kirchhoff's law and Ohm's law to a circuit, we obtain the following equations:

$$\begin{aligned} 3I_1 - 5I_2 + 3I_3 &= 7,5 \\ 2I_1 + I_2 - 7I_3 &= -17,5 \\ 10I_1 + 4I_2 + 5I_3 &= 16 \end{aligned}$$

where I_1, I_2 and I_3 represent current. Find the values of I_1, I_2 and I_3 by using

(a) Gaussian elimination (10)

(b) the inverse matrix method (10)

(c) Cramer's rule (10)

[50]

You should now be able to solve simultaneous equations using Gaussian elimination.

We can now move to the second unit on linear algebra and study how to determine eigenvalues and eigenvectors of a matrix.

MODULE 5
LINEAR ALGEBRA

LEARNING UNIT 2
EIGENVALUES AND EIGENVECTORS

OUTCOMES

At the end of this learning unit you should be able to

- determine the eigenvalues of a 2 by 2 and a 3 by 3 matrix
- determine the eigenvectors of a 2 by 2 and a 3 by 3 matrix

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

2.1 INTRODUCTION

In many applications of matrices to technical problems involving coupled oscillations and vibrations, equations of the form

$$\mathbf{A} \bullet \mathbf{x} = \lambda \mathbf{x}$$

occur, where **A** is a square matrix and λ is a number. Clearly, **x** = 0 is a solution for any value of λ and is not normally useful. For non-trivial solutions, that is **x** ≠ 0, the values of λ are called the eigenvalues of the matrix **A** and the corresponding solutions of the given equation $\mathbf{A} \bullet \mathbf{x} = \lambda \mathbf{x}$ are called the eigenvectors of **A**.

Before continuing:

- Refer to myUnisa to refresh your memory on determinants and matrices.
- Refer to Tutorial Letter 101 for the reference to your prescribed book.
- Take your prescribed book and study the part on eigenvalues and eigenvectors.

Now study the following examples:

2.2 EIGENVALUES

EXAMPLE 1

Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

SOLUTION

Characteristic determinant: $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix}$

Characteristic equation: $|\mathbf{A} - \lambda \mathbf{I}| = 0$

Thus

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda) - (-1)(2) = 0$$

$$4 - 5\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 2$$

EXAMPLE 2

Find the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 4 & -2 \\ 2 & 10 & -5 \end{bmatrix}$.

SOLUTION

The characteristic determinant is $\begin{vmatrix} 2-\lambda & 3 & -2 \\ 1 & 4-\lambda & -2 \\ 2 & 10 & -5-\lambda \end{vmatrix}$.

Using the co-factor method to calculate the determinant and expanding along the first row we get:

$$= (2-\lambda)\{(4-\lambda)(-5-\lambda)-(-20)\} - 3\{(1)(-5-\lambda)-(-2)(2)\} + (-2)\{(10)-(4-\lambda)(2)\}$$
$$= (2-\lambda)\{-20+\lambda+\lambda^2+20\} - 3\{-5-\lambda+4\} - 2\{10-8+2\lambda\}$$
$$= (2-\lambda)\{\lambda^2+\lambda\} - 3\{-\lambda-1\} - 2\{2+2\lambda\}$$
$$= (2-\lambda).\lambda.(\lambda+1) + 3(\lambda+1) - 4(\lambda+1)$$
$$= (\lambda+1)\left[(2-\lambda).\lambda + 3 - 4\right]$$
$$= (\lambda+1)\left[2\lambda - \lambda^2 - 1\right]$$
$$= -(\lambda+1)\left[\lambda^2 - 2\lambda + 1\right]$$
$$= -(\lambda+1)(\lambda-1)^2$$

From the characteristic equation $-(\lambda+1)(\lambda-1)^2 = 0$

it follows that $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 1$.

2.3 EIGENVECTORS

The eigenvalues for a matrix is unique. However, there are many possibilities for the corresponding eigenvectors, as can be seen from the following example.

EXAMPLE 3

Find the eigenvectors for the matrix $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$.

