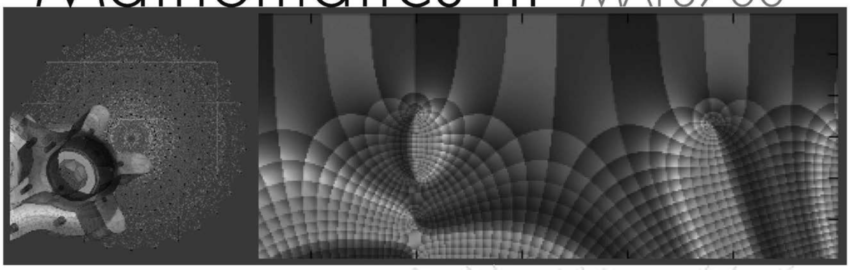
Mathematics III

STUDY GUIDE 1 FOR

MAT3700



LE Greyling

Department of Mathematical Sciences

UNIVERSITY OF SOUTH AFRICA
PRETORIA

Printed and published by the
University of South Africa
Muckleneuk, Pretoria

MAT3700/1/2018–2020

70525137

Pdf_Florida Campus

CONTENTS Page

MODULE 2

LEARNING UNIT 1: SOLVING SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

LEARNING UNIT 2: METHOD OF UNDETERMINED COEFFICIENTS

MODULE 3

LEARNING UNIT 1: INTRODUCTION TO DIFFERENTIAL OPERATORS

LEARNING UNIT 0: INTRODUCTION

This material has been compiled to serve the mathematical needs of students engaged in engineering studies. A student entering this course should have studied at least two semesters of engineering mathematics.

To achieve success requires discipline and hard work. You need to buy the prescribed book named in Tutorial Letter 101 to use with these notes. You must have access to myUnisa, where additional notes and explanations will be posted. You must also be prepared to consult other resources for a better understanding of the subject matter. To restrict your studies to the study guides alone will not be sufficient.

Where do you start?

Learn by heart

- your student number
- the module (subject) code

Organise your workspace. You need

- a place to write
- paper and a pen to try examples and do activities and exercises
- a scientific calculator
- the tutorial letter for MAT3700
- the prescribed book (Purchase the text book, details in tutorial letter 101, as soon as possible.)

Consult your tutorial letter to obtain information on

- assignments
- how to obtain a year mark
- examinations
- due dates
- prescribed and recommended books (you must buy the prescribed book)
- contact details of your lecturer or tutor

Set up a study plan:

- Identify your goals.
- Make a weekly timetable.
- Indicate work and family responsibilities on your timetable.
- Remember to allow time for rest and relaxation.
- Evaluate the time available for studying.

Before you complete your plan, we must first explain the format of the study material.

Format of study material

- ## Study guides

The work is divided into two study guides consisting of modules. Note: The corresponding chapters in your textbook are given in tutorial letter 101.

- ## Modules

The work is divided into modules. Each module deals with a major subject area. Each module is divided into learning units. The end of each learning unit provides a natural break. This enables you to plan your time. You will find a complete list of modules and learning units in the list of contents. Your tutorial letter will give guidance as to the importance of each module and each learning unit.

- ## Learning units

Each learning unit starts with OUTCOMES. These outcomes list what you should be able to do after you have mastered the content of the learning unit. To explain the content, you will find examples. An example gives both the question and the answer. Do the example yourself and add steps for your own understanding. Not all the thinking steps are written down in every example.

An average student should be able to study a learning unit and a half per week. The lecturer/tutor should be contacted regularly by telephone or e-mail, if necessary, to clarify any points. Contact details are given in the tutorial letter.

- ## Self-test

At the end of each module you need to assess your progress. The self-test should be used for self-evaluation. These questions include questions from past examination papers and questions similar to what you can expect in future examination papers. The solutions to all the questions are given on myUnisa and should be used to mark your own work. An average student should be able to complete the test in an evening.

Try to do the test without referring to your study notes. Warning: Do not look at the answers before attempting a solution!

When answering a question, your writing should be clear. That is, the marker or any other person reading your answer must be able to follow your reasoning. In mathematics we are not only interested in the correct answer, but also in the method you used to obtain the answer. Pay attention to the correct use of symbols.

- **Use of computer software**

We do not mention any specific programs in the notes because of the rapid development in this field, which results in any reference becoming outdated very quickly. Please read your tutorial letter carefully as we hope to introduce such programs in future.

USEFUL INFORMATION

MATHEMATICAL SYMBOLS

$+$	plus
$-$	minus
$\pm$	plus or minus
$\times$	multiply by
$\cdot$	multiply by
$\div$	divide by
$=$	is equal to
$\equiv$	is identically equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$>$	is greater than

$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$n!$	factorial $n = 1 \times 2 \times 3 \times \ldots \times n$
$\lvert k \rvert$	modulus of k, that is the size of k irrespective of the sign
$\in$	is a member of set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\therefore$	therefore
∞	infinity
e	base of natural logarithms (2,718…)
ln	natural logarithm
log	logarithm to base 10
$\sum$	sum of terms
$\lim\limits_{n \to \infty}$	limiting value as $n \to \infty$
$\int$	integral
$\frac{dy}{dx}$	derivative of y with respect to x

GREEK ALPHABET

Greek letter		**Greek name**
Α	α	Alpha
Β	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
Ε	ε	Epsilon
Ζ	ζ	Zeta
Η	η	Eta

Θ	θ	Theta
Ι	ι	Iota
Κ	κ	Kappa
Λ	λ	Lambda
Μ	μ	Mu
Ν	ν	Nu
Ξ	ξ	Xi
Ο	ο	Omicron
Π	π	Pi
Ρ	ρ	Rho
Σ	σ	Sigma
Τ	τ	Tau
Υ	υ	Upsilon
Φ	ϕ	Phi
Χ	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

FORMULA SHEETS

The following pages contain the formula sheets and tables that will be included with the examination paper.

FORMULA SHEETS

ALGEBRA

Laws of indices

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $\left(a^m\right)^n = a^{mn} = \left(a^n\right)^m$
4. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
5. $a^{-n} = \dfrac{1}{a^n}$ and $a^n = \dfrac{1}{a^{-n}}$
6. $a^0 = 1$
7. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Logarithms

Definitions: If $y = a^x$, then $x = \log_a y$.
If $y = e^x$, then $x = \ell n\, y$.

1. $\log(A \times B) = \log A + \log B$
2. $\log\left(\dfrac{A}{B}\right) = \log A - \log B$
3. $\log A^n = n \log A$
4. $\log_a A = \dfrac{\log_b A}{\log_b a}$
5. $a^{\log_a f} = f \quad \therefore \quad e^{\ell n f} = f$

Factors

$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

Partial fractions

$$\frac{f(x)}{(x+a)(x+b)(x+c)} = \frac{A}{(x+a)} + \frac{B}{(x+b)} + \frac{C}{(x+c)}$$

$$\frac{f(x)}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

$$\frac{f(x)}{\left(ax^2 + bx + c\right)(x+d)} = \frac{Ax+B}{\left(ax^2 + bx + c\right)} + \frac{C}{(x+d)}$$

Quadratic formula

If $ax^2 + bx + c = 0$

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right)$$

SERIES

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots$$

and $|b| < |a|$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

and $-1 < x < 1$

Maclaurin's Theorem

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{n-1}(0)}{(n-1)!}x^{n-1} + \ldots$$

Taylor's Theorem

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \ldots$$

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \ldots$$

COMPLEX NUMBERS

1. $z = a + bj = r(\cos\theta + j\sin\theta) = r\underline{|\theta} = re^{j\theta}$,

   where $j^2 = -1$

   Modulus: $r = |z| = \sqrt{(a^2 + b^2)}$

   Argument: $\theta = \arg z = arc\tan\frac{b}{a}$

2. Addition:

   $(a + jb) + (c + jd) = (a + c) + j(b + d)$

3. Subtraction:

   $(a + jb) - (c + jd) = (a - c) + j(b - d)$

4. If $m + jn = p + jq$, then $m = p$ and $n = q$

5. Multiplication: $z_1 z_2 = r_1 r_2 \underline{|(\theta_1 + \theta_2)}$

6. Division: $\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{|(\theta_1 - \theta_2)}$

7. De Moivre's Theorem

   $\left[r\underline{|\theta}\right]^n = r^n \underline{|n\theta} = r^n(\cos n\theta + j\sin n\theta)$

8. $z^{\frac{1}{n}}$ has n distinct roots:

   $z^{\frac{1}{n}} = r^{\frac{1}{n}} \underline{\left|\frac{\theta + k360^\circ}{n}\right.}$ with $k = 0, 1, 2, \ldots, n-1$ <

9. $re^{j\theta} = r(\cos\theta + j\sin\theta)$

   $\therefore \Re\left(re^{j\theta}\right) = r\cos\theta$ and $\Im\left(re^{j\theta}\right) = r\sin\theta$

10. $e^{a+jb} = e^a(\cos b + j\sin b)$

11. $\ell n\, re^{j\theta} = \ell n\, r + j\theta$

GEOMETRY	**MENSURATION**

GEOMETRY

1. Straight line: $y = mx + c$

$y - y_1 = m(x - x_1)$

Perpendiculars, then $m_1 = \dfrac{-1}{m_2}$

2. Angle between two lines:

$\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

3. Circle:

$x^2 + y^2 = r^2$

$(x-h)^2 + (y-k)^2 = r^2$

4. Parabola:

$y = ax^2 + bx + c$

axis at $x = \dfrac{-b}{2a}$

5. Ellipse:

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

6. Hyperbola:

$xy = k$

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (round x - axis)

$-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (round y - axis)

STATISTICS

$\overline{x} = \dfrac{\Sigma f_i x_i}{n}$

$s = \sqrt{\dfrac{\Sigma\ f_i\left(x_i - \overline{x}\right)^2}{n-1}}$

$Me = L + \dfrac{\frac{n}{2} - n_L}{f_{me}} \cdot c$

$M_0 = L + \left(\dfrac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}}\right) \cdot c$

MENSURATION

1. Circle: (θ in radians)

Area $= \pi r^2$

Circumference $= 2\pi r$

Arc length $\ell = r\theta$

Sector area $= \dfrac{1}{2} r^2 \theta = \dfrac{1}{2}\ell r$

Segment area $= \dfrac{1}{2} r^2 (\theta - \sin\theta)$

2. Ellipse:

Area $= \pi ab$

Circumference $= \pi(a+b)$

3. Cylinder:

Volume $= \pi r^2 h$

Surface area $= 2\pi rh + 2\pi r^2$

4. Pyramid:

Volume $= \dfrac{1}{3}$ area base $\times$ height

5. Cone:

Volume $= \dfrac{1}{3}\pi r^2 h$

Curved surface $= \pi r \ell$

6. Sphere:

$A = 4\pi r^2$

$V = \dfrac{4}{3}\pi r^3$

7. Trapezoidal rule:

$\dfrac{1}{2}\left[\dfrac{b-a}{n}\right]\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$

8. Simpson's rule:

$\dfrac{1}{3}\left[\dfrac{b-a}{n}\right][f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) +$

$2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$

9. Mid-Ordinate rule

$\left[\dfrac{b-a}{n}\right]\left[f(m_1) + f(m_2) + \ldots + f(m_{n-1}) + f(m_n)\right]$

HYPERBOLIC FUNCTIONS

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Identities:

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \text{sech}^2 x$$

$$\coth^2 x - 1 = \text{cosech}^2 x$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= 2\cosh^2 x - 1$$

$$= 1 + 2\sinh^2 x$$

TRIGONOMETRY

Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta\ \theta$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

$$\sin(-\theta) = -\sin\theta\ \theta$$

$$\cos(-\theta) = +\cos\theta\ \theta$$

$$\tan(-\theta) = -\tan\theta\ \theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Compound angle addition and subtraction formulae:

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double angles:

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin^2 A = ½(1 - \cos 2A)$

$\cos^2 A = ½(1 + \cos 2A)$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products of sines and cosines into sums or differences:

$\sin A \cos B = ½(\sin(A + B) + \sin(A - B))$

$\cos A \sin B = ½(\sin(A + B) - \sin(A - B))$

$\cos A \cos B = ½(\cos(A + B) + \cos(A - B))$

$\sin A \sin B = -½(\cos(A + B) - \cos(A - B))$

Sums or differences of sines and cosines into products:

$$\sin x + \sin y = 2\sin\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\sin x - \sin y = 2\cos\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

$$\cos x + \cos y = 2\cos\left[\frac{x+y}{2}\right]\cos\left[\frac{x-y}{2}\right]$$

$$\cos x - \cos y = -2\sin\left[\frac{x+y}{2}\right]\sin\left[\frac{x-y}{2}\right]$$

DIFFERENTIATION

1. $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

2. $\dfrac{d}{dx} k = 0$

3. $\dfrac{d}{dx} ax^n = anx^{n-1}$

4. $\dfrac{d}{dx} f.g = f.g' + g.f'$

5. $\dfrac{d}{dx}\dfrac{f}{g} = \dfrac{g.f' - f.g'}{g^2}$

6. $\dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.f'(x)$

7. $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dv}.\dfrac{dv}{dx}$

8. Parametric equations

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

9. Maximum/minimum

For turning points: $f'(x) = 0$

Let $x = a$ be a solution for the above.

If $f''(a) > 0$, then a minimum.

If $f''(a) < 0$, then a maximum.

For points of inflection: $f''(x) = 0$

Let $x = b$ be a solution for the above.

Test for inflection: $f(b - h)$ and $f(b + h)$

Change sign or $f'''(b) \neq 0$ if $f'''(b)$ exists.

10. $\dfrac{d}{dx}\sin^{-1} f(x) = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$

11. $\dfrac{d}{dx}\cos^{-1} f(x) = \dfrac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

12. $\dfrac{d}{dx}\tan^{-1} f(x) = \dfrac{f'(x)}{1+[f(x)]^2}$

13. $\dfrac{d}{dx}\cot^{-1} f(x) = \dfrac{-f'(x)}{1+[f(x)]^2}$

14. $\dfrac{d}{dx}\sec^{-1} f(x) = \dfrac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$

15. $\dfrac{d}{dx}\text{cosec}^{-1} f(x) = \dfrac{-f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$

16. $\dfrac{d}{dx}\sinh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{[f(x)]^2+1}}$

17. $\dfrac{d}{dx}\cosh^{-1} f(x) = \dfrac{f'(x)}{\sqrt{[f(x)]^2-1}}$

18. $\dfrac{d}{dx}\tanh^{-1} f(x) = \dfrac{f'(x)}{1-[f(x)]^2}$

19. $\dfrac{d}{dx}\coth^{-1} f(x) = \dfrac{f'(x)}{1-[f(x)]^2}$

20. $\dfrac{d}{dx}\text{sech}^{-1} f(x) = \dfrac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$

21. $\dfrac{d}{dx}\text{cosech}^{-1} f(x) = \dfrac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$

22. Increments: $\delta z = \dfrac{\partial z}{\partial x}.\delta x + \dfrac{\partial z}{\partial y}.\delta y + \dfrac{\partial z}{\partial w}.\delta w$

23. Rate of change:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt} + \frac{\partial z}{\partial w}.\frac{dw}{dt}$$

INTEGRATION

1. By parts: $\int udv = uv - \int vdu$

2. $S = \displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

3. Mean value $= \dfrac{1}{b-a}\displaystyle\int_a^b y\, dx$

4. R.M.S. $= \sqrt{\dfrac{1}{b-a}\displaystyle\int_a^b y^2\, dx}$

TABLE OF INTEGRALS

1. $\int ax^n\,dx$	$=\dfrac{ax^{(n+1)}}{n+1}+c,$	$n\neq -1$
2. $\int [f(x)]^n .f'(x)\,dx$	$=\dfrac{[f(x)]^{n+1}}{n+1}+c,$	$n\neq -1$
3. $\int \dfrac{f'(x)}{f(x)}dx$	$=\ell n\lvert f(x)\rvert+c$	
4. $\int f'(x).e^{f(x)}\,dx$	$=e^{f(x)}+c$	
5. $\int f'(x).a^{f(x)}\,dx$	$=\dfrac{a^{f(x)}}{\ell n\, a}+c$	
6. $\int f'(x).\sin f(x)\,dx$	$=-\cos f(x)+c$	
7. $\int f'(x).\cos f(x)\,dx$	$=\sin f(x)+c$	
8. $\int f'(x).\tan f(x)\,dx$	$=\ell n\,\lvert\sec f(x)\rvert+c$	
9. $\int f'(x).\cot f(x)\,dx$	$=\ell n\,\lvert\sin f(x)\rvert+c$	
10. $\int f'(x).\sec f(x)\,dx$	$=\ell n\,\lvert\sec f(x)+\tan f(x)\rvert+c$	
11. $\int f'(x).\mathrm{cosec}\, f(x)\,dx$	$=\ell n\,\lvert\mathrm{cosec}\, f(x)-\cot f(x)\rvert+c$	
12. $\int f'(x).\sec^2 f(x)\,dx$	$=\tan f(x)+c$	
13. $\int f'(x).\mathrm{cosec}^2 f(x)\,dx$	$=-\cot f(x)+c$	
14. $\int f'(x).\sec f(x).\tan f(x)\,dx$	$=\sec f(x)+c$	
15. $\int f'(x).\mathrm{cosec}\, f(x).\cot f(x)\,dx$	$=-\mathrm{cosec}\, f(x)+c$	
16. $\int f'(x).\sinh f(x)\,dx$	$=\cosh f(x)+c$	
17. $\int f'(x).\cosh f(x)\,dx$	$=\sinh f(x)+c$	
18. $\int f'(x).\tanh f(x)\,dx$	$=\ell n \cosh f(x)+c$	

19. $\int f'(x).\coth f(x)\, dx = \ell n \sinh f(x) + c$

20. $\int f'(x).\text{sech}^2 f(x)\, dx = \tanh f(x) + c$

21. $\int f'(x).\text{cosech}^2 f(x)\, dx = -\coth f(x) + c$

22. $\int f'(x).\text{sech} f(x).\tanh f(x)\, dx = -\text{sech } f(x) + c$

23. $\int f'(x).\text{cosech } f(x).\coth f(x)\, dx = -\text{cosech } f(x) + c$

24. $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = arc\sin\left(\frac{f(x)}{a}\right) + c$

25. $\int \frac{f'(x)}{[f(x)]^2 + a^2} dx = \frac{1}{a} arc\tan\left(\frac{f(x)}{a}\right) + c$

26. $\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}} dx = arc\sinh\left(\frac{f(x)}{a}\right) + c$

27. $\int \frac{f'(x)}{\sqrt{[f(x)]^2 - a^2}} dx = arc\cosh\left(\frac{f(x)}{a}\right) + c$

28. $\int \frac{f'(x)}{a^2 - [f(x)]^2} dx = \frac{1}{a} arc\tanh\left(\frac{f(x)}{a}\right) + c$

29. $\int \frac{f'(x)}{[f(x)]^2 - a^2} dx = -\frac{1}{a} arc\coth\left(\frac{f(x)}{a}\right) + c$

30. $\int f'(x)\sqrt{a^2 - [f(x)]^2}\, dx = \frac{a^2}{2} arc\sin\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{a^2 - [f(x)]^2} + c$

31. $\int f'(x)\sqrt{[f(x)]^2 + a^2}\, dx = \frac{a^2}{2} arc\sinh\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{[f(x)]^2 + a^2} + c$

32. $\int f'(x)\sqrt{[f(x)]^2 - a^2}\, dx = -\frac{a^2}{2} arc\cosh\left(\frac{f(x)}{a}\right) + \frac{f(x)}{2}\sqrt{[f(x)]^2 - a^2} + c$

TABLE OF LAPLACE TRANSFORMS

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$
e^{bt}	$\frac{1}{s-b}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n e^{bt}$	$\frac{n!}{(s-b)^{n+1}}, \quad n = 1, 2, 3, \ldots$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sinh at$	$\frac{2as}{(s^2-a^2)^2}$
$t \cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
$e^{bt} \sin at$	$\frac{a}{(s-b)^2+a^2}$
$e^{bt} \cos at$	$\frac{(s-b)}{(s-b)^2+a^2}$
$e^{bt} \sinh at$	$\frac{a}{(s-b)^2-a^2}$
$e^{bt} \cosh at$	$\frac{(s-b)}{(s-b)^2-a^2}$
$H(t-c)$	$\frac{e^{-cs}}{s}$
$H(t-c).F(t-c)$	$e^{-cs}.f(s)$
$\delta(t-a)$	e^{-as}

$\mathcal{L}\{f(t)\} = F(s)$
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$

MODULE 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

LEARNING UNIT 1

SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS

OUTCOMES

At the end of this learning unit you should be able to

- find the order of a differential equation
- solve first-order differential equations by using
  - direct integration
  - separation of variables
  - an integrating factor
- identify and solve linear, Bernoulli and homogeneous first-order differential equations

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

1.1 INTRODUCTION

Equations that involve differential coefficients are called differential equations. They play an important role in almost every branch of science and engineering, since most of the problems of the physical world can be formulated as differential equations of one type or another.

For EXAMPLE:

$$y' = f(x) \tag{1.1}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} = 0 \tag{1.2}$$

$$\sqrt{\left(\frac{d^3y}{dx^3}\right)^2 + y = e^x} \tag{1.3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.4}$$

are all differential equations. The first three of these are called ordinary differential equations since they involve the ordinary derivatives of the unknown function *y*. In each case there is only one independent variable *x*. Equation (1.4) contains partial derivatives of the unknown *u* and hence is known as a partial differential equation.

In this course we shall be concerned only with ordinary differential equations and their solutions. By **solution** we mean that we shall try to eliminate the differential coefficients from the equation by means of integration. We shall then be left with a relation between the two variables *x* and *y*. In practice, however, it is not always possible to integrate the functions in question and so, in general, it is customary to regard a differential equation as solved when it can be reduced to expressions of the form $\int f(x)\,dx$ and $\int \phi(y)\,dy$.

In equation (1.1) *y*' is used to denote $\frac{dy}{dx}$ and this notation is frequently used. It can be extended to higher order derivatives so that $y'' = \frac{d^2y}{dx^2}$ and so on.

You will recall from an earlier course that the **order** of a differential equation is the order of the highest derivative occurring in it. Thus, equation (1.1) is of first order, (1.2) is of second order and (1.3) is of third order. A differential equation of order *n* will have *n* arbitrary constants in its **solution**. The **degree** of a differential equation is the degree of the highest derivative contained in the equation after radicals and fractions have been cleared. Thus equation (1.1) is of first degree, (1.2) is of third degree and (1.3) is of second degree.

1.2 FIRST-ORDER DIFFERENTIAL EQUATIONS

A large number of first-order differential equations, including many which are of practical importance, can be classified into various types and the solutions can then be found by established methods. We shall now study some of these.

1.2.1 **Direct integration** – equations depending on *x* only (without a *y*-term)

Equations of the form $\boxed{\dfrac{dy}{dx} = f(x)}$ can be solved by direct integration.

The solution is $\boxed{y = \int f(x)\,dx + c}$

where *c* is the constant of integration and must always be included.

EXAMPLE 1

Solve the equation: $\dfrac{dy}{dx} + \dfrac{1}{x^2} = 0$

SOLUTION

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$dy = -\frac{1}{x^2}\,dx$$

$$y = -\int \frac{1}{x^2}\,dx + c$$

$$y = \frac{1}{x} + c$$

1.2.2 Equations with separable variables

These are equations of the form

$$\boxed{f(x)\,dx = f(y)\,dy}$$

where $f(x)$ is a function of *x* only and $f(y)$ a function of *y* only.

In this case the variables are said to be **separable**. This type of equation can also be solved directly, its solution being:

$$\boxed{\int f(x)\,dx = \int f(y)\,dy + c}$$

If the equation is not given in the above form, it is sometimes possible to rearrange the terms into two groups, each containing only one variable.

EXAMPLE 2

Solve the equation: $x\,dy + y\,dx = 0$

SOLUTION

Divide through by $\frac{1}{xy}$ to separate the variables:

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = C$$

$$\ell n\, y + \ell n\, x = C$$

Since *C* is an arbitrary constant, we can set it equal to $\ell n\, k$. We therefore have:

$$\ell n\, y + \ell n\, x = \ell n\, k$$

$$xy = k$$

The factor $\frac{1}{xy}$ used to multiply through to separate the variables is called an **integrating factor**. Different integrating factors will be needed for different examples.

1.2.3 Exact differential equations

An equation of the form

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

is said to be exact if it has been formed from its primitive by direct differentiation and without cancelling out any common factor.

For example, if the primitive is $x^3 + 3x^2 y + y^3 = c$,

then by differentiation we get $3x^2 dx + 6xy \quad dx + 3x^2 dy + 3y^2 dy = 0$

$$\left(3x^2 + 6xy\right)dx + \left(3x^2 + 3y^2\right)dy = 0$$

which is an exact equation.

Equations that are not exact can often be made so by multiplying through by an integrating factor. In practice, however, this factor is not always easy to find. The next examples will illustrate how we solve exact differential equations. The solution for an exact differential equation will be a function of *x* and *y*.

EXAMPLE 3

Solve the exact differential equation:

$$\left(6x^2 - 10xy + 3y^2\right)dx - \left(5x^2 - 6xy + 3y^2\right)dy = 0$$

SOLUTION

Solve: $\left(6x^2 - 10xy + 3y^2\right)dx - \left(5x^2 - 6xy + 3y^2\right)dy = 0$

$$\left(6x^2 - 10xy + 3y^2\right)dx + \left(-5x^2 + 6xy - 3y^2\right)dy = 0$$

Check if it is exact:

$\dfrac{\partial M}{\partial y} = -10x + 6y$ and $\dfrac{\partial N}{\partial x} = -10x + 6y$

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and it is an exact differential equation.

The signs in the check is very important. If the signs of the partial derivatives are not the same, the check fails and we must look for another method to solve the equation.

We can find the solution, starting with

$$\frac{\partial f}{\partial x} = M(x, y) = 6x^2 - 10xy + 3y^2$$

Direct integration to x gives

$$f(x, y) = 2x^3 - 5x^2 y + 3xy^2 + f(y)$$

$$\frac{\partial f}{\partial y} = N(x, y) = -5x^2 + 6xy - 3y^2$$

Direct integration to y gives

$$f(x, y) = -5x^2 y + 3xy^2 - y^3 + f(x)$$

By comparing the two functions for $f(x, y)$ we find $f(y) = -y^3$, and $f(x) = 2x^3$

Therefore the solution is $2x^3 - 5x^2 y + 3xy^2 - y^3 = C$

EXAMPLE 4

Solve the differential equation $xdy - ydx = x^2 dx$ given the integrating factor $\frac{1}{x^2}$.

SOLUTION

This is an example of a differential equation which is not exact that can be converted into an exact one by using a suitable integrating factor. There is no general rule for finding this factor. In this course the integrating factor will be given as part of the question.

Multiply the given equation with the given integrating factor of $\frac{1}{x^2}$.

$$\frac{1}{x}dy - \frac{y}{x^2}dx = dx$$

Rearrange and group the terms in dx and dy together:

$\left(-\frac{y}{x^2} - 1\right)dx + \frac{1}{x}dy \quad = 0$ which compares to the form of an exact equation

$M(x, y)dx + N(x, y)dy = 0.$

Check if it is exact:

$$\frac{\partial M}{\partial y} = -\frac{1}{x^2} \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -\frac{1}{x^2}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and it is an exact differential equation.

We can find the solution, starting with

$$\frac{\partial f}{\partial x} = M\left(x, y\right) = -\frac{y}{x^2} - 1$$

Direct integration gives $f\left(x, y\right) = \frac{y}{x} - x + f\left(y\right)$

Differentiate this answer for $f\left(x, y\right)$ to find $\frac{\partial f}{\partial y} = \frac{1}{x} + f'\left(y\right)$

By comparing with N in the given equation we find $f'\left(y\right) = 0,$ so that $f(y) = c$

Therefore the solution is $\frac{y}{x} - x = c.$

1.2.4 Linear equations

An equation of the form $\boxed{\frac{dy}{dx} + Py = Q}$ (1.5)

where P and Q are functions of x only, is called a linear differential equation. It is so called because the dependent variable y and its derivatives are of the first degree.

The solution of $$\frac{dy}{dx} + Py = 0 \qquad (1.6)$$

$$\frac{dy}{y} = -Pdx$$

is $$\ell n\, y = -\int Pdx + c$$

$$\ell n\, y = -\int Pdx + \ell n \;\; k$$

$$y = e^{-\int Pdx + \ell n\, k}$$

$$= e^{-\int Pdx} . e^{\ell n\, k}$$

$$\therefore\; y = ke^{-\int Pdx}$$

or $$ye^{\int Pdx} = k$$

On differentiating $ye^{\int P\,dx} = k$, that is $\frac{d}{dx}\left(y^{\;\int Pdx}\right) = \frac{d}{dx}\left(k\right)$,

we obtain $\frac{dy}{dx}e^{\int Pdx} + yPe^{\int Pdx} = 0$ using the product rule.

$$\therefore dy\, e^{\int Pdx} + yPe^{\int Pdx}dx = 0$$

$$\therefore e^{\int Pdx}(dy + Pydx) \qquad = 0$$

This shows that $e^{\int Pdx}$ is an integrating factor of equation (1.6).

We must therefore multiply both sides of our original equation (1.5) by the integrating factor to obtain the equation:

$$e^{\int P dx}\left(\frac{dy}{dx}+Py\right)=Q\,e^{\int P dx}$$

The left-hand side can be replaced by $\frac{d}{dx}\left(ye^{\int P dx}\right)$ using the product rule in reverse, giving us the equation:

$$\frac{d}{dx}\left(ye^{\int P dx}\right)=Qe^{\int P dx}$$

$$\therefore\ ye^{\int P dx}=\int Qe^{\int P dx}dx+c \qquad \text{integrating both sides}$$

and $$y=e^{-\int P dx}\left(\int Qe^{\int P dx}dx+c\right) \qquad (1.7)$$

We can solve first-order linear equations either by substituting for *P*, *Q* and $\int P dx$ in this formula or by multiplying both sides of the given equation by the integrating factor $e^{\int P dx}$. We may omit the constant of integration from $\int P dx$ since this cancels out.

EXAMPLE 5

Solve: $\cos x\frac{dy}{dx}+y\sin x=1$

SOLUTION

We divide through by cos *x* to get:

$$\frac{dy}{dx}+y\tan x=\sec x$$

This is of the form of a linear equation (refer to equation (1.5)) with *P* = tan *x* and *Q* = sec *x*.

Now $\int P dx=\int \tan x\, dx=\ell n\sec x$

so the integrating factor is

$$e^{\int P dx}=e^{\ell n\sec x}=\sec x \text{ (See note.)}$$

We substitute for *P*, *Q* and $e^{\int P dx}$ in (1.7) to obtain:

$$y = \frac{1}{\sec x}\left(\int \sec x.\sec x\,dx + c\right)$$

$$= \frac{1}{\sec x}\left(\tan x + c\right)$$

$$= \sin x + c\cos x$$

NOTE:

Here we have used the fact that $e^{\ell n \sec x} = \sec x$

because if $y = \ell n\, x$ then $x = e^{y}$ and it follows that $x = e^{\ell n\, x}$.

This holds for any function of *x*, for example: $e^{-\ell n\, x} = e^{\ell n\, x^{-1}} = x^{-1} = \frac{1}{x}$ and

$$e^{4\,\ell n\, x} = e^{\ell n\, x^{4}} = x^{4} \text{ and}$$

$$e^{-\ell n \sin x} = \frac{1}{\sin x}$$

1.2.5 Equations reducible to linear form – Bernoulli equations

An equation of the form $\boxed{\frac{dy}{dx} + Py = Qy^{n}}$ (1.8)

is known as a **Bernoulli equation** and is the same as the linear equation (1.5) except for the presence of the factor y^n. The above equation can be reduced to the linear type by eliminating y^n using substitution as follows:

Multiply (1.8) by y^{-n} to obtain: $\frac{1}{y^{n}}\ \frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$ (1.9)

Let $z = y^{1-n}$.

Then $\frac{dz}{dx} = (1-n)\,y^{-n}\,\frac{dy}{dx}$

Multiply equation (1.9) by $(1-n)$: $(1-n)\frac{1}{y^{n}}\ \frac{dy}{dx} + (1-n)\frac{P}{y^{n-1}} = (1-n)Q$

Substitute z and $\frac{dz}{dx}$ in this equation: $\frac{dz}{dx} + (1-n)Pz = (1-n)Q$

which is a linear differential equation.

This can be solved in the usual way for linear equations, remembering to substitute back, that is, put $z = y^{1-n}$ in the final solution.