SOLUTION

The characteristic equation is: $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$$(4-\lambda)(2-\lambda) - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda_1 = 1 \text{ and } \lambda_2 = 5$$

For $\lambda_1 = 1$ the equation $\mathbf{A} \bullet \mathbf{x} = \lambda\mathbf{x}$ becomes:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use matrix multiplication rules to find a set of simultaneous equations:

$$4x_1 + x_2 = x_1$$
$$3x_1 + 2x_2 = x_2$$

From either of these equations we can obtain the relationship $x_2 = -3x_1$.

So if we choose $x_1 = 1$, then the value of x_2 will be -3.

Then $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

Depending on our choice we will get different answers which will all be correct eigenvectors.

So if we choose $x_2 = 1$, then the value of x_1 will be $\frac{-1}{3}$.

Then $\mathbf{x} = \begin{bmatrix} \frac{-1}{3} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

Both of the above answers are correct and there are many more depending on the choice of one of the variables. We always try to make our choice as simple as possible.

For $\lambda_2 = 5$ the equation $\mathbf{A} \bullet \mathbf{x} = \lambda\mathbf{x}$ becomes:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use matrix multiplication rules to find a set of simultaneous equations:

$$4x_1 + x_2 = 5x_1$$
$$3x_1 + 2x_2 = 5x_2$$

From either of these equations we can obtain the relationship $x_2 = x_1$.

So if we choose $x_2 = 1$, then the value of x_1 will also be 1.

We always try to make our choice as simple as possible.

Then $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 5$.

EXAMPLE 4

If $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ find the eigenvalues of **A** and the eigenvector that corresponds with one of the eigenvalues $\lambda = 5$.

SOLUTION

The characteristic equation is $\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

Develop along the first column

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda_1 = 3 \text{ , } \lambda_2 = 2 \text{ and } \lambda_3 = 5$$

It is often more convenient to use the form $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ instead of $\mathbf{A} \bullet \mathbf{x} = \lambda\mathbf{x}$.

For $\lambda_3 = 5$ the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ becomes:

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \overline{x} = \overline{0}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use matrix multiplication to obtain the set of simultaneous equations:

$-2x_1 + x_2 + 4x_3 = 0 \qquad (1)$

$-3x_2 + 6x_3 = 0 \qquad (2)$

$0 = 0$ (the last one can be omitted as it does not lead to a solution)

From equation (2) $\quad x_2 = 2x_3 \quad (A)$

into equation $(1): -2x_1 + 2x_3 + 4x_3 = 0$

$$2x_1 = 6x_3$$

$$x_1 = 3x_3 \quad (B)$$

Equation (A) and (B) gives us the relationships between x_1, x_2 and x_3.

Choose $x_3 = 1$ and the eigenvector corresponding to $\lambda_2 = 5$ is $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

2.4 POST TEST: MODULE 5 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 90 minutes

For the coefficient matrix **A** given in each case, determine the eigenvalues and an eigenvector corresponding to each eigenvalue:

1. $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$ (12)

2. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ (12)

3. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$ (12)

4. $\mathbf{A} = \begin{bmatrix} 1 & -4 & -2 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (12)

5. $\mathbf{A} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ (12)

[60]

You should now be able to determine the eigenvalues and eigenvectors of a 2 by 2 and a 3 by 3 matrix.

We shall now continue to the next module on Fourier series.

MODULE 6
FOURIER SERIES

LEARNING UNIT 1
FOURIER SERIES FOR PERIODIC FUNCTIONS OF PERIOD 2π

OUTCOMES

At the end of this learning unit you should be able to

- describe a Fourier series
- understand periodic functions
- state the formula for a Fourier series and Fourier coefficients
- obtain Fourier series for a given function

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

1.1 INTRODUCTION

Many functions can be expressed in the form of an infinite series. Problems involving various forms of oscillations are common in all fields of modern technology and Fourier series enable us to represent a periodic function as an infinite trigonometrical series in sine and cosine terms. One important advantage of Fourier series is that it can represent a function containing discontinuities, whereas Maclaurin and Taylor's series require the function to be continuous throughout.