The general method for solving an equation in the form of a **Bernoulli equation** is illustrated in the following example:

EXAMPLE 6

$$\frac{dy}{dx} + \frac{1}{x}.y = xy^2$$

Step 1: Divide by the power of y on the right-hand side, which is y^2 in this example:

$$y^{-2}\frac{dy}{dx} + \left(\frac{1}{x}\right)y^{-1} = x$$

Step 2: Now put $z = y^{1-n}$, in this case $z = y^{-1}$:

$$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

Step 3: Multiply throughout by -1:

$$-y^{2}\frac{dy}{dx} - \left(\frac{1}{x}\right)y^{-1} = -x$$

$$\frac{dz}{dx} - \left(\frac{1}{x}\right)z = -x$$

Step 4: This is a linear equation with $P = \left(-\frac{1}{x}\right)$ and $Q = (-x)$. Now solve it:

$$\int P dx = -\int \frac{1}{x} dx = -\ell n \quad x$$

$$e^{\int P dx} = e^{-\ell n\, x} = e^{\ell n\, x^{-1}} = x^{-1} = \frac{1}{x}$$

Now from equation (1.7) the solution to the linear equation writing z for y is

$$\begin{aligned} z &= e^{-\int P dx}\left(\int Q e^{\int P dx} dx + c\right) \\ &= x\left(\int (-x)\left(\frac{1}{x}\right) dx + c\right) \\ &= x\left(\int (-1)\, dx + c\right) \\ &= x(-x + c) \\ &= -x^2 + cx \end{aligned}$$

Step 5: Convert back to x and y, substitute $z = y^{-1} = \dfrac{1}{y}$:

$$\frac{1}{y} = -x^2 + cx$$
$$y = (-x^2 + cx)^{-1}$$

EXAMPLE 7

Solve the equation: $\dfrac{dy}{dx} - y = xy^5$

SOLUTION

Divide by y^5: $$y^{-5}\frac{dy}{dx} - y^{-4} = x \qquad (1.10)$$

Let $z = y^{-4}$, then $\dfrac{dz}{dx} = -4y^{-5}\dfrac{dy}{dx}$

Multiply equation 1.10 by –4:

$$-4y^{-5}\frac{dy}{dx} + 4y^{-4} = -4x$$

Now substitute to z : $\dfrac{dz}{dx} + 4z = -4x$

Integrating factor is: $\int P dx = \int 4\, dx = 4x$

$$e^{\int P dx} = e^{4x}$$

Therefore: $z = e^{-4x}\left(-4\int xe^{4x}dx + c\right)$

Use integration by parts to solve the integral:

$$z = e^{-4x}\left(-xe^{4x} + \frac{1}{4}e^{4x} + c\right)$$
$$= -x + \frac{1}{4} + ce^{-4x} \qquad \left(\text{Remember that } e^{-4x}.e^{4x} = e^0 = 1.\right)$$

Substitute back: $y^{-4} = -x + \dfrac{1}{4} + ce^{-4x}$

EXAMPLE 8

Solve: $\dfrac{dy}{dx} + \dfrac{1}{x}.y = x^2y^6$

SOLUTION

Divide by y^6: $y^{-6}\dfrac{dy}{dx}+\dfrac{1}{x}y^{-5}=x^2$ (1.11)

Put $z=y^{-5}$, then $\dfrac{dz}{dx}=-5y^{-6}\dfrac{dy}{dx}$

Multiply equation (1.11) by –5 and substitute: $\dfrac{dz}{dx}-\dfrac{5}{x}.z=-5x^2$

Integrating factor: $\int Pdx=\int -\dfrac{5}{x}dx=-5\ell nx$

$$e^{\int Pdx}=e^{-5\,\ell n\,x}=\frac{1}{x^5}$$

Therefore:
$$z=x^5\left(\int\left(-5x^2\right).\left(\frac{1}{x^5}\right)dx+c\right)$$
$$=x^5\left(\int -\frac{5}{x^3}dx+c\right)$$
$$=x^5\left(\frac{5}{2}x^{-2}+c\right)$$
$$y^{-5}=\frac{5}{2}x^3+cx^5$$

EXAMPLE 9

Solve: $\dfrac{dy}{dx}+\dfrac{1}{x}y=x^5y^5$

SOLUTION

Multiply through by y^{-5} to get:

$$y^{-5}\frac{dy}{dx}+\frac{y^{-4}}{x}=x^5$$

Using the substitution $z=y^{-4}$

we have $\dfrac{dz}{dx}=-4y^{-5}\dfrac{dy}{dx}$

First multiply by –4: $-4y^{-5}\dfrac{dy}{dx}-4\dfrac{y^{-4}}{x}=-4x^5$

The equation to solve then becomes:

$$\frac{dz}{dx}+\left(-\frac{4}{x}\right)z=-4x^5$$

This is a linear equation with $P = \left(-\frac{4}{x}\right)$ and $Q = (-4x^5)$.

$$\int P dx = -\int \frac{4}{x} dx = -4\ell n\, x = \ell n\, x^{-4}$$

We substitute into equation (1.7) writing *z* for *y*:

$$z = e^{\ell n\, x^4}\left(\int \left(-4x^5\right).\left(e^{\ell n\, x^{-4}}\right) dx + c\right)$$

$$= x^4\left(\int -4x^5.x^{-4}\, dx + c\right)$$

$$= x^4\left(\int -4x\, dx + c\right)$$

$$= x^4\left(-2x^2 + c\right)$$

$$\therefore\ y^{-4} = -2x^6 + cx^4$$

1.2.6 Homogeneous equations

These are equations of the form $\boxed{\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}}$ (1.12)

where *P* and *Q* are homogeneous functions of the same degree in both *x* and *y*.

This means that the total degree for each term in the equation is the same.

The expression $\frac{x^2y^2 + y^4}{xy^3}$ is homogeneous, since each term is of the fourth degree.

If we make the substitution $y = v.x$, where *v* is a function of *x*, then:

$$dy = vdx + xdv$$

and equation (1.12) will reduce to the form

$$v + x\frac{dv}{dx} = F(v) \qquad (1.13)$$

where we have put $F(v) = \frac{P(x, vx)}{Q(x, vx)}$

Equation (1.13) is now expressed in terms of *v* and *x*, and can be solved by separating the variables.

EXAMPLE 10

Solve: $\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$

SOLUTION

Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

and the given equation becomes:

$$v + x\frac{dv}{dx} = \frac{x^3 + v^3x^3}{x^3v^2}$$

$$x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v$$

$$x\frac{dv}{dx} = \frac{1+v^3 - v^3}{v^2}$$

$$x\frac{dv}{dx} = \frac{1}{v^2}$$

$$\int v^2 dv = \int \frac{dx}{x}$$

$$\frac{v^3}{3} = \ell n\, x + c$$

Thus $\frac{y^3}{3x^3} = \ell n\ x + c$

Put $k = \ell n\ c$

$$kx = e^{\frac{y^3}{3x^3}}$$

1.3 PARTICULAR SOLUTIONS

Thus far we have considered the constants of integration to be completely arbitrary, that is, we have not assigned any particular values to them. A solution containing a number of arbitrary constants, which is equal to the order of the equation, is called a **general solution**.

However, when certain information about the function is given, for example a value of y for a value of x, we can assign particular values to the constants. This gives us a **particular solution**. These given values of x and y are known as **initial conditions**.

EXAMPLE 11

Solve the differential equation $\frac{dy}{dx} + y\cot x = \cos x$ given that $y = 0$ when $x = 0$.

(The condition can also be written in functional notation as $y(0) = 0$)

SOLUTION

This equation is linear with P = cot x and Q = cos x.

$\int P dx = \int \cot x\, dx = \ell n \sin x$

And $e^{\int P dx} = e^{\ell n \sin x} = \sin x$

We substitute these values in equation (1.7) to get:

$$\begin{aligned} y &= \frac{1}{\sin x}\left(\int \cos x \sin x\, dx + c\right) \\ &= \frac{1}{\sin x}\left(\frac{1}{2}\sin^2 x + c\right) \\ &= \frac{1}{2}\sin x + c\,\text{cosec}x \end{aligned}$$

This is the **general solution** of the given equation. The initial conditions for this problem are y = 0 when x = 0.

Substituting these conditions in the general solution gives:

$$\begin{aligned} &0 = \frac{1}{2}\sin 0 + c\,\text{cosec}0 \\ &0 = \frac{1}{2}(0) + c(0) \\ &\therefore c = 0 \end{aligned}$$

The **particular solution** is obtained by inserting this value for c in the general solution so that

$$y = \frac{1}{2}\sin x.$$

1.4 POST-TEST: MODULE 1 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 90 minutes

1. Solve the following differential equations:

   (a) $\left(1+x^2\right)y\,dx+\left(1-y^2\right)x\,dy=0$ (8)

   (b) $\sin x\cos y\,dx=\sin y\cos x\,dy$ (5)

2. Solve the following exact differential equation:

   $\left(2xy-y^2+2x\right)dx+\left(x^2-2xy+2y\right)dy=0$ (5)

3. Solve the following linear equation:

   $x\dfrac{dy}{dx}-y=x^2$ (7)

4. Solve the following differential equation:

   $x^2+y^2=2xy\dfrac{dy}{dx}$ (14)

5. Solve the following equation:

   $\dfrac{dy}{dx}+y=e^x y^4$ (11)

6. Solve: $-2x\dfrac{dy}{dx}+y=x(x+1)y^3$ (10)

7. Solve: $\dfrac{dy}{dx}+\dfrac{2}{x}.y=3x^2y^{\frac{4}{3}}$ (9)

8. Solve: $\dfrac{dy}{dx}+y\cos x=y^n\sin 2x$ (13)

[82]

You have now completed learning unit 1 and you should be able to

- find the order of a differential equation
- solve first-order differential equations by using
    - direct integration
    - separation of variables
    - an integrating factor
- identify and solve linear, Bernoulli and homogeneous first-order differential equations

We now move on to learning unit 2: Practical applications of first-order differential equations.

MODULE 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

LEARNING UNIT 2

PRACTICAL APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

OUTCOMES

At the end of this learning unit you should be able to solve practical problems in the areas of

- growth and decay
- cooling
- mixtures
- falling bodies with resistance

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

We will now apply our knowledge of first-order differential equations to certain areas of science and technology. We will approach this by doing an example in each area.

2.1 GROWTH AND DECAY

Suppose that we have a function $N(t)$ representing the amount of a substance that is either growing or decaying. $N(t)$ represents the amount that is present at any time *t*. We will assume that $\frac{dN}{dt}$, the rate of change of this substance at any time, t, is proportional to the amount of the substance present at that time.

Then $\frac{dN}{dt} = kN$ or $\frac{dN}{dt} - kN = 0$ where *k* is the constant of proportionality.

EXAMPLE 1

Radioactive material decays at a rate which is proportional to the amount present at a given time. The half-life τ of such material is defined as the time required for half of the original material to decay. If A_0 is the amount of radioactive material present originally and A_t the amount present after a time *t*,

(a) find an equation to describe the radioactive process

(b) solve this equation

(c) calculate τ

SOLUTION

(a) $\frac{dA}{dt} = -kA$ *k* a constant

(b) $\int_{A_0}^{A_t} \frac{dA}{A} = \int_0^t -k\,dt$

$$\int_{A_0}^{A_t} \frac{dA}{A} = \int_0^t -k\,dt$$

$$\ell n\,A\Big|_{A_0}^{A_t} = -kt\Big|_0^t$$

$$\ell n\,A_t - \ell n\,A_0 = -kt + k(0)$$

$$\ell n\left(\frac{A_t}{A_0}\right) = -kt$$

$$\frac{A_t}{A_0} = e^{-kt}$$

$$A_t = A_0 e^{-kt}$$

(c) At $t = \tau$, then $A = \frac{1}{2}A_0$.

Thus $\frac{1}{2}A_0 = A_0 e^{-k\tau}$

$$\frac{1}{2} = e^{-k\tau}$$

$$-k\tau = \ell n\,2^{-1}$$

$$= -\ell n\,2$$

$$\tau = \frac{1}{k}\ell n\,2$$

2.2 COOLING

Newton's law of cooling states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and the surrounding medium. Let T represent the temperature of the body and T_m the temperature of its surrounding medium. The time rate of change of the temperature of the body is $\frac{dT}{dt}$. **Newton's law of cooling** can be written as $\frac{dT}{dt} = -k\left(T - T_m\right)$, where k is a positive constant.

EXAMPLE 2

According to Newton's law the rate of cooling of a substance in moving air is proportional to the difference between the temperature of the substance and that of the air. The temperature of the air is 30 °C and a substance cools from 100 °C to 70 °C in 15 minutes.

(a) Write down the differential equation which illustrates Newton's law of cooling in this case.

(b) Find when the temperature will be 45 °C.

SOLUTION

(a) $\frac{dT}{dt} = -k(T-30)$, where k = constant, T = temperature and t = time.

(b) First find the value of the constant k:

Separate the variables $\frac{dT}{T-30} = -k\,dt$.

Integrate between the limits t = 0 and t = 15, T = 100 and T = 70.

$$\int_0^{15} -k dt = \int_{100}^{70} \frac{dT}{T-30}$$

$$-k\Big|_0^{15} = \ell n\left(T-30\right)\Big|_{100}^{70}$$

$$-15k = \ell n\,40 - \ell n\,70$$

$$15k = \ell n \frac{7}{4}$$

$$\therefore\ k = 0.0373$$

Now integrate between the limits t = 0, T = 100 and t = t, T = 45:

$$\int_{100}^{45} \frac{dT}{T-30} = -0.0373\int_0^t dt$$

$$\ell n 15 - \ell n 70 = -0.0373t$$

$$\therefore\ t = 41.3 \text{ minutes}$$

2.3 MIXTURES

Suppose you have a tank originally holding a quantity V_0 of a solution that contains some quantity *a* of a dissolved chemical. Another solution containing quantity *b* of the same dissolved chemical is added to the tank at the rate of r_1, while at the same time the well-mixed solution is emptied at the rate r_2. We want to find the quantity of the chemical in the tank at any time *t*.

If we let $Q(t)$ represent the amount of the chemical at any time, then $\frac{dQ}{dt}$ represents the difference between the rate at which the chemical enters the tank and the rate at which it leaves. The chemical enters the tank at the rate of br_1 per unit of time. It leaves at the rate of ar_2 per unit of time. The volume in the tank at any time is $V_0 + r_1t - r_2t$, where V_0 represents the volume at $t = 0$. Thus, the concentration of the chemical at any time is $\frac{Q}{V_0 + r_1t - r_2t}$ and so it leaves at a rate of $r_2\left(\frac{Q}{V_0 + r_1t - r_2t}\right)$ per unit of time.

This means that: $\frac{dQ}{dt} = br_1 - r_2\left(\frac{Q}{V_0 + r_1t - r_2t}\right)$

or $\frac{dQ}{dt} + r_2\left(\frac{Q}{V_0 + r_1t - r_2t}\right) = br_1$

EXAMPLE 3

A certain chemical dissolves in water at a rate proportional to the product of the amount undissolved and the difference between the concentration in the saturated solution and the concentration in the actual solution. 50 grams of the substance is dissolved in 100 grams of a saturated solution. If, when 30 grams of the chemical are agitated with 100 grams of water, 10 grams are dissolved in two hours, how much will be dissolved in 5 hours?

SOLUTION

Let *x* denote the number of grams of chemical undissolved after *t* hours. At this time the concentration of the actual solution is $\frac{30 - x}{100}$ and that of the saturated solution is $\frac{50}{100}$.

Then: $\dfrac{dx}{dt} = kx\left(\dfrac{50}{100} - \dfrac{30-x}{100}\right)$

$$= kx\left(\frac{x+20}{100}\right)$$

$$\frac{dx}{x(x+20)} = \frac{kdt}{100}$$

Now resolve the left-hand side into partial fractions and we eventually obtain:

$$\frac{dx}{x} - \frac{dx}{x+20} = \frac{k}{5}dt$$

To find the value of k, integrate between $t = 0$ and $t = 2$, $x = 30$ and $x = 30 - 10 = 20$.

$$\int_{30}^{20}\frac{dx}{x} - \int_{30}^{20}\frac{dx}{x+20} = \frac{k}{5}\int_{0}^{2}dt$$

$$\ell n\, x\Big|_{30}^{20} - \ell n\left(x+20\right)\Big|_{30}^{20} = \frac{k}{5}t\Big|_{0}^{2}$$

$$\left(\ell n20 - \ell n30\right) - \left(\ell n40 - \ell n50\right) = \frac{k}{5}(2)$$

$$\left(\ell n20 + \ell n50\right) - \left(\ell n40 + \ell n30\right) = \frac{2k}{5}$$

$$\therefore\ \ell n\frac{(20)(50)}{(40)(30)} = \frac{2k}{5}$$

$$k = \frac{5}{2}\ell n\frac{5}{6}$$

$$= -0.46$$

Now integrate between $t = 0$ and $t = 5$, $x = 30$ and $x = x$:

$$\int_{30}^{x}\frac{dx}{x} - \int_{30}^{x}\frac{dx}{x+20} = \frac{k}{5}\int_{0}^{5}dt$$

$$\ell n\ \frac{5x}{3(x+20)} = k$$

$$\frac{x}{x+20} = \frac{3}{5}e^{-0,46} = 0.38$$

$$\therefore\ x = 12$$

The amount dissolved after 5 hours is 30 – 12 = 18 grams.