- Refer to myUnisa and revise integration by parts.
- Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

You will not be able to master the outcomes for this learning unit without referring to your prescribed book.

1.2 PERIODIC FUNCTIONS

Periodic functions repeat themselves at regular intervals. The trigonometric functions are examples of periodic functions.

A function $f(x)$ is said to be **periodic** if $f(x+T)=f(x)$ for all values of *x*, where *T* is a positive number. *T* is the interval between two successive repetitions and is called the **period of the function** $f(x)$.

For example, $y=\sin x$ is periodic in *x* with period 2π since
$\sin(x)=\sin(x+2\pi)=\sin(x+4\pi)$ and so on.

1.3 FOURIER SERIES FOR PERIODIC FUNCTIONS OF PERIOD 2π

The basis of Fourier series is that all functions, which are defined in the interval $-\pi \le t \le \pi$, can be expressed in terms of a convergent trigonometric series of the form:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where for the range $-\pi$ to π:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$(n = 1, 2, 3, \ldots)$$

a_0, a_n and b_n are called the Fourier coefficients of the series.

If you refer to a book other than the prescribed book, make sure that you know the meaning of the symbols used. Whichever reference you use, the final Fourier series will be the same.

Note that the limits of integration may be replaced by any interval of length 2π, such as from 0 to 2π, and need not always be from –π to π.

EXAMPLE 1

A function $f(x)$ is defined by $f(x) = \begin{cases} -x & -\pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}$

and $\left[f(x) = f(x + 2\pi) \right]$.

Determine the Fourier series of $f(x)$.

SOLUTION

Sketch the given function:

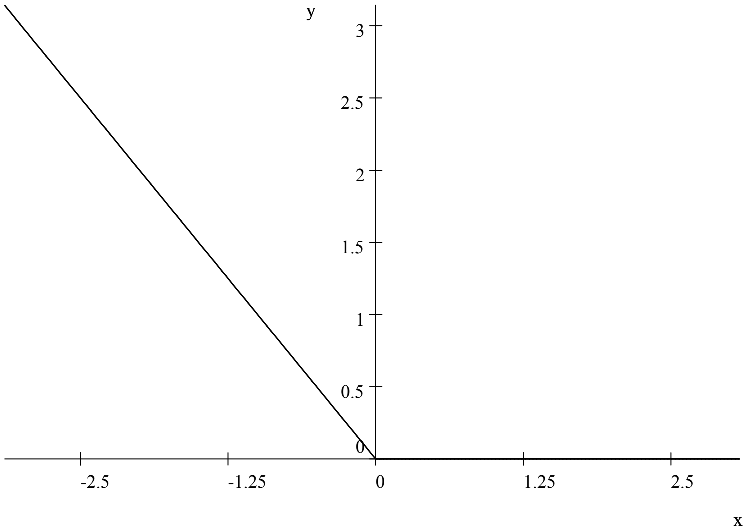


Figure 1

Now calculate the Fourier coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

The integral must be split over the two pieces over which f is defined

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} 0\,dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{-x^2}{2}\right]_{-\pi}^{0} + 0$$

$$= \frac{1}{2\pi}\left\{0 + \frac{\pi^2}{2}\right\}$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\,dx$$

[Use integration by parts let $u = -x$ and $dv = \cos nx dx$

$$\int dv = \int \cos nx\,dx$$

$$v = \frac{1}{n}\sin nx]$$

$$a_n = \frac{1}{\pi}\left[-\frac{x}{n}\sin nx + \frac{1}{n}\int \sin nx \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[-\frac{x}{n}\sin nx - \frac{1}{n^2}\cos nx \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n^2} - \frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi \right]$$

Remember n is a natural number.