EXAMPLE 4

A 100 ℓ tank is filled with a brine solution which contains 25 kg of salt. Water flows into this tank at a rate of 8 litres per minute and the mixture, which is well stirred, flows away at the same rate. How much salt will there be in the tank after 1 hour?

SOLUTION

Suppose the amount of salt in the tank after *t* minutes is *a* kg. During the interval *dt*, 8 litres *dt* water will flow into the tank and

$$\frac{8a}{100}dt = \frac{2a}{25}dt$$

of salt solution will flow out of the tank. The change in the amount of salt thus is:

$$da = -\frac{2a}{25}dt$$

$$\frac{da}{a} = -\frac{2}{25}dt$$

$$\ell na = -\frac{2}{25}\ t + c$$

$$a\ = c_1 e^{-\frac{2}{25}t}$$

If $t = 0$ then $a = 25$

$\therefore c_1 = 25$ and

$$a = 25e^{-\frac{2}{25}t}$$

If $t = 60$ then

$$a = 25e^{-\frac{120}{25}}$$

$$= 0.2 \text{ kg}$$

2.4 FALLING BODIES WITH RESISTANCE

The force *F* acting on a freely falling body in a vacuum is given by $F = mg$ where *m* is the mass and *g* is the force of gravity. Most of the time when an object falls, it is affected by a resisting medium such as air. The resistance depends on the velocity and the size and shape of the object. At relatively low velocities, the resistance appears to be proportional to the velocity of *kv*, where $k > 0$.

Newton's second law of motion states that the net force acting on a body is equal to the time rate of change of the momentum of the body. For an object of constant mass, this means that $F = m\frac{dv}{dt}$, where F is the net force of the body and v its velocity, both at time t. Because the resistance opposes the velocity, the net force is $F = mg - kv$. Substituting this into the equation $F = m\frac{dv}{dt}$, we obtain: $mg - kv = m\frac{dv}{dt}$

$$\text{or } g = \frac{dv}{dt} + \frac{kv}{m}$$

EXAMPLE 5

A parachutist is falling with a speed of 50 m/s when his parachute opens. If the air resistance is $\frac{Wv^2}{16}N$ where W is the total weight of the man and the parachute, find his speed as a function of the time t after the parachute has opened.

SOLUTION

Net force on system = weight of system – air resistance

$$\frac{W}{g}\frac{dv}{dt} = W - \frac{Wv^2}{16}$$

$$\frac{1}{g}\frac{dv}{dt} = 1 - \frac{v^2}{16}$$

$$\frac{1}{g}\frac{dv}{dt} = \frac{16 - v^2}{16}$$

$$\frac{dv}{v^2 - 16} = -\frac{gdt}{16} \quad (g = 10 \text{ m/s}^2)$$

Now integrate between the limits t = 0 and v = 50, t = t and v = v:

$$\int_{50}^{v} \frac{dv}{v^2 - 16} = -\frac{10}{16}\int_{0}^{t} dt$$

Use partial fractions to find $\frac{1}{v^2-16} = \frac{1}{8(v-4)} - \frac{1}{8(v+4)} = \frac{1}{8}\left(\frac{1}{v-4} - \frac{1}{v+4}\right)$

$$\frac{1}{8}\int_{50}^{v}\left(\frac{1}{v-4} - \frac{1}{v+4}\right) = -\frac{10}{16}\int_{0}^{t} dt$$

$$\int_{50}^{v}\left(\frac{1}{v-4} - \frac{1}{v+4}\right) = -5\int_{0}^{t} dt$$

$$\ell n \frac{v-4}{v+4}\Bigg|_{50}^{v} = -5t \Bigg|_{0}^{t}$$

$$\ell n \frac{v-4}{v+4} - \ell n \frac{46}{54} = -5t$$

$$\ell n \left(\frac{v-4}{v+4} . \frac{54}{46}\right) = -5t$$

$$\frac{27}{23} . \frac{v-4}{v+4} = e^{-5t}$$

We can show that $v = 4\left(\frac{27 + 23e^{-5t}}{27 - 23e^{-5t}}\right)$

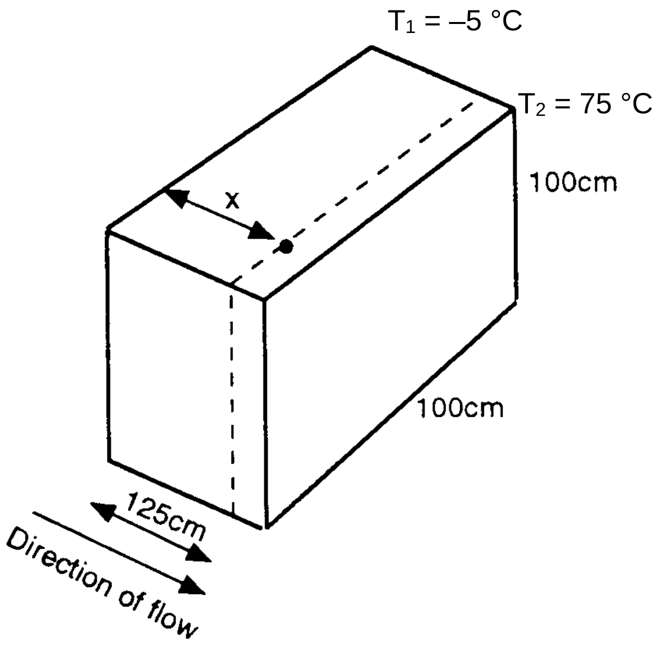
2.5 OTHER APPLICATIONS

EXAMPLE 6

Under certain conditions the constant quantity *J* joules/second of heat flowing through a wall is given by $Q = -kA\frac{dT}{dx}$

where *k* is the conductivity of the material, *A* is the area of the face of the wall perpendicular to the direction of flow and *T* is the temperature 10*x* mm from that face such that *T* increases as *x* increases. Find the number of Joules of heat per hour flowing through one square metre of the wall of a refrigerator room 125 cm thick for which $k = 0.0025$ if the temperature of the inner face is –5 °C and that of the outer face is 75 °C.

SOLUTION



Let x denote the distance of a point within the wall from the outer face:

$$dT = -\frac{Q}{kA}dx$$

Integrate within the limits x = 0 to 125, T_2 = 75 and T_1 = –5:

$$\int_{75}^{-5} dT \quad = -\frac{Q}{kA}\int_{0}^{125} dx$$

$$t\Big|_{75}^{-5} = \left(-\frac{Q}{kA}\right) x\Big|_{0}^{125}$$

$$(-5-75) = \left(-\frac{Q}{kA}\right)(125-0)$$

$$80 = \frac{Q}{kA}(125)$$

$$Q = \frac{80kA}{125}\text{ joule}$$

$$= 0.64\ \ kA\ \ \text{joule}$$

$$= (0.64)(0.0025)(1)\ \ \text{joule}$$

$$= 0.0016\ \ \text{joule}$$

2.6 POST-TEST: MODULE 1 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 30 minutes

1. A body with mass m_a falls from rest in a medium for which the resistance (N) is proportional to the square of the velocity. If the terminal velocity is 50 m/s, find
   (a) the velocity at the end of 2 seconds
   (b) the time required for the velocity to become 30 m/s (14)

2. Tank A initially contains 500 ℓ of a brine solution with a concentration of 0.5 kg per litre and tank B contains 500 ℓ of water. Solution is pumped from tank A to tank B at a rate of 4 ℓ/min and from tank B to tank A at a rate of 8 ℓ/min. The tanks are kept well stirred. Let the number of kilograms of salt in tank A be A_1 after t minutes and B_1 in tank B.
   (a) Write down equations showing, respectively, the concentrations C_1 and C_2 in the tanks after t minutes. (2)
   (b) Write down two differential equations to show how A_1 and B_1 vary with time. (2)
   (c) What is the relation between A_1 and B_1? (1)
   (d) Eliminate B_1 between the two equations in (b) and solve the resulting equation for A_1. (8)
   (e) What is the concentration in tank A after 10 minutes? (1)

[28]

You have completed learning unit 2 and should be able to solve practical problems in the areas of

- growth and decay
- cooling
- mixtures
- falling bodies with resistance

We can now move to learning unit 3 to learn how to solve first-order linear differential equations using numerical methods.

MODULE 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

LEARNING UNIT 3

NUMERICAL METHODS

OUTCOMES

At the end of this learning unit you should be able to

- state the reason for solving first-order differential equations using numerical methods
- obtain a numerical solution to first-order differential equations by using
  - Euler's metod
  - Euler-Cauchy method
  - Runge-Kutta method

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

3.1 INTRODUCTION

A numerical method is a procedure that gives approximate solutions. We can often measure certain points but would like to know the values at other points. With numerical methods we can find these values.

The basic operations addition, subtraction, division and multiplication and the ability to use a formula are used in numerical methods. To get the desired solution, a great deal of calculations is often needed, which makes this ideal for calculator and computer programs.

Always evaluate the answer, as this is an important skill for you as an engineer combined with your common sense.

You have used analytical methods to solve differential equations in the previous modules. Sometimes it is difficult to find the solution using these methods. In these cases we can still get a solution if we know some numerical methods. These methods use repeated calculations so that the use of calculators and computer packages becomes very handy. You must, however, be able to do it by hand first.

3.2 EULER’S METHOD

This method approximates the solution of a differential equation by using tangent lines. Remember that the first derivative at a point gives the slope or gradient of the tangent line at that point. This section must be studied from other sources. Refer to Tutorial Letter 101 and myUnisa for the references to your prescribed book.

3.3 EULER-CAUCHY METHOD

This section must be studied from other sources. Refer to Tutorial Letter 101 and myUnisa for the references to your prescribed book.

3.4 RUNGE-KUTTA METHOD

This section must be studied from other sources. Refer to Tutorial Letter 101 and myUnisa for the references to your prescribed book.

3.5 POST-TEST: MODULE 1 (LEARNING UNIT 3)

(*Solutions on myUnisa under additional resources*)

Time: 45 minutes

1. Use Euler's method to find a numerical solution for the equation $\frac{dy}{dx} = 1 + xy$ given the initial conditions that $x = 0$ when $y = 1$ for the range $x = 0$ to $x = 0.5$ with intervals of 0.1. (10)

2. Obtain a numerical solution for the differential equation $\frac{dy}{dx} = 3 - \frac{y}{x}$ for the range $1.0(0.1)1.5$, given the initial conditions $x = 1$ when $y = 2$. (15)

[25]

You should now be able to

- state the reason for solving first-order differential equations using numerical methods
- obtain a numerical solution to first-order differential equations by using
  - Euler's metod
  - Euler-Cauchy method
  - Runge-Kutta method

We can now move to module 2, learning unit 1 to learn how to solve second-order linear differential equations with constant coefficients.

MODULE 2

SECOND-ORDER DIFFERENTIAL EQUATIONS

LEARNING UNIT 1

SOLVING SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Outcome

At the end of this learning unit you should be able to

- identify and solve the auxiliary equation of a second-order differential equation
- solve second-order differential equations of the form $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=0$

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

1.1 INTRODUCTION

The simplest second-order linear differential equations are those of the form

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q} \qquad (1.1)$$

where Q is equal to zero, a constant or a function of x only, and a, b and c are constants. This type of equation is frequently used to solve practical problems in the field of engineering.

In this unit we shall find the solution of equation (1.1) when $Q = 0$, and in the following unit we shall deal with the cases for $Q = k$, a constant, and $Q = f(x)$.

1.2 SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS AND *Q* = 0

When $Q = 0$, equation (1.1) reduces to $\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}$ (1.2)

which is a homogenous equation. In finding the solution of this equation we shall make use of Theorem 1.1.

THEOREM 1.1

If $y = y_1(x)$ and $y = y_2(x)$ are any two solutions of equation (1.2), then the linear combination $c_1y_1(x)+c_2y_2(x)$ with c_1 and c_2 constants is also a solution of (1.2).

By a linear combination we mean that $c_1y_1(x)+c_2y_2(x)\neq 0$ for $c_1\neq 0$ and $c_2\neq 0$.

This theorem can be extended to the general case for *n*th-order homogeneous linear equations.

We suppose that y_1 and y_2 are two solutions of equation (1.2).

Then $a\frac{d^2y_1}{dx^2}+b\frac{dy_1}{dx}+cy_1=0$

and multiplying by c_1 we get: $c_1a\frac{d^2y_1}{dx^2}+c_1b\frac{dy_1}{dx}+c_1cy_1=0$ (1.3)

Also, $a\dfrac{d^2y_2}{dx^2}+b\dfrac{dy_2}{dx}+cy_2=0$

which, on being multiplied by c_2, becomes:

$$c_2a\frac{d^2y_2}{dx^2}+c_2b\frac{dy_2}{dx}+c_2cy_2=0 \qquad (1.4)$$

Adding equations (1.3) and (1.4) we find that:

$$a\left(c_1\frac{d^2y_1}{dx^2}+c_2\frac{d^2y_2}{dx^2}\right)+b\left(c_1\frac{dy_1}{dx}+c_2\frac{dy_2}{dx}\right)+c\left(c_1y_1+c_2y_2\right)=0$$

But $\left(c_1\dfrac{d^2y_1}{dx^2}+c_2\dfrac{d^2y_2}{dx^2}\right)=\dfrac{d^2}{dx^2}\left(c_1y_1+c_2y_2\right)$

and $\left(c_1\dfrac{dy_1}{dx}+c_2\dfrac{dy_2}{dx}\right)=\dfrac{d}{dx}\left(c_1y_1+c_2y_2\right)$.

Thus $a\dfrac{d^2}{dx^2}\left(c_1y_1+c_2y_2\right)+b\dfrac{d}{dx}\left(c_1y_1+c_2y_2\right)+c\left(c_1y_1+c_2y_2\right)=0$

which is the same as equation (1.2) with y replaced by $\left(c_1y_1+c_2y_2\right)$.

In other words, if y_1 and y_2 are solutions of (1.2), so also is $c_1y_1+c_2y_2$.

Let us now assume a solution of equation (1.2) to be $y=e^{mx}$.

Then $$\frac{dy}{dx}=me^{mx}$$

and $$\frac{d^2y}{dx^2}=m^2e^{mx}.$$

We substitute these values in (1.2) to obtain:

$$am^2e^{mx}+bme^{mx}+ce^{mx}=0$$

Dividing through by e^{mx} gives

$$am^2+bm+c=0$$

which is a quadratic equation in m. As we know, this must have two roots which we shall call m_1 and m_2. It is clear then that $y=e^{m_1x}$ and $y=e^{m_2x}$ are two solutions of the equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

provided that m_1 and m_2 are roots of the equation

$$am^2+bm+c=0$$

which is called the **auxiliary equation**.

By **Theorem 1.1** it follows that

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

is also a solution of the given equation and it is called the **general solution** provided that

$$c_1 e^{m_1 x} + c_2 e^{m_2 x} \neq 0 .$$

The roots m_1 and m_2 may be real or complex, and they may be different ($m_1 \neq m_2$) or equal ($m_1 = m_2$). The method of solution of an equation varies according to the type of its roots.

1.2.1 Roots real and different

In this case the auxiliary equation has two real and different roots m_1 and m_2, and hence $y = e^{m_1 x}$ and $y = e^{m_2 x}$ are two solutions of the given equation. Since m_1 and m_2 are not equal, we can see that $c_1 e^{m_1 x} + c_2 e^{m_2 x} \neq 0$.

(We assume here that the constants $c_1, c_2, ..., c_n$ are not all equal to zero.)

Therefore the linear combination

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad (1.5)$$

is the general solution of the equation. c_1 and c_2 are the **two** arbitrary constants that we would expect to find in the solution of a **second**-order differential equation.

EXAMPLE 1

Find the general solution of $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$.

SOLUTION

First we form the auxiliary equation by writing m^2 for $\dfrac{d^2y}{dx^2}$, m for $\dfrac{dy}{dx}$ and 1 for y.

We have $m^2 + m - 2 = 0$

which we factorise as $(m - 1)(m + 2) = 0$.

The roots are therefore $m = 1$ and $m = -2$.

Substitution of these values in equation (1.5) will give us the general solution

$y = c_1 e^{x} + c_2 e^{-2x}$.

EXAMPLE 2

Solve: $$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = 0$$

SOLUTION

The auxiliary equation is: $m^2 - 3\ \ m + 2 = 0$.

$$\therefore\ (m-1)(m-2) = 0$$

The roots are therefore *m* = 1 and *m* = 2, giving the general solution:

$y = c_1 e^x + c_2 e^{2x}$

1.2.2 Roots real and equal

Consider the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

The auxiliary equation $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

has two equal roots *m* = 2.

We would therefore expect the solution of the differential equation to be:

$$y = c_1 e^{2x} + c_2 e^{2x} = (c_1 + c_2)e^{2x} \qquad (1.6)$$

But $c_1 e^{2x} + c_2 e^{2x}$ is not a linear combination, since for $c_1 = -c_2$ this expression is equal to zero. Also, equation 1.6 contains only one constant $c_1 + c_2 = c_3$, say, and a second-order differential equation must have two constants in its solution. We therefore need another term containing a second constant.

Let us assume that $y = xe^{2x}$ is also a solution.

Then $$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$$

and $$\frac{d^2y}{dx^2} = 2e^{2x} + 2e^{2x} + 4xe^{2x}.$$

We substitute these in the original equation to obtain

$$2e^{2x} + 2e^{2x} + 4xe^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x} = 0$$

so that $y = xe^{2x}$ is indeed a solution, and the general solution is $y = c_1 e^{2x} + c_2 xe^{2x}$ where the right-hand side is obviously a linear combination.

In general, the differential equation (1.2) has the solution $y = c_1 e^{m_1 x} + c_2 xe^{m_2 x}$ when the roots of the auxiliary equation are real and equal.

EXAMPLE 3

Solve: $$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

SOLUTION

We see that the auxiliary equation is $4m^2 + 4m + 1 = 0$

$$\therefore \left(2m+1\right)^2 = 0$$

so that the two equal roots are $m = -\frac{1}{2}$.

Hence the general solution is $y = c_1 e^{-\frac{x}{2}} + c_2 xe^{-\frac{x}{2}}$.

EXAMPLE 4

Solve: $$\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 49y = 0$$

SOLUTION

In this case the auxiliary equation is $m^2 - 14m + 49 = 0$

$$\therefore \left(m-7\right)^2 = 0$$

giving the two equal roots $m = 7$.

The general solution is $y = c_1 e^{7x} + c_2 xe^{7x}$.

1.2.3 Roots complex

When the auxiliary equation cannot be solved by factorising as we have done in the previous examples, we use the quadratic formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1.7)$$

If the quantity within the square root sign is positive, the roots are real and the general solution can be found as before. However, we sometimes have a minus sign within the square root. For example, the quadratic equation

$$m^2 + 2\,m + 2 = 0$$

cannot be solved by direct factorisation, so we use the formula (1.7) which becomes

$$\begin{aligned} m &= \frac{-2 \pm \sqrt{4-8}}{2} \\ &= -1 \pm \frac{\sqrt{-4}}{2} \end{aligned}$$

which in terms of $i = \sqrt{-1}$ can be written as $m = -1 \pm i$.

We recall that the differential equation (1.2) has the general solution

$$y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

when the roots m_1 and m_2 of the associated equation are real and different. For the case of complex roots, $m_1 = a + ib$ and $m_2 = a - ib$, and so the general solution is:

$$y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x} \qquad (1.8)$$

From the theory of complex numbers we have the identities

$$e^{ix} = \cos x + i \sin x$$

and $\quad e^{-ix} = \cos x - i \sin x$

so that $e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}\left(\cos bx + i \sin bx\right)$

and $e^{(a-ib)x} = e^{ax} e^{-ibx} = e^{ax}\left(\cos bx - i \sin bx\right)$.

Substitution of these values in (1.8) gives

$$\begin{aligned} y &= d_1 e^{ax}\left(\cos bx + i \sin bx\right) + d_2 e^{ax}\left(\cos bx - i \sin bx\right) \\ &= e^{ax}\left[\left(d_1 + d_2\right)\cos bx + i\left(d_1 - d_2\right)\sin bx\right] \qquad (1.9) \\ &= e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right) \end{aligned}$$

where $c_1 = d_1 + d_2$ and $c_2 = i\left(d_1 - d_2\right)$.

Thus equation (1.9) is the general solution of the differential equation (1.2) when the auxiliary equation has complex roots.

EXAMPLE 5

Solve the equation $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0$.

SOLUTION

The auxiliary equation $m^2 + 4m + 8 = 0$ cannot be factorised, so we use formula (1.7) to obtain:

$$\begin{aligned} m &= \frac{-4 \pm \sqrt{16-32}}{2} \\ &= -2 \pm \frac{\sqrt{-16}}{2} \\ &= -2 \pm 2i \end{aligned}$$

Here $a = -2$ and $b = 2$, so the general solution (1.9) is:

$$y = e^{-2x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

EXAMPLE 6

Solve: $$8\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

SOLUTION

The auxiliary equation is $8m^2 + 4m + 1 = 0$.

We use the quadratic formula to obtain:

$$\begin{aligned} m &= \frac{-4 \pm \sqrt{16-32}}{16} \\ &= -\frac{1}{4} \pm \frac{\sqrt{-16}}{16} \\ &= -\frac{1}{4} \pm \frac{i}{4} \end{aligned}$$

The general solution is therefore $y = e^{-\frac{x}{4}}\left(c_1 \cos\dfrac{x}{4} + c_2 \sin\dfrac{x}{4}\right)$.

EXAMPLE 7

Solve: $$\frac{d^2y}{dx^2} + 7y = 0$$

SOLUTION

The auxiliary equation is $m^2 + 7 = 0$

giving the roots $m = \pm\sqrt{-7}$

$= \pm\sqrt{7}\,i$

and the solution is $y = c_1 \cos\sqrt{7}x + c_2 \sin\sqrt{7}x$.

EXAMPLE 8

Solve: $$\frac{d^2y}{dx^2} - 9y = 0$$

SOLUTION

The auxiliary equation is $m^2 - 9 = 0$

$\therefore m^2 = 9$

and $m = \pm 3$

giving the general solution $y = c_1e^{3x} + c_2e^{-3x}$.

If we use the hyperbolic identities

$e^{3x} = \cosh 3x + \sinh 3x$

$e^{-3x} = \cosh 3x - \sinh 3x$

then the solution can be written as

$y = c_1(\cosh 3x + \sinh 3x) + c_2(\cosh 3x - \sinh 3x)$

$= (c_1 + c_2)\cosh 3x + (c_1 - c_2)\sinh 3x$

$= d_1\cosh 3x + d_2\sinh 3x$

where we have set $d_1 = c_1 + c_2$

$d_2 = c_1 - c_2$.

In other words, $y_1 = d_1\cosh 3x + d_2\sinh 3x$ is also a general solution of $\frac{d^2y}{dx^2} - 9y = 0$.

1.3 POST-TEST: MODULE 2 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 60 minutes

Solve the following differential equations:

1. $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} - 24y = 0$ (5)

2. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 6y = 0$ (6)

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (5)

4. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$ (5)

5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ (5)

6. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 11y = 0$ (5)

7. $\frac{d^2y}{dx^2} + 4y = 0$ (5)

8. $\frac{d^2y}{dx^2} - 16y = 0$ (4)

[40]

You should now be able to identify and solve the auxiliary equation of a second-order differential equation and solve second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$.

We will now move to learning unit 2 and solve second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q$ with $Q \neq 0$, using the method of undetermined coefficients.

MODULE 2

SECOND-ORDER DIFFERENTIAL EQUATIONS

LEARNING UNIT 2

METHOD OF UNDETERMINED COEFFICIENTS

OUTCOME

At the end of this learning unit you should be able to solve second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q$ when Q =

- a constant
- a function of the first degree in x
- a function of the second degree in x
- an exponential function
- a trigonometric function
- a hyperbolic function
- a sum of functions

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

2.1 SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS AND $Q = f(x)$ OR $Q = k$, A CONSTANT

You will remember from the previous unit that the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

has a general solution of the type

$$y = c_1 y_1(x) + c_2 y_2(x) \qquad (2.1)$$

where the exact form of $y_1(x)$ and $y_2(x)$ depends on the roots of the auxiliary equation, and c_1 and c_2 are constants. Now consider the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q$$

where $Q = f(x)$ or $Q = k$, a constant. If the solution (2.1) is substituted in this equation, the left-hand side (L.H.S.) will be equal to zero. Clearly another term is needed in the solution to make the L.H.S. equal to Q ($Q \neq 0$). To find this term we make use of:

THEOREM 2.1

If $c_1\, y_1(x)$ and $c_2\, y_2(x)$ is the general solution of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ (2.2)

and if $y_0(x)$ is any one particular solution of the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q \qquad (2.3)$$

then the complete general solution of the equation is:

$$y = y_0(x) + c_1\, y_1(x) + c_2\, y_2(x)$$

The part $c_1\, y_1(x) + c_2\, y_2(x)$ of the solution is known as the **complementary function** (C.F.), while $y_0(x)$ is called the **particular integral** (P.I.). The **general solution** is thus the sum of the complementary function and the particular integral.

To solve a linear differential equation of the form (2.3), we first find the general solution of the equation with $Q = 0$ (that is, an equation of the form (2.2)) using the methods studied in learning unit 1 of this module. This gives us the C.F. Next we determine the P.I. by assuming a trial solution of the general form of Q. The method of assuming a trial solution is called the method of undetermined coefficients.

For example, if $Q = 2x^2 + 3x + 6$ we assume $y = Ax^2 + Bx + C$.

This assumed solution is then substituted in the given equation and coefficients of like terms are equated to obtain the values of the constants. We shall now discuss the various forms of Q, showing the solutions obtained in each case.

2.1.1 Q = a constant

When $Q = k$, a constant, we assume $y = A$, a constant.

Q = a constant $\rightarrow$ y = a constant

EXAMPLE 1

Solve: $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4$$

SOLUTION

The auxiliary equation is found from the homogeneous equation:

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1 \quad m = 2$$

The roots $m = 1$, $m = 2$ are real and different, so the C.F. is

$$y = c_1e^x + c_2e^{2x}.$$

To find the P.I., assume the general form $y = A$.

Then: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$

Substituting these values in the given equation we obtain:

$$2A = 4$$
$$A = 2$$

The P.I. is therefore $y = 2$

and the general solution is $y = c_1e^x + c_2e^{2x} + 2$.

2.1.2 Q = a function of the first degree in x

If $Q = k_1 x + k_2$ where k_1 and k_2 are constants, we use the trial solution $y = Ax + B$ and substitute in the given equation. On equating coefficients we are able to find the values of A and B. We can then determine the P.I. and finally the general solution.

$$\boxed{Q = k_1 x + k_2 \;\Rightarrow\; y = Ax + B}$$

EXAMPLE 2

Solve: $$\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = 54x + 18$$

SOLUTION

To find the C.F. solve: $m^2 - 6m + 9 = 0$

$$(m-3)^2 = 0$$

$$m = 3 \quad m = 3$$

The C.F. is $y = c_1 e^{3x} + c_2 x e^{3x}$

Since $f(x) = 54x + 18$, we assume the general form $y = Ax + B$

Then $\frac{dy}{dx} = A$ and $\frac{d^2 y}{dx^2} = 0$

Substitution of these values in the original equation gives:

$$-6A + 9Ax + 9B = 54x + 18$$

$$\therefore 9Ax + (-6A + 9B) = 54x + 18$$

We now equate coefficients of like terms:

x-terms: $9A = 54$

$A = 6$

Constant terms: $-6A + 9B = 18$

$-36 + 9B = 18$

$9B = 54$

$B = 6$

We have found that $A = 6$ and $B = 6$, so (2.4) becomes $y = 6x + 6$ which is the P.I.

The general solution is therefore:

$$y = c_1 e^{3x} + c_2 x e^{3x} + 6x + 6$$

2.1.3 Q = a function of the second degree in x

If $Q = k_1x^2 + k_2x + k_3$, we assume: $y = Ax^2 + Bx + C$ (2.5)

Note that k_2 and/or k_3 can be zero. For example, if $k_2 = 0$, then Q will have the form $Q = k_1x^2 + k_3$.

However, we still use (2.5) as a substitution.

$$\boxed{Q = k_1x^2 + k_2x + k_3 \Rightarrow y = Ax^2 + Bx + c}$$

EXAMPLE 3

Solve: $\dfrac{d^2y}{dx^2} + 25y = 5x^2 + x$

SOLUTION

The auxiliary equation $m^2 + 25 = 0$

has the roots $m = \pm\sqrt{-25}$

$= \pm 5i$

so that the C.F. is $y = c_1 \cos 5x + c_2 \sin 5x$

Let $y = Ax^2 + Bx + C$

Then $\dfrac{dy}{dx} = 2Ax + B$

and $\dfrac{d^2y}{dx^2} = 2A$

These values are substituted in the given equation to get:

$$2A + 25Ax^2 + 25Bx + 25C = 5x^2 + x$$

$$25Ax^2 + 25Bx + (2A + 25C) = 5x^2 + x$$

Equating coefficients of like powers gives:

x^2-terms: $25A = 5$

$A = \dfrac{1}{5}$

x-terms: $25B = 1$

$B = \dfrac{1}{25}$

Constant terms: $2A + 25C = 0$

$$\frac{2}{5} = -25C$$

$$C = -\frac{2}{125}$$

The P.I. is therefore given by

$$y = \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$$

and the general solution is:

$$y = c_1 \cos 5x + c_2 \sin 5x + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$$

2.1.4 *Q* = an exponential function

When $Q = ke^{ax}$, we assume the general form $Q = ke^{ax}$.

$$\boxed{Q = ke^{ax} \Rightarrow y = Ae^{ax}}$$

EXAMPLE 4

Solve: $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = 8e^{4x}$

SOLUTION

The auxiliary equation is:

$$m^2 - 6m + 8 = 0$$
$$(m-2)(m-4) = 0$$
$$m = 2 \quad m = 4$$

The C.F. is thus $y = c_1e^{2x} + c_2e^{4x}$.

Let $y = Ae^{4x}$ be the trial solution.

Then $\dfrac{dy}{dx} = 4Ae^{4x}$

and $\dfrac{d^2y}{dx^2} = 16Ae^{4x}$.

Substitution in the original equation gives $16Ae^{4x} - 24Ae^{4x} + 8Ae^{4x} = 8e^{4x}$

$$8e^{4x} = 0$$

which is clearly not possible.

The usual method of determining the P.I. has failed in this case because the general form of the right-hand side is already included in the C.F. In other words, the term e^{4x} on the right-hand side is also a solution of the homogeneous equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0 .$$

To overcome this difficulty we multiply the assumed general form of the P.I. by x and proceed as usual. Thus in the given example we assume $y = Axe^{4x}$.

Then $\quad \dfrac{dy}{dx} = A(e^{4x} + 4xe^{4x})$

and $\dfrac{d^2y}{dx^2} = A(4e^{4x} + 4e^{4x} + 16xe^{4x})$.

When these values are substituted in the original equation we obtain:

$$8Ae^{4x} + 16Axe^{4x} - 6Ae^{4x} - 24Axe^{4x} + 8Axe^{4x} = 8e^{4x}$$
$$(8A - 6A)e^{4x} = 8e^{4x}$$

We equate the terms in e^{4x} to get:

$$8A - 6A = 8$$
$$A = 4$$

The P.I. is therefore $y = 4xe^{4x}$

giving the general solution $y = c_1e^{2x} + c_2e^{4x} + 4xe^{4x}$

In all cases where the general form of the right-hand side is included in the C.F., remember to multiply the trial solution by x and continue as usual. If this term is also included in the C.F., then multiply again by x.

2.1.5 Q = a trigonometric function

For the trigonometric function $Q = k \sin ax$ or $Q = k \cos ax$ the general form is $y = A\cos ax + B\sin ax$.

$Q = k \sin ax$ or $k \cos ax \Rightarrow y = A \cos ax + B \sin ax$

EXAMPLE 5

Solve: $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4\sin x$$

SOLUTION

We read off the auxiliary equation which is: $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1 \quad m = 1$$

The C.F. is $y = c_1e^x + c_2xe^x$.

Although the R.H.S. is $f(x) = 4\sin x$, it is necessary to assume the full general form

$$y = A\cos x + B\sin x$$

since in finding the differential coefficients, the cosine term will also give rise to sin x.