Investigate the behaviour of the trigonometric functions for different values of n. Do the same for $(-1)^n$. We will use the result to simplify a_n and b_n.

n	$\sin n\pi$	$\cos n\pi$	$(-1)^n$
1	0	–1	–1
2	0	1	1
3	0	–1	–1
4	0	1	1
5	0	–1	–1

Now $a_n = \frac{1}{\pi}\left[0 - \frac{1}{n^2} - 0 + \frac{(-1)^n}{n^2} \right] = \frac{-1+(-1)^n}{\pi n^2}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin nx dx$$

Integrate by parts let $u = x$ and $v = \int -\sin nx\, dx = \cos nx$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cos nx - \frac{1}{n}\int \cos nx \right]$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cos nx - \frac{1}{n^2}\sin n\pi \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cos nx - 0 \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\cos nx \right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[0 + \frac{\pi}{n}\cos n\pi \right]$$ Remember $\cos(-n\pi) = \cos(n\pi)$

$$= \frac{(-1)^n}{n}$$

We are now ready to write down the final answer.

The Fourier series expansion is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

We calculated the coefficients to be $a_0 = \frac{\pi}{4}$, $a_n = \frac{-1+(-1)^n}{\pi n^2}$ and $b_n = \frac{(-1)^n}{n}$

Thus $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{\pi n^2} \cos(nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$

The answer may be left in sigma notation or expanded substituting the values of *n*

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + ...\right) - \left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right)$$

1.4 POST-TEST: MODULE 6 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 30 minutes

Find the Fourier series for the following functions:

1. $f(x) = 1 - \frac{x}{\pi} \qquad 0 < x < 2\pi$

   $f(x) = f\left(x + 2\pi\right)$

   up to and including the third harmonic (10)

2. $$f(x) = \begin{cases} \pi + x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$ (10)

   $f(x) = f\left(x + 2\pi\right)$

[20]

You should now be able to describe a Fourier series, understand periodic functions and state the formula for a Fourier series and Fourier coefficients. You should also be able to obtain the Fourier series for a given function with period 2π.

We shall now continue to expand our knowledge of Fourier series and study Fourier series for a non-periodic function over range 2π in the next learning unit.

MODULE 6
FOURIER SERIES

LEARNING UNIT 2
FOURIER SERIES FOR NON-PERIODIC FUNCTIONS OVER A RANGE OF 2π

OUTCOMES

At the end of this learning unit, you should be able to

- understand that Fourier expansions of non-periodic functions have a limited range
- determine Fourier series for non-periodic functions over a range of 2π

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

2.1 INTRODUCTION

You will not be able to master the outcomes for this learning unit without referring to your prescribed book. Refer to Tutorial Letter 101 for the relevant chapter.

If a function $f(x)$ is not periodic, then it cannot be expanded in a Fourier series for **all** values of x. However, given a non-periodic function we can construct a new function by taking the values for $f(x)$ in the given range and then repeating them outside of the range at intervals of 2π. By our construction, this new function is periodic with period 2π and can thus be expanded in a Fourier series as described in learning unit 1 for all values of x.

2.2 POST-TEST: MODULE 6 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 30 minutes

1. Determine the Fourier series for the function defined by:

$$f(x) = \begin{cases} 1 - x & -\pi < x < 0 \\ 1 + x & 0 < x < \pi \end{cases}$$ (10)

2. Find the Fourier series for the function $f(x) = x + \pi$ within the range $-\pi < x < \pi$.

(10)

[20]

You should now be able to determine Fourier series for non-periodic functions over a range of 2π.

We shall now continue to expand our knowledge of Fourier series and study half-range Fourier series in the next learning unit.

MODULE 6

FOURIER SERIES

LEARNING UNIT 3

EVEN AND ODD FUNCTIONS AND HALF-RANGE FOURIER SERIES

OUTCOMES

At the end of this learning unit you should be able to

- recognise even and odd functions
- determine Fourier cosine series and Fourier sine series
- determine Fourier half-range cosine series and Fourier half-range sine series