We have $$\frac{dy}{dx} = -A\sin x + B\cos x$$

and $$\frac{d^2y}{dx^2} = -A\cos x - B\sin x.$$

These values are substituted in the given equation so that:

$$-A\cos x - B\sin x + 2A\sin x - 2B\cos x + A\cos x + B\sin x = 4\sin x$$

$$(-B + 2A + B)\sin x + (-A - 2B + A)\cos x = 4\sin x$$

$$2A\sin x - 2B\cos x = 4\sin x$$

Equating coefficients of like terms gives:

sin terms: $2A = 4$

$A = 2$

cos terms: $-2B = 0$

$B = 0$

The P.I. is therefore $y = 2 \cos x$

and the general solution is $y = c_1e^x + c_2xe^x + 2\cos x$

2.1.6 Q = a hyperbolic function

In the case of the hyperbolic functions

$$Q = k \sinh ax \text{ or } Q = k \cosh ax$$

we assume the trial solution

$$y = A \cosh ax + B \sinh ax.$$

$Q = k \sinh ax$ or $k \cosh ax \Rightarrow y = A \cosh ax + B \sinh ax$

EXAMPLE 6

Solve: $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2\cosh x$$

SOLUTION

The auxiliary equation $m^2 - 4m + 4 = 0$

has the two equal roots $m = 2$ and $m = 2$.

The complementary function is therefore:

$$y = c_1e^{2x} + c_2xe^{2x}$$

To find the P.I., let $y = A\cosh x + B\sinh x$.

Then $\dfrac{dy}{dx} = A\sinh x + B\cosh x$

and $\dfrac{d^2y}{dx^2} = A\cosh x + B\sinh x$

and the given equation becomes:

$$\begin{aligned} A\cosh x + B\sinh x - 4A\sinh x - 4B\cosh x + 4A\cosh x + 4B\sinh x &= 2\cosh x \\ (A - 4B + 4A)\cosh x + (B - 4A + 4B)\sinh x &= 2\cosh x \\ (5A - 4B)\cosh x + (5B - 4A)\sinh x &= 2\cosh x \end{aligned}$$

We equate coefficients of like terms to get:

sinh terms: $5B = 4A$

$$B = \frac{4}{5}A$$

cosh terms: $5A - 4B = 2$

$$5A - \frac{16}{5}A = 2$$

$$\frac{9}{5}A = 2$$

and $$A = \frac{10}{9}$$

$$B = \frac{8}{9}$$

The P.I. is therefore $y = \frac{10}{9}\cosh x + \frac{8}{9}\sinh x$

and the general solution is $y = c_1 e^{2x} + c_2 x e^{2x} + \frac{10}{9}\cosh x + \frac{8}{9}\sinh x$.

2.1.7 *Q* = a sum of functions

If Q is the sum of two or more functions, we assume the sum of the general form of each. If, for example, $Q = x + e^x$

the trial solution is $y = Ax + B + Ce^x$.

$Q = x + e^x \Rightarrow y = Ax + B + Ce^x$

EXAMPLE 7

Solve: $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2 + e^x$$

SOLUTION

The auxiliary equation $m^2 - 3m + 2 = 0$

has the roots $m = 2$ and $m = 1$

and so the C.F. is $y = c_1 e^x + c_2 e^{2x}$.

Note that the term e^x is contained in the C.F., so we assume $y = A + Bxe^x$.

Then $\frac{dy}{dx} = B(e^x + xe^x)$

and $\frac{d^2y}{dx^2} = B(e^x + e^x + xe^x)$.

Thus the given equation becomes:

$$2Be^x + Bxe^x - 3Be^x - 3Bxe^x + 2A + 2Bxe^x = 2 + e^x$$

$$\left(B - 3B + 2B\right)xe^x + (2B - 3B)e^x + 2A = 2 + e^x$$

Equating coefficients of like terms we get:

e^x terms: $\quad -B = 1$

$\quad B = -1$

Constant terms: $\quad 2A = 2$

$\quad A = 1$

The P.I. is therefore $y = 1 - xe^x$

and the general solution is $y = c_1e^x + c_2e^{2x} + 1 - xe^x$.

2.2 PARTICULAR SOLUTIONS

The general solution of a second-order differential equation contains two arbitrary constants c_1 and c_2, say. If we are given extra information that allows us to assign values to c_1 and c_2, we then have a **particular solution**. The values of the constants can be found only from the general solution and not from the complementary function.

EXAMPLE 8

Solve the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 3\sin x$ given that when $x = 0$, $y = -0.9$ and $\dfrac{dy}{dx} = -0.7$.

SOLUTION

The auxiliary equation is $m^2 + 3m + 2 = 0$

$$(m + 1)(m + 2) = 0$$

giving the roots -1 and -2.

The C.F. is therefore $y = c_1e^{-x} + c_2e^{-2x}$.

The R.H.S. is a trigonometric function, so in finding the P.I. we assume $y = A\cos x + B\sin x$

so that $\dfrac{dy}{dx} = -A\sin x + B\cos x$ and $\dfrac{d^2y}{dx^2} = -A\cos x - B\sin x$.

Substitution in the original equation gives:

$$-A\cos x - B\sin x - 3A\sin x + 3B\cos x + 2A\cos x + 2B\sin x = 3\sin x$$

$$(-B - 3A + 2B)\ \sin x + (-A + 3B + 2A)\ \cos x = 3\sin x$$

We equate coefficients of like terms to get:

cos terms: $-A + 3B + 2\text{A} = 0$

$$A = -3B$$

sin terms: $-B - 3A + 2B = 3$

$$-B + 9B + 2B = 3$$

$$B = \frac{3}{10}$$

and $A = -\frac{9}{10}$

The P.I. is $y = -\frac{9}{10}\cos x + \frac{3}{10}\sin x$

and the general solution is: $y = c_1 e^{-x} + c_2 e^{-2x} - \frac{9}{10}\cos x + \frac{3}{10}\sin x$ (2.6)

We can now find the values of c_1 and c_2 using the given information.

At $x = 0$, $y = -0.9$, so we insert these values into (2.6) which becomes:

$$-\frac{9}{10} = c_1 + c_2 - \frac{9}{10}$$

$$c_1 = -c_2$$

To find $\frac{dy}{dx}$ we differentiate (2.6) obtaining:

$$\frac{dy}{dx} = -c_1 e^{-x} - 2c_2 e^{-2x} + \frac{9}{10}\sin x + \frac{3}{10}\cos x$$

We now substitute $\frac{dy}{dx} = -0.7$ and $x = 0$ and $c_1 = -c_2$ in this to get:

$$-\frac{7}{10} = c_2 - 2c_2 + \frac{3}{10}$$

$$-c_2 + 2c_2 = 1$$

$$\therefore\ c_2 = 1 \text{ and } c_1 = -1$$

The particular solution is $y = e^{-2x} - e^{-x} - \frac{9}{10}\cos x + \frac{3}{10}\sin x$.

2.3 POST-TEST: MODULE 2 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 105 minutes

Solve the differential equations.

1. $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 2x - 3$ (14)

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-2x}$ (10)

3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = x - 2e^{-2x}$ (13)

4. $\frac{d^2y}{dx^2} + 4y = 2 - 4x^2$ (13)

5. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 85\sin 3x$

   given that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = -20$ (20)

[70]

You should now be able to solve second-order differential equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = Q$ when Q =

- a constant
- a function of the first degree in x
- a function of the second degree in x
- an exponential function
- a trigonometric function
- a hyperbolic function
- a sum of functions

We can now move to module 3, learning unit 1, an introduction to differential operators.

MODULE 3

DIFFERENTIAL OPERATORS

LEARNING UNIT 1

INTRODUCTION TO DIFFERENTIAL OPERATORS

OUTCOMES

At the end of this learning unit you should be able to apply the

- differential operator D to a function
- inverse differential operator $\frac{1}{D}$ to a function

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

1.1 THE DIFFERENTIAL OPERATOR *D*

The symbol $\frac{dy}{dx}$ means that the operation of differentiation with respect to x is performed on the function y. Thus $\frac{d}{dx}$ is an operator and for convenience we can replace it with the symbol D. In this notation $\frac{dy}{dx}$ is written as Dy. Similarly,

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= D(Dy)\\ &= D^2 y\end{aligned}$$

and in general:

$$\boxed{\frac{d^n y}{dx^n} = D^n y}$$

The usual rules of differentiation still apply, for example:

$$\begin{aligned}D\left\{x^2\right\} &= 2x\\ D\left\{e^{2x}\right\} &= 2e^{2x}\\ D\left\{x\sin 2x\right\} &= \sin 2x + 2x\cos 2x\\ D\left\{\frac{e^x}{x}\right\} &= \frac{e^x(x-1)}{x^2}\\ D^3\left\{5x^4\right\} &= D^2\left\{20x^3\right\}\\ &= D\{60x^2\}\\ &= 120x\end{aligned}$$

The symbol D, when combined with constants, obeys many of the ordinary laws of algebra.

If a and b are constants, then:

1. $\boxed{D(ay) = (Da)y = aDy}$

   For example: $D\left\{10e^{3x}\right\} = 10D\left\{e^{3x}\right\}$

   $= 30e^{3x}$

2. $\boxed{(D+a)\,y = Dy + ay}$

For example: $(D+5)\{\sinh x\} = D\{\sinh x\} + 5\sinh x$

$$= \cosh x + 5\sinh x$$

3. $\boxed{D(u+v) = Du + Dv}$

For example: $D\{4x^2 + 3\cos 3x\} = D\{4x^2\} + D\{3\cos 3x\}$

$$= 8x - 9\sin 3x$$

4. $\boxed{D^m(D^n y) = D^n(D^m y) = D^{m+n} y}$

For example: $D\{D^2 3x^3\} = 18$

$$D^2\{D3x^3\} = 18$$

$$D^3\{3x^3\} = 18$$

5. $\boxed{\begin{aligned}(D+a)(D+b)y &= \left(D^2 + aD + bD + ab\right)y \\ &= \left(D^2 + (a+b)D + ab\right)y\end{aligned}}$

For example: $(D+2)(D-4)\{\sin 2x\}$

$$\begin{aligned}
&= \left(D^2 - 2D - 8\right)\{\sin 2x\} \\
&= D^2\{\sin 2x\} - 2D\{\sin 2x\} - 8\sin 2x \\
&= -4\sin 2x - 4\cos 2x - 8\sin 2x \\
&= -12\sin 2x - 4\cos 2x
\end{aligned}$$

1.2 THE INVERSE OPERATOR $\frac{1}{D}$

Since we have defined the operator D as being one of differentiation, it follows that its inverse $\frac{1}{D}$ will indicate the process of integration with respect to x. However, we make one stipulation, namely that the arbitrary constant of integration be omitted at each stage of integration.

The usual rules of integration apply, so that, for example:

$$\frac{1}{D}\left(3x^2+2x-9\right)=x^3+x^2-9x$$

$$\frac{1}{D^2}\left(e^{2x}+\cos x\right)=\frac{1}{D}\left(\frac{e^{2x}}{2}+\sin x\right)$$

$$=\frac{e^{2x}}{4}-\cos x$$

1.3 IMPORTANT THEOREMS

THEOREM 1.1

$$F(D)\left\{e^{ax}\right\}=e^{ax}F(a)$$

where $F(D)$ is any function of D and a is a constant.

Note that the result is the original expression with D replaced by a.

This applies to any function of D operating on e^{ax}.

For example, let $F(D) = D^2 - 1$.

Then: $\left(D^2-1\right)\left\{e^{ax}\right\}=e^{ax}(a^2-1)$

Check by using algebraic methods: $\left(D^2-1\right)\left\{e^{ax}\right\}=D^2\left\{e^{ax}\right\}-e^{ax}$

$$=D\left\{ae^{ax}\right\}-e^{ax}$$

$$=a^2e^{ax}-e^{ax}$$

$$=e^{ax}(a^2-1)$$

Thus a replaced D in the given function $\left(D^2-1\right)$.

In particular, the theorem holds for $F(D)=\frac{1}{f(D)}$

so that $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$.

For example: $\frac{1}{D^2+4}\left\{e^{-3x}\right\}=e^{-3x}\frac{1}{(-3)^2+4}=\frac{e^{-3x}}{13}$

THEOREM 1.2

$$F(D)\left\{e^{ax}.\,y\right\} = e^{ax}F(D+a)\{y\}$$

where *F(D)* is any function of *D*, *y* a non-exponential function of *x* and *a* a constant.

In general, $D^n(e^{ax}y) = e^{ax}(D+a)^n y$

and in particular, when $F(D) = \dfrac{1}{f(D)}$, we have

$$\frac{1}{f(D)}\left\{e^{ax}.y\right\} = e^{ax}\,\frac{1}{f(D+a)}\{y\}.$$

For example, let *F(D)* = *D* + 2.

Then: $(D+2)\left\{e^{ax}y\right\} = e^{ax}\left(\left(D+a\right)+2\right)y$

Check by algebra: $(D+2)\left\{e^{ax}y\right\} = D\left(e^{ax}y\right) + 2e^{ax}y$

$$= ae^{ax}y + e^{ax}Dy + 2e^{ax}y$$

$$= e^{ax}(a+D+2)y$$

$$= e^{ax}\left(\left(D+a\right)+2\right)y$$

so that (*D* + *a*) replaces *D* in the original function and e^{ax} moves to the left, leaving only *y* to be acted on by D.

THEOREM 1.3

$$F(D^2)\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} = F\left(-\left(a^2\right)\right)\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}$$

Thus when a function of D^2 operates on $\sin ax$ or on $\cos ax$ (or on both), the $\sin ax$ or $\cos ax$ does not change but D^2 is replaced by $\left(-\left(a^2\right)\right)$ everywhere.

Note that this applies only to D^2 and not to *D*.

For example, find: $\dfrac{1}{D^2-5}\ \{\sin 2x - 3\cos x\}$

$$= \frac{1}{D^2 - 5}\{\sin 2x\} + \frac{1}{D^2 - 5}\{-3\cos x\}$$

$$= \frac{1}{-4-5}\{\sin 2x\} + \frac{1}{-1-5}\{-3\cos x\}$$

$$= \frac{\sin 2x}{-9} + \frac{3\cos x}{6}$$

$$= -\frac{\sin 2x}{9} + \frac{\cos x}{2}$$

When the operator form of D contains D terms as well as D^2 terms, we can handle the solution as follows:

For example, find: $\frac{1}{D^2 + 10D + 25}\{4\cos 3x\}$

We can substitute $\left(-a^2\right)$ for D^2 and obtain:

$$\frac{1}{D^2 + 10D + 25}\{4\cos 3x\}$$

$$= \frac{1}{-9 + 10D + 25}\{4\cos 3x\}$$

$$= \frac{1}{10D + 16}\{4\cos 3x\}$$

$$= \frac{1}{2}.\frac{1}{5D + 8}\{4\cos 3x\}$$

Multiply the expression by $\frac{5D-8}{5D-8}$:

$$= \frac{1}{2}.\frac{(5D-8)}{25D^2 - 64}\{4\cos 3x\}$$

$$= \frac{1}{2}.\frac{(5D-8)}{(25)(-9) - 64}\{4\cos 3x\}$$

$$= \frac{1}{2}.\frac{(5D-8)}{-289}\{4\cos 3x\}$$

$$= -\frac{1}{578}(5D-8)\{4\cos 3x\}$$

$$= \frac{60}{578}\sin 3x + \frac{32}{578}\cos 3x$$

$$= \frac{30}{289}\sin 3x + \frac{16}{289}\cos 3x$$

1.4 POST-TEST: MODULE 3 (LEARNING UNIT 1)

(*Solutions on myUnisa under additional resources*)

Time: 35 minutes

Determine:

1. $\left(D^3 - D^2 - D + 1\right)\{5\}$ (3)
2. $\left(2D^2 + 4D + 3\right)\{\sin x\}$ (3)
3. $D\left\{\dfrac{\cos 5x}{x^2+1}\right\}$ (3)
4. $\dfrac{1}{D}\{\cosh 4x\}$ (3)
5. $\dfrac{1}{D^2}\left\{5x^2 + \sin 3x\right\}$ (3)
6. $\left(D^2 - 7D + 3\right)\left\{e^{-2x}\right\}$ (3)
7. $\dfrac{1}{(D-2)(D+4)}\left\{e^{3x}\left(2x^2 - 1\right)\right\}$ (5)
8. $\dfrac{1}{(D+3)(D+1)}\left\{e^{-2x}\left(x^2 - 3\cos 4x\right)\right\}$ (5)

[28]

You should now be able to apply the differential operator D and the inverse differential operator $\dfrac{1}{D}$ to a function.

We will now move to the next unit and learn how to solve differential equations using differential operators.