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

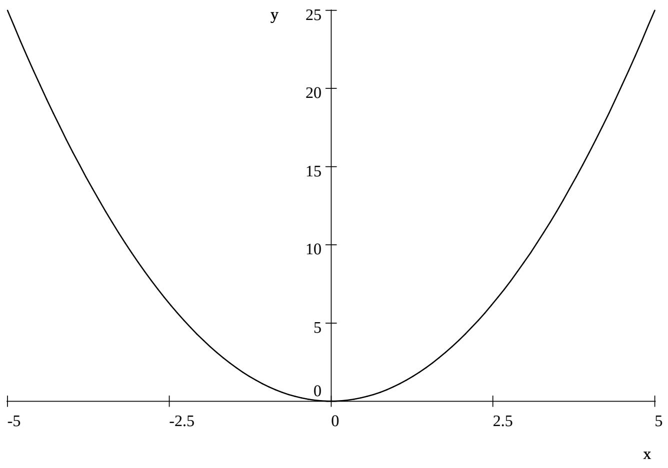
3.1 INTRODUCTION

You will not be able to master the outcomes for this learning unit without referring to your prescribed book.

3.2 REVISION: EVEN AND ODD FUNCTIONS

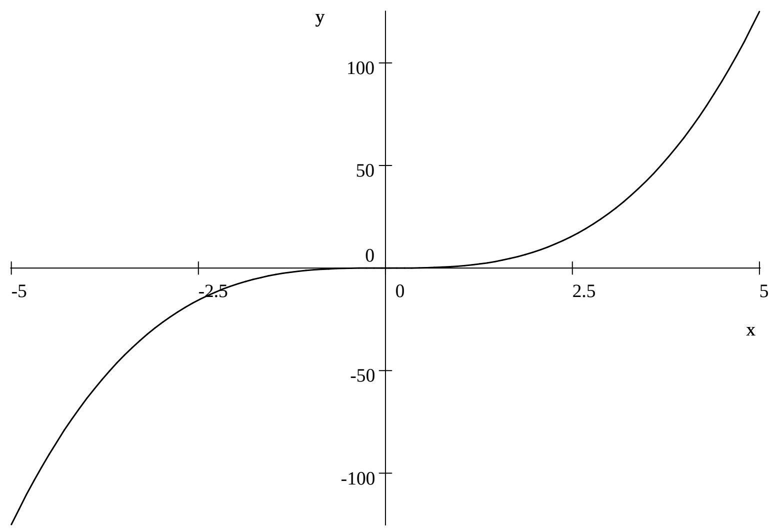
A function is said to be even if it is symmetrical around the y-axis. The easiest way to check if a function is even is to make a sketch and fold on the y-axis. The two pieces should be identical.

Mathematically $f(-x) = f(x)$, for example the parabola $y = x^2$:



A function is said to be odd if it is symmetrical around the origin.

Mathematically $f(-x) = -f(x)$, for example $y = x^3$:



EXAMPLE 1

Determine whether each of the following functions is odd, even or neither:

(a) $f\left(x\right)=2x+3$ (1)

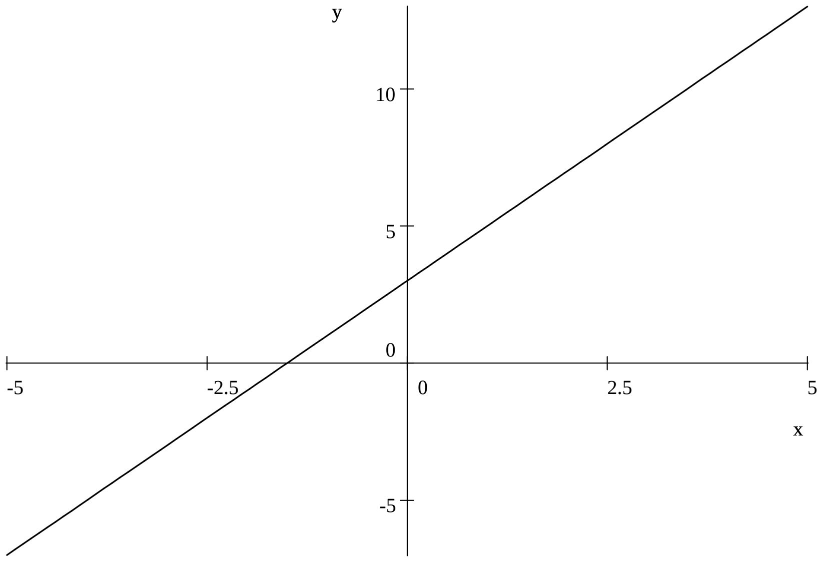
(b) $f\left(x\right)=\cos x$ (1)

(c) $f(t)=\begin{cases} 3 & -\dfrac{\pi}{2}<t<0 \\ -3 & 0<t<\dfrac{\pi}{2} \end{cases}$ (1)

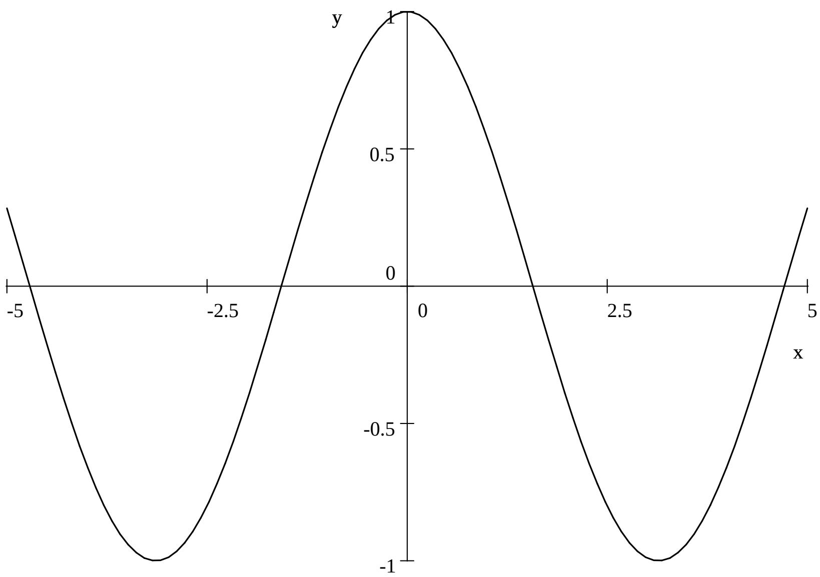
SOLUTION

Sketch the given functions:

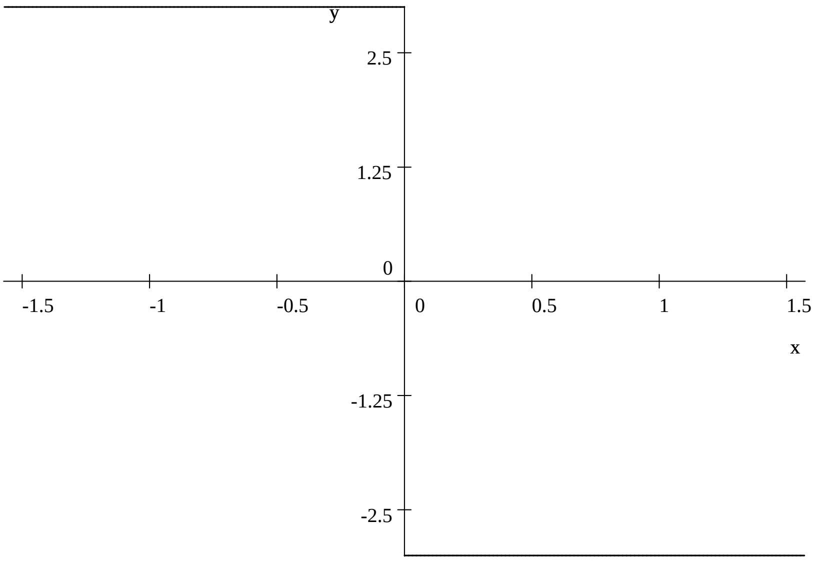
(a) Straight line



(b) cosine function



(c)



Answers

(a) Neither

(b) Even

(c) Odd

3.3 FOURIER SERIES

EXAMPLE 2

A function $f(t)$ is defined by:

$$f(t) = \begin{cases} 3 & -\pi < t < 0 \\ -3 & 0 < t < \pi \end{cases}$$

$$[f(t) = f(t+2\pi)]$$

Determine the Fourier expansion of $f(t)$.