MODULE 3

DIFFERENTIAL OPERATORS

LEARNING UNIT 2

DIFFERENTIAL EQUATIONS AND D-OPERATOR METHODS

OUTCOME

At the end of this learning unit you should be able to solve second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q$, using D-operator methods when Q =

- a constant
- a polynomial in x
- a trigonometric function
- a hyperbolic function
- a product of functions

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

2.1 USE OF THE D-OPERATOR IN SOLVING SECOND-ORDER DIFFERENTIAL EQUATIONS OF THE FORM $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=Q$ WHEN *Q* = *f*(*x*)

2.1.1 When *Q* is a constant

When *Q* is a constant, we have an equation of the form $f(D)\{y\}=k$

so that the P.I. is given by $y=\dfrac{1}{f(D)}k$.

Since $e^{0x}=e^0=1$, it follows that *k* can be multiplied by this factor without changing its value. We then have $y=\dfrac{1}{f(D)}\left\{ke^{0x}\right\}$

which, by **Theorem 1.1**, page 59, becomes:

$$y=\frac{1}{f(0)}k \qquad\qquad \text{* (In this case } a = 0.)$$

EXAMPLE 1

Solve $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4$$

SOLUTION

Written in *D* notation this is $(D^2+3D+2)y=4$

To find the C.F. we first solve the equation $(D^2+3D+2)y=0$

On factorising we obtain $(D+2)(D+1)y=0$

so that the C.F. is $y=c_1e^{-x}+c_2e^{-2x}$

The P.I. is found by solving $y=\dfrac{1}{D^2+3D+2}.\{4\}$

which, on being multiplied by e^{0x}, becomes $y=\dfrac{1}{D^2+3D+2}.\left\{4e^{0x}\right\}$

We can now apply **Theorem 1.1**, page 59, to obtain $y=\dfrac{1}{0+0+2}.4$

where the function of D has become the same function of a which, in this case, is zero.

Thus the P.I. is $y = 2$

and the general solution is $y = c_1 e^{-x} + c_2 e^{-2x} + 2$

The above method of solution breaks down when $f(0) = 0$, since then we have $y = \frac{1}{0}.k$ which is undefined. We can get around this problem by first factoring $f(D)$ and then applying **Theorem 1.1**, page 59, to only one of the operators.

EXAMPLE 2

Evaluate $$y = \frac{1}{D^2 + 2D}.\{5\}$$

SOLUTION

If we were to introduce the factor e^{0x} and proceed as above, we would get $y = \frac{1}{0}.5$

which is undefined. Instead we factorise $f(D)$ to obtain

$$\begin{aligned} y &= \frac{1}{D(D+2)}.\{5\} \\ &= \frac{1}{D}.\frac{1}{D+2}.\{5e^{0x}\} \end{aligned}$$

where we have multiplied by a factor e^{0x}.

We can now apply **Theorem 1.1**, page 59, to only the second operator, so that:

$$\begin{aligned} y &= \frac{1}{D}\left(\frac{1}{D+2}.\{5e^{0x}\}\right) \\ &= \frac{1}{D}\left(5.\frac{1}{0+2}\right) \\ &= \frac{1}{D}\left(\frac{5}{2}\right) \\ &= \frac{5x}{2} \end{aligned}$$

2.1.2 When *Q* is a polynomial in *x*

When *F(D)* = x^n, the particular integral has the form:

$$y = \frac{1}{f(D)}.\left\{x^n\right\}$$

We expand $\frac{1}{f(D)}$ in a series of ascending powers of *D* by the ordinary methods of algebra. Then, since more than *n* differentiations of x^n gives zero, $\frac{1}{f(D)}.\left\{x^n\right\}$ can be found. This method can be extended to any polynomial in *x*.

EXAMPLE 3

Solve $$\frac{d^3y}{dx^3} + \frac{dy}{dx} = x^3$$

SOLUTION

We write this in terms of D-operators as $\left(D^3 + D\right) y = x^3$

The C.F. is found from the equation:

$$(D^3 + D)y = 0$$
$$D(D^2 + 1)y = 0$$

The roots are $D = 0$ and $D = \pm i$

so that the C.F. is: $y = c_1 e^{0x} + c_2 \cos x + c_3 \sin x$

$$= c_1 + c_2 \cos x + c_3 \sin x$$

To find the P.I. we solve

$$y = \frac{1}{D^3 + D}.\left\{x^3\right\}$$
$$= \frac{1}{D(D^2 + 1)}.\left\{x^3\right\}$$
$$= \frac{1}{D}.\left(1 + D^2\right)^{-1}.\left\{x^3\right\}$$

We can use the binomial theorem to expand ($(1 + D^2)^{-1}$ as:

$$(1 + D^2)^{-1} = 1 - D^2 + D^4 - D^6 + \ldots$$

$$y = \frac{1}{D}\left(1 - D^2 + D^4 - D^6 + ...\right)\left\{x^3\right\}$$
$$= \left(\frac{1}{D} - D + D^3 - D^5 + ...\right)\left\{x^3\right\}$$

But $\frac{1}{D}\left\{x\right\}^3 = \frac{x^4}{4}$

$D\left\{x^3\right\} = 3x^2$

$D^2\left\{x^3\right\} = 6x$

$D^3\left\{x^3\right\} = 6$

$D^4\left\{x^3\right\} = 0$

so that the P.I. is $y = \frac{x^4}{4} - 3x^2 + 6$

and the general solution is $y = c_1 + c_2 \cos x + c_3 \sin x + \frac{x^4}{4} - 3x^2 + 6$

EXAMPLE 4

Solve: $\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = x^2 + 2x$

SOLUTION

The equation may be written as: $\left(D^2 + 3D + 2\right) y = x^2 + 2x$

The homogenous form of this is $\left(D^2 + 3D + 2\right) y = 0$

$\left(D + 2\right)\left(D + 1\right) y = 0$

so that the C.F. is $y = c_1 e^{-x} + c_2 e^{-2x}$.

The P.I. is found by solving $y = \frac{1}{D^2 + 3D + 2}\left(x^2 + 2x\right)$.

We divide 1 by $D^2 + 3D + 2$ by ordinary algebraic long division as follows:

Rearrange divisor: $2 + 3D + D^2$

$$\begin{array}{r|l} & \frac{1}{2}-\frac{3}{4}D+\frac{7}{8}D^2+\text{ higher powers} \\ \hline 2+3\mathrm{D}+\mathrm{D}^2 & \quad 1 \end{array}$$

$$\begin{array}{l} 1+\frac{3}{2}D+\frac{1}{2}D^2 \\ \hline -\frac{3}{2}D-\frac{1}{2}D^2 \\ -\frac{3}{2}D-\frac{9}{4}D^2-\frac{3}{4}D^3 \\ \hline +\frac{7}{4}D^2+\frac{3}{4}D^3 \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

We can stop here because the derivatives of the polynomial in the example becomes zero after the second derivative.

Thus
$$\begin{aligned} y &= \left(\frac{1}{2}-\frac{3}{4}D+\frac{7}{8}D^2+\ldots\right)\left\{x^2+2x\right\} \\ &= \left(\frac{1}{2}-\frac{3}{4}D+\frac{7}{8}D^2+\ldots\right)\left\{x^2\right\}+\left(\frac{1}{2}-\frac{3}{4}D+\frac{7}{8}D^2+\ldots\right)\left\{2x\right\} \\ &= \left[\frac{1}{2}x^2-\frac{3}{4}(2x)+\frac{7}{8}(2)\right]+\left[\frac{1}{2}(2x)-\frac{3}{4}(2)+\frac{7}{8}(0)\right] \\ &= \frac{1}{2}x^2-\frac{3}{2}x+\frac{7}{4}+x-\frac{3}{2} \\ &= \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4} \end{aligned}$$

The general solution is $y=c_1e^{-x}+c_2e^{-2x}+\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}$

Note: Using long division is only applicable when *Q* is a polynomial in *x*.

2.1.3 When *Q* is an exponential function

When $Q=ke^{ax}$, the particular integral has the form:

$$y=\frac{1}{f(D)}\left\{ke^{ax}\right\} \qquad (2.1)$$

We can apply **Theorem 1.1**, page 59, to this provided that $f(a)\neq 0$.

If $f(a)=0$, we are not able to use **Theorem 1.1** since in that case equation (2.1) would have the form $y=\frac{1}{f(D)}\left\{ke^{ax}\right\}=\frac{1}{f(a)}\left\{ke^{ax}\right\}=\frac{1}{0}ke^{ax}$.

However, if we multiply (2.1) by the factor 1, we do not alter its value and it then becomes

$$y = \frac{1}{f(D)}\left\{ke^{ax}.1\right\}$$

on which we can use **Theorem 1.2**, page 60, to obtain:

$$y = ke^{ax}\frac{1}{f(D+a)}.\{1\}$$

If necessary, $f(D + a)$ can then be factorised and **Theorem 1.1**, page 59, applied to only one operator.

EXAMPLE 5

Solve $$\frac{d^2y}{dx^2} + y = 3e^x + 5e^{2x}$$

SOLUTION

The D-operator form of this is $\left(D^2+1\right)y = 3e^x + 5e^{2x}$

and the C.F. is found from the equation $\left(D^2+1\right)y = 0$

The roots are $D = \pm i$

and the C.F. is $y = c_1 \cos x + c_2 \sin x$

To find the P.I., we evaluate

$$\begin{aligned} y &= \frac{1}{D^2+1}\left\{3e^x + 5e^{2x}\right\} \\ &= \frac{1}{D^2+1}\left\{3e^x\right\} + \frac{1}{D^2+1}\left\{5e^{2x}\right\} \end{aligned}$$

We now apply **Theorem 1.1**, page 59, with $a = 1$ in the first term and $a = 2$ in the second:

$$\begin{aligned} y &= (3e^x)\frac{1}{1^2+1} + (5e^{2x})\frac{1}{2^2+1} \\ &= \frac{3}{2}e^x + e^{2x} \end{aligned}$$

The general solution is $y = c_1 \cos x + c_2 \sin x + \frac{3}{2}e^x + e^{2x}$.

EXAMPLE 6

Solve $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$$

SOLUTION

The C.F. is found from: $\left(D^2 - 4D + 4\right) y = 0$

$$\left(D-2\right)^2 y = 0$$

$$D = 2$$

Thus $y = c_1 e^{2x} + c_2 x e^{2x}$

To find the P.I., solve: $$y = \frac{1}{\left(D-2\right)^2}\left\{e^{2x}\right\} \qquad (2.2)$$

On applying **Theorem 1.1**, page 59, to this we obtain

$$y = \frac{1}{0}\left\{e^{2x}\right\}$$

which is undefined. So since *f(a)* = 0, **Theorem 1.1** breaks down.

However, if we multiply equation 2.2 by 1, we do not alter its value and we get:

$$y = \frac{1}{\left(D-2\right)^2}\left\{e^{2x}.1\right\}$$

By **Theorem 1.2**, page 60, this becomes: $y = e^{2x}\dfrac{1}{\left(D+2-2\right)^2}\{1\}$

$$= e^{2x}\frac{1}{D^2}\{1\}$$

$$= e^{2x}\frac{1}{D}.x$$

$$= e^{2x}\frac{x^2}{2}$$

The general solution is therefore $y = c_1 e^{2x} + c_2 x e^{2x} + \dfrac{x^2}{2}e^{2x}$.

2.1.4 When *Q* is a trigonometric function

Remember: $D\{\sin x\} = \cos x$

$D\{\tan x\} = \sec^2 x$

$D^2\{3 \sin x + \cos 4x\}$

$= D(3 \cos x - 4 \sin 4x)$

$= -3 \sin x - 16 \cos 4x$

Furthermore: $(D-3)\{\sin 2x\}$

$= D\{\sin 2x\} - 3\{\sin 2x\}$

$= 2\cos 2x - 3\sin 2x$

and $(D^2 - 7D + 3)\{\sin 3x + 2\cos 3x\}$

$$= D^2\{\sin 3x + 2\cos 3x\} - 7D\{\sin 3x + 2\cos 3x\} + 3\sin 3x + 6\cos 3x$$
$$= D\{3\cos 3x - 6\sin 3x\} - 7\{3\cos 3x - 6\sin 3x\} + 3\sin 3x + 6\cos 3x$$
$$= -9\sin 3x - 18\cos 3x - 21\cos 3x + 42\sin 3x + 3\sin 3x + 6\cos 3x$$
$$= 36\sin 3x - 33\cos 3x$$

The inverse operator $\frac{1}{D}$ indicates integration with respect to x while the arbitrary constant is omitted.

Thus: $\frac{1}{D^2}\{\sin 5x - 2\cos 3x\}$

$$= \frac{1}{D}\left\{\frac{-\cos 5x}{5} - \frac{2\sin 3x}{3}\right\}$$
$$= -\frac{\sin 5x}{25} + \frac{2\cos 3x}{9}$$

Also refer to **Theorem 1.3**, page 60.

EXAMPLE 7

Solve $$\frac{d^2y}{dx^2} + 4y = \sin 3x$$

SOLUTION

The equation $\left(D^2+4\right)\{y\} = 0$

has the roots: $D = \pm\sqrt{-4}$

$D = \pm 2i$

Thus the complementary function is $y = c_1\cos 2x + c_2\sin 2x$.

The particular integral is determined by the equation: $y = \frac{1}{D^2+4}\sin 3x$

$$= \frac{1}{-9+4}\sin 3x$$
$$= \frac{\sin 3x}{-5}$$

The general solution is $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{\sin 3x}{5}$.

EXAMPLE 8

Solve $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 2\sin 3x$

SOLUTION

The complementary function is: $D^2 - 4D + 13 = 0$

$$D = \frac{4 \pm \sqrt{16-52}}{2}$$
$$= 2 \pm 3j$$

Thus: $y = e^{2x}(A\cos 3x + B\sin 3x)$

The particular integral is:

$$\begin{aligned} y &= \frac{1}{\left(D^2 - 4D + 13\right)}\{2\sin 3x\} \\ &= \frac{1}{(-9 - 4D + 13)}\{2\sin 3x\} \\ &= \frac{1}{(4 - 4D)}\{2\sin 3x\} \\ &= \frac{1}{2}\frac{1}{(1-D)}\{\sin 3x\} \\ &= \frac{1}{2}\frac{1+D}{(1-D^2)}\{\sin 3x\} \\ &= \frac{1}{2}\frac{\left(1+D\right)}{(1-(-9))}\{\sin 3x\} \\ &= \frac{1}{20}\left(\sin 3x + 3\cos 3x\right) \end{aligned}$$

Thus the general solution is $y = e^{2x}\{A\cos 3x + B\sin 3x\} + \dfrac{1}{20}\{\sin 3x + 3\cos 3x\}$.

The case when $F(-a^2) = 0$:

We know that $e^{j\theta} = \cos\theta + j\sin\theta$

Therefore cos θ is the real part of $e^{j\theta}$. Let us write it as $R\left\{e^{j\theta}\right\}$. Likewise it follows that sin θ is the imaginary part of $e^{j\theta}$ and therefore we write $I\left\{e^{j\theta}\right\}$.

Let us now do an example where we have to find:

$$y = \frac{1}{D^2 + 9} \left\{\sin 3x\right\}$$

If we use the ordinary method we obtain:

$$y = \frac{1}{0}\sin 3x$$

Therefore we use the following method:

$$e^{j3x} = \cos 3x + j\sin 3x$$
$$I\left\{e^{j3x}\right\} = \sin 3x$$

Now $\quad y = I\,\frac{1}{D^2+9}\{e^{j3x}.1\}$

$$= I \quad e^{3jx}\,\frac{1}{(D+3j)^2+9}\{1\}$$

$$= I \quad e^{j3x}\,\frac{1}{D(D+6j)}\{e^{0x\}}$$

$$= I \quad e^{3jx}.\frac{1}{D}e^{0x}.\frac{1}{0+6j}$$

$$= I \quad e^{3jx}.\frac{1}{D}\left\{\frac{1}{6j}\right\}$$

$$= I \quad e^{3jx}.\frac{x}{6j}$$

$$= \frac{1}{6} \;\; I \;\; \frac{x}{j}(\cos 3x + j\sin 3x)$$

$$= \frac{1}{6} \;\; I \;\; \left(x\sin 3x - jx\cos 3x\right)$$

$$= -\frac{x\cos 3x}{6}$$

2.1.5 When *Q* is a hyperbolic function

For the hyperbolic functions $k\cosh ax$ and $k\sinh ax$ we make use of the exponential forms. In this case we use the identities:

$$k\cosh ax \quad = \quad k\left(\frac{e^{ax}+e^{-ax}}{2}\right)$$

$$k\sinh ax \quad = \quad k\left(\frac{e^{ax}-e^{-ax}}{2}\right)$$

Thus: $\frac{1}{f(D)}\{k\cosh ax\}$

$$= k \; \frac{1}{f(D)}\left\{\frac{e^{ax}+e^{-ax}}{2}\right\}$$

$$= \frac{k}{2}\left(\frac{1}{f(a)}e^{ax}+\frac{1}{f(-a)}e^{-ax}\right)$$

and $\frac{1}{f(D)}\;\{k\sinh ax\}$

$$= k \; \frac{1}{f(D)}\left\{\frac{e^{ax}-e^{-ax}}{2}\right\}$$

$$= \frac{k}{2}\left(\frac{1}{f(a)}e^{ax}-\frac{1}{f(-a)}e^{-ax}\right)$$

EXAMPLE 9

Solve $\quad \frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=\cosh 3x$

SOLUTION

The C.F. is the general solution of the equation: $\left(D^2+4D+4\right)y=0$

$$(D+2)^2\,y=0$$

The C.F. is given by $y=c_1e^{-2x}+c_2xe^{-2x}$.