SOLUTION

We have already found the function to be odd in example 1, thus $a_0 = a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

$$= \frac{2}{\pi}\left[\int_{-\pi}^{0} 3\sin(nt)dt - \int_{0}^{\pi} 3\sin(nt)dt\right]$$

(No integration by parts necessary as 3 is a constant number)

$$= \frac{2}{\pi}\left[-\frac{3}{n}\cos(nt)\Big|_{-\pi}^{0} + \left(\frac{3}{n}\cos(nt)\right)_{0}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{3}{n} + \frac{3}{n}\cos n\pi + \frac{3}{n}\cos n\pi - \frac{3}{n}\right]$$

$$= \frac{2}{\pi}\left[-\frac{6}{n} + \frac{6}{n}(-1)^n\right]$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[-\frac{6}{n} + \frac{6}{n}(-1)^n\right]\sin nt$$

$$= -\frac{12}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$$

3.4 POST-TEST: MODULE 6 (LEARNING UNIT 3)

(*Solutions on myUnisa under additional resources*)

Time: 90 minutes

1. Find the Fourier series for the following function:

$$f(x) = \begin{cases} a & 0 < x < \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < x < \frac{2\pi}{3} \\ -a & \frac{2\pi}{3} < x < \pi \end{cases} \qquad (10)$$

$$f(x) = f(x+\pi)$$

2. If $f(x)$ is defined by

$$f(x) = x(\pi - x) \qquad 0 < x < \pi,$$

express $f(x)$ as a half-range cosine series. (10)

[40]

You should now be able to recognise even and odd functions, determine Fourier cosine series and Fourier sine series and determine Fourier half-range cosine series and Fourier half-range sine series.

MODULE 6

FOURIER SERIES

LEARNING UNIT 4

FOURIER SERIES OVER ANY RANGE

OUTCOMES

At the end of this learning unit, you should be able to

- determine the Fourier series of a periodic function of period $2L$
- determine the half-range Fourier series for functions of period $2L$

Refer to Tutorial Letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS PAGE

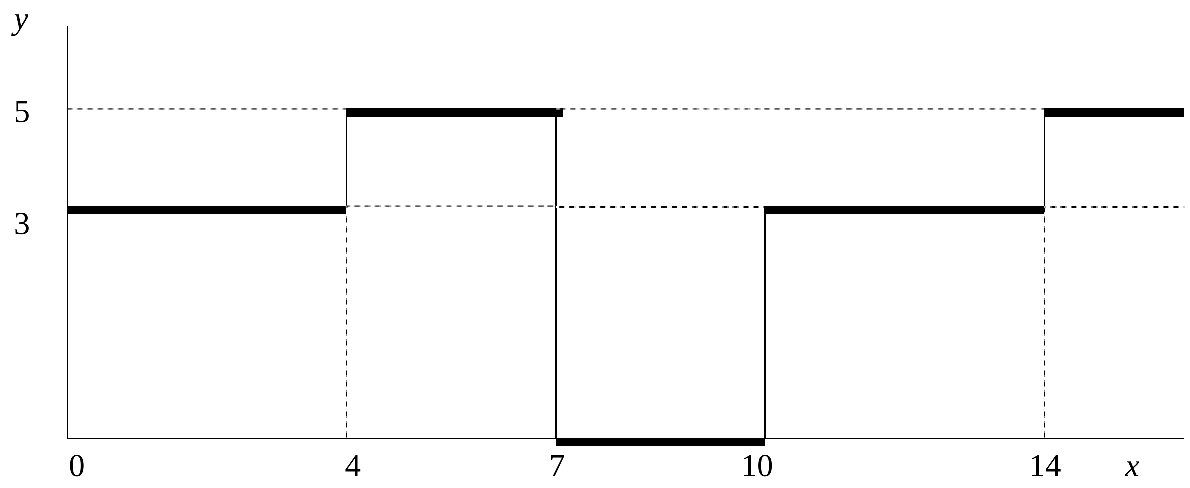
4.1 INTRODUCTION

You will not be able to master the outcomes for this learning unit without referring to your prescribed book.

4.2 REVISION: DEFINITION OF PIECEWISE FUNCTIONS

EXAMPLE 1

Analytically define the periodic function shown:



SOLUTION

$$f(x)=\begin{cases}3 & 0<x<4\\5 & 4<x<7\\0 & 7<x<10\end{cases}$$

$$f(x+10)=f(x)$$

Remember to indicate the period.

Also note that this function is neither odd nor even.

4.3 FOURIER SERIES OVER RANGE 2*L*

Definition:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

where $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(t)\,dt$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Note that L is defined as half the period of the function.
So if it is given that $f(t) = f(t + 2\pi)$, it means that the function repeats itself over the length 2π. Thus the period is 2π and $L = \pi$.

Whichever book you use as a reference, make sure that you know the meaning of the symbols in the formula. For instance, if the book you are referring to defined the function over a period of L, then you have to replace L with $\dfrac{L}{2}$ in the definition above.

The limits of integration for calculating the Fourier coefficients may be replaced by any interval equal to the period. In the above definition, we could have integrated from 0 to $2L$.

EXAMPLE 2

A function $f(t)$ is defined by:

$$f(t) = \begin{cases} 3 & -\dfrac{\pi}{2} < t < 0 \\ -3 & 0 < t < \dfrac{\pi}{2} \end{cases}$$

$$\left[f(t) = f(t + \pi) \right]$$

Determine the Fourier expansion of $f(t)$.

SOLUTION

We found this function to be odd in learning unit 3, thus $a_0 = a_n = 0$.

$$b_n = \frac{1}{\left(\frac{\pi}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) \sin \frac{2\pi nt}{\pi} dt$$

$$= \frac{2}{\pi} \left[\int_{-\pi/2}^{0} 3\sin(2nt)\, dt - \int_{0}^{\pi/2} 3\sin(2nt)\, dt \right]$$

(No integration by parts necessary as 3 is a number)

$$= \frac{2}{\pi} \left[-\frac{3}{2n}\cos(2nt)\Big|_{-\pi/2}^{0} + \left(\frac{3}{2n}\cos(2nt)\right)\Big|_{0}^{\pi/2} \right]$$

$$= \frac{2}{\pi} \left[-\frac{3}{2n} + \frac{3}{2n}\cos n\pi + \frac{3}{2n}\cos n\pi - \frac{3}{2n} \right]$$

$$= \frac{2}{\pi} \left[-\frac{6}{2n} + \frac{6}{2n}(-1)^n \right]$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi} \left[-\frac{6}{2n} + \frac{6}{2n}(-1)^n \right] \sin(2nt)$$

$$f(t) = -\frac{12}{\pi}\left(\sin 2t + \frac{1}{3}\sin 6t + \ldots\ldots \right)$$

4.4 POST-TEST: MODULE 6 (LEARNING UNIT 4)

(*Solutions on myUnisa under additional resources*)

Time: 30 minutes

1. Find the Fourier series for the following functions:

$$f(x) = \begin{cases} a & 0 < x < \frac{\pi}{3} \\ 0 & \frac{\pi}{3} < x < \frac{2\pi}{3} \\ -a & \frac{2\pi}{3} < x < \pi \end{cases} \qquad (10)$$

$$f(x) = f(x + \pi)$$

2. If $f(x)$ is defined by

$$f(x) = x(\pi - x) \qquad 0 < x < \pi,$$

express $f(x)$ as a half-range cosine series. (10)

[20]

You should now be able to determine the Fourier series of a periodic function of period $2L$ and the half-range Fourier series for functions of period $2L$.

This brings us to the end of study guide 2 and concludes the work necessary to successfully complete MAT3700.