We find the P.I. by solving $y = \frac{1}{(D+2)^2} \{\cosh 3x\}$

$$= \frac{1}{(D+2)^2}\left(\frac{e^{3x}+e^{-3x}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(D+2)^2} e^{3x} + \frac{1}{(D+2)^2} e^{-3x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(3+2)^2} e^{3x} + \frac{1}{(-3+2)^2} e^{-3x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{25} e^{3x} + e^{-3x}\right)$$

$$= \frac{1}{50} e^{3x} + \frac{1}{2} e^{-3x}$$

The general solution is therefore $y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{50} e^{3x} + \frac{1}{2} e^{-3x}$

For the hyperbolic functions we can also use the following theorem:

THEOREM 2.1

$$F(D^2)\begin{Bmatrix}\sinh ax\\ \cosh ax\end{Bmatrix} = F(a^2)\begin{Bmatrix}\sinh ax\\ \cosh ax\end{Bmatrix}$$

EXAMPLE 10

Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \cosh 3x$

SOLUTION

The C.F.: $(D+2)^2 y = 0$

$$\therefore y = c_1 e^{-2x} + c_2 x e^{-2x}$$

The P.I.: $y = \frac{1}{D^2+4D+4}\cosh 3x$

$$= \frac{1}{4D+13}\cosh 3x$$

$$= \frac{(4D-13)}{16D^2-169}\cosh 3x$$

$$= \frac{(4D-13)}{-25}\cosh 3x$$

$$= \frac{12\sinh 3x - 13\cosh 3x}{-25}$$

The general solution: $y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{12\sinh 3x - 13\cos\ \ 3x}{-25}$

(As an exercise show that this is the same answer as obtained in example 9.)

2.1.6 When *Q* is a product of functions

When Q is the product of two or more functions, the method of evaluating $\frac{1}{f(D)}\{Q\}$ depends, of course, on the types of function in question. For example, if Q has the form $k\,e^{ax} f(x)$, then the particular integral is found by evaluating:

$$y = \frac{1}{f(D)}\{ke^{ax}.f(x)\}$$

On applying **Theorem 1.2**, page 60, we obtain

$$y = ke^{ax}\frac{1}{f(D+a)}\{f(x)\}$$

which is then solved in the usual way.

EXAMPLE 11

Solve $$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = xe^x + e^x \sin x$$

SOLUTION

The equation $D^2 + D + 1 = 0$ cannot be solved by factorising, so we use the quadratic formula to obtain:

$$D = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

The C.F. is therefore $y = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$.

To determine the P.I. we evaluate:

$$y = \frac{1}{D^2 + D + 1}\left\{xe^x + e^x \sin x\right\}$$
$$= \frac{1}{D^2 + D + 1}\left\{e^x (x + \sin x)\right\}$$

By **Theorem 1.2**, page 60, this becomes:

$$y = e^x \frac{1}{(D+1)^2 + (D+1) + 1}\left\{x + \sin x\right\}$$
$$= e^x \frac{1}{D^2 + 2D + 1 + D + 1 + 1}\left\{x + \sin x\right\}$$
$$= e^x \frac{1}{D^2 + 3D + 3}\left\{x + \sin x\right\}$$
$$= e^x \left(\frac{1}{D^2 + 3D + 3}\left\{x\right\} + \frac{1}{D^2 + 3D + 3}\left\{\sin x\right\}\right)$$

For convenience, we shall evaluate each of the two terms in brackets separately.

Let $\qquad A = \frac{1}{D^2 + 3D + 3}\, x$

and $\qquad B = \frac{1}{D^2 + 3D + 3}\left\{\sin x\right\}$

so that $\qquad y = e^x (A + B) \qquad\qquad (*)$

To find A we first divide 1 by $D^2 + 3D + 3$ as follows:

$$\begin{array}{r|l} & \frac{1}{3} - \frac{1}{3}D + \text{higher powers} \\ \hline 3 + 3D + D^2 & 1 \\ & 1 + D + \frac{1}{3}D^2 \\ \hline & -D - \frac{1}{3}D^2 \\ & -D - D^2 - \frac{1}{3}D^3 \\ \hline \end{array}$$

We can stop here because we do not need higher powers than D since

$$D\left\{x\right\} = 1, \; D^2\left\{x\right\} = 0.$$

Thus: $$A = \left(\frac{1}{3} - \frac{1}{3}D\right)\{x\}$$
$$= \frac{1}{3}x - \frac{1}{3}$$

To evaluate B: $$B = \frac{1}{D^2 + 3D + 3}\{\sin x\}$$
$$= \frac{1}{3D + 2}\{\sin x\}$$
$$= \frac{(3D - 2)}{9D^2 - 4}\{\sin x\}$$
$$= -\frac{1}{13}(3\cos x - 2\sin x)$$

We insert the values for A and B in equation (*), so that the P.I. is

$$y = e^x\left[\frac{1}{3}x - \frac{1}{3} - \frac{1}{13}(3\cos x - 2\sin x\right]$$
$$= \frac{e^x}{3}(x - 1) - \frac{e^x}{13}(3\cos x - 2\sin x)$$

and the general solution is

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + \frac{e^x}{3}(x - 1) - \frac{e^x}{13}(3\cos x - 2\sin x)\,.$$

2.2 POST-TEST: MODULE 2 (LEARNING UNIT 2)

(*Solutions on myUnisa under additional resources*)

Time: 120 minutes

Solve the following differential equations using D-operator methods:

1. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 5$ (7)

2. $2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x$ (8)

3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = x^3$ (10)

4. $\left(D^2 - 4D + 4\right)y = x^2$ (6)

5. $\left(D^2 - 2D + 3\right)\left(D^2 - 2D - 3\right)y = x + x^2$ (10)

6. $\left(D-2\right)^3\left(D+1\right)^2\left(D^2 - 4D + 9\right)^2\ y = 0$ (9)

7. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin\ 2x$ (10)

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$ (10)

9. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x \sin x$ (10)

10. $\frac{d^2y}{dx^2} + y = 3\cos x$ (10)

[90]

You should now be able to solve second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q$, using D-operator methods when Q =

- a constant
- a polynomial in x
- a trigonometric function
- a hyperbolic function
- a product of functions

We can now move on to the next learning unit and use D-operator methods to solve simultaneous differential equations.

MODULE 3

DIFFERENTIAL OPERATORS

LEARNING UNIT 3

SIMULTANEOUS DIFFERENTIAL EQUATIONS

OUTCOMES

At the end of this learning unit you should be able to solve simultaneous differential equations using D-operator methods by

- elimination
- determinants and Cramer's rule

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

3.1 INTRODUCTION

We consider equations with more than two variables, but only one of the variables is independent. Furthermore the equations are linear with constant coefficients. An example of such a system of equations is:

$$2\frac{dx}{dt}+\frac{dy}{dt}-6x-y=e^t$$

$$\frac{dx}{dt}\quad +3x+y=0$$

If we write D for $\dfrac{d}{dt}$, we can also express the above equations as follows:

$$2\left(D-3\right)x+\left(D-1\right)y=e^t$$

$$\left(D+3\right)x\quad +y=0$$

We want to solve the system of equations for *x* and *y* where both are functions of *t*.

Second example: $$\frac{dx}{dt}+\frac{dy}{dt}+y=1$$

$$\frac{dx}{dt}-\frac{dz}{dt}+2x+z=1$$

$$\frac{dy}{dt}+\frac{dz}{dt}+y+2z=0$$

or

$$Dx+\left(D+1\right)y=1$$

$$\left(D+2\right)x-\left(D-1\right)z=1$$

$$\left(D+1\right)y+\left(D+2\right)z=0$$

We want to solve this system of equations for *x, y* and *z* where all are functions of *t*.

3.2 BASIC PROCEDURES TO BE FOLLOWED TO OBTAIN A SOLUTION

Consider the following simultaneous equations:

$$\frac{dx}{dt}+\frac{dy}{dt}+x+y=0 \qquad (3.1)$$

$$\frac{dy}{dt}+5x+3y=0 \qquad (3.2)$$

We write the equations in a D-operator form:

$$(D+1)x+(D+1)y=0 \tag{3.3}$$

$$5x \quad +(D+3)y \ =0 \tag{3.4}$$

Now eliminate *x* between these two equations (regard *D* as an algebraic multiplier).

Multiply (3.3) by 5 and (3.4) by (*D* + 1):

$$5(D+1)x \qquad\qquad +5(D+1)y=0$$

$$5(D+1)x \quad +(D+1)(D+3)y \ =0$$

The right-hand side remains zero as we multiply 0 by 5 and by (*D* + 1), respectively.

Now subtract these equations to eliminate *x* and proceed to find *y*:

$$5(D+1)y-(D+1)(D+3)y=0$$

$$\left(-D^2+D+2\right)y=0$$

$$\left(D^2-D-2\right)y=0$$

$$\therefore \quad (D-2)(D+1)=0$$

$$D=2 \quad or \quad D=-1$$

Thus $$y=Ae^{2t}+Be^{-t} \tag{3.5}$$

Substitute (3.5) in (3.4) to obtain *x*:

$$5x+(D+3)\left\{Ae^{2t}+Be^{-t}\right\}=0$$

$$5x+2Ae^{2t}-Be^{-t}+3Ae^{2t}+3Be^{-t} \ =0$$

Thus: $$x=-\frac{1}{5}\left(5Ae^{2t}+2Be^{-t}\right) \tag{3.6}$$

This method is known as elimination.

Let us do another example:

Consider the system:

$$2\frac{dx}{dt}+\frac{dy}{dt}-6x-y=e^t \tag{3.7}$$

$$\frac{dx}{dt}+3x+y=0 \tag{3.8}$$

We can write this system as:

$$2(D-3)x+(D-1)y=e^t \tag{3.9}$$

$$(D+3)x \quad +y=0 \tag{3.10}$$

Eliminate y: Multiply (3.10) by $(D-1)$:

$$2(D-3)x+(D-1)y=e^t$$
$$(D-1)(D+3)x \quad +(D-1)y=0$$

Subtract these equations:

$$\{2(D-3)-(D-1)(D+3)\}x=e^t$$
$$(-D^2-3)x=e^t \tag{3.11}$$
$$(D^2+3)x=-e^t$$

The complementary function of the differential equation (3.11) is:

$$x=A\cos\sqrt{3}t+B\sin\sqrt{3}t$$

For the particular integral:

$$x=\frac{1}{D^2+3}\left(-e^t\right)$$
$$=-\frac{e^t}{4}$$

Thus the general solution is $x=A\cos\sqrt{3}t+B\sin\sqrt{3}t-\frac{e^t}{4}$.

Substitute (as you are used to do) this value of x in (3.10) to obtain:

$$y=-(D+3)\left\{A\cos\sqrt{3}t+B\sin\sqrt{3}t-\frac{e^t}{4}\right\}$$
$$=\sqrt{3}A\sin\sqrt{3}t-\sqrt{3}B\cos\sqrt{3}t-3A\cos\sqrt{3}t-3B\sin\sqrt{3}t+\frac{e^t}{4}+\frac{3e^t}{4}$$
$$=\left(\sqrt{3}A-3B\right)\sin\sqrt{3}t-\left(\sqrt{3}B+3A\right)\cos\sqrt{3}t+e^t$$

3.3 THE USE OF DETERMINANTS TO OBTAIN A SOLUTION

Consider the system:

$$2(D-2)x+(D-1)y=e^t$$
$$(D+3)x \quad +y=0$$

Use Cramer's rule for two variables: $x=\frac{\Delta_x}{\Delta}$ and $y=\frac{\Delta_y}{\Delta}$

Rewrite as: $\Delta.x=\Delta_x$ and $\Delta.y=\Delta_y$

The solution of the above system for *x* expressed in terms of determinants is:

$$\begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} .\ x \ = \ \begin{vmatrix} e^t & D-1 \\ 0 & 1 \end{vmatrix}$$

Thus $\quad \left[2(D-2)-(D-1)(D+3)\right]x = e^t - 0$

$$\left[2D-4-\left(D^2+2D-3\right)\right]x = e^t$$

$$\left(2D-4-D^2-2D+3\right)x = e^t$$

$$-\left(D^2+1\right)x = e^t$$

$$\left(D^2+1\right)x = -e^t$$

Solving for *x* by using the usual methods we obtain

$$x = c_1 \cos t + c_2 \sin t - \frac{1}{2}e^t$$

and the solution of the above system for *y* expressed in terms of determinants is:

$$\begin{vmatrix} 2(D-2) & D-1 \\ D+3 & 1 \end{vmatrix} .y = \begin{vmatrix} 2(D-2) & e^t \\ D+3 & 0 \end{vmatrix}$$

Thus $\quad \left[2(D-2)-(D-1)(D+3)\right]y = -(D+3)e^t - 0$

$$-\left(D^2+1\right)y = -\left(e^t + 3e^t\right)$$

$$\left(D^2+1\right)y = 4e^t$$

Solving for *y* by using the usual methods we obtain:

$$y = c_3 \cos t + c_4 \sin t + 2e^t$$

We can also solve for *y* by substituting the solution for *x* in one of the given equations.

EXAMPLE 1

Solve the following system:

$$\left(D^2-2\right)x - 3y = e^{2t}$$

$$\left(D^2+2\right)y + x = 0$$

Also find a particular solution if $x = y = 1$, $Dx = Dy = 0$ if $t = 0$.

SOLUTION

$$\begin{vmatrix} D^2-2 & -3 \\ 1 & D^2+2 \end{vmatrix} . \, x = \begin{vmatrix} e^{2t} & -3 \\ 0 & D^2+2 \end{vmatrix}$$

Thus $\left(D^4-1\right)x = 6e^{2t}$.

Complementary function: $D^4-1=0$

$$(D-1)(D+1)\left(D^2+1\right)=0$$

Thus $x = Ae^{t} + Be^{-t} + C\cos t + E\sin t$.

Particular function:

$$\begin{aligned} x &= \frac{1}{D^4-1}6.e^{2t} \\ &= \frac{6e^{2t}}{15} \\ &= \frac{2}{5}e^{2t} \end{aligned}$$

The general solution for *x* is thus:

$$x = Ae^{t} + Be^{-t} + C\cos t + E\sin t + \frac{2}{5}e^{2t}$$

Substitute for *x* in the first equation:

$$\begin{aligned} y &= \frac{1}{3}\left\{(D^2-2)x - e^{2t}\right\} \\ &= \frac{1}{3}\left\{\left(D^2-2\right)\left(Ae^{t}+Be^{-t}+C\cos\ t+E\sin\ t+\frac{2}{5}e^{2t}\right)-e^{2t}\right\} \\ &= \frac{1}{3}\left\{Ae^{t}+Be^{-t}-C\cos\ t-E\sin\ t+\frac{8}{5}e^{2t}-2Ae^{t}-2Be^{-t}-2C\cos\ t-2E\sin\ t-\frac{4}{5}e^{2t}-e^{2t}\right\} \\ &= -\frac{1}{3}\left(Ae^{t}+Be^{-t}\right)-\left(C\cos\ t+E\sin\ t\right)-\frac{1}{15}e^{2t} \end{aligned}$$

If $t = 0$, then: $x = A+B+C+\frac{2}{5}=1$

and $Dx = A-B+E+\frac{4}{5}=0$

while $y = -\frac{1}{3}\left(A+B\right)-C-\frac{1}{15}=1$

and $Dy = -\frac{1}{3}\left(A-B\right)-E-\frac{2}{15}=0$

You can now solve for A, B, C and E to obtain:

$$A = \frac{3}{4}, \qquad B = \frac{7}{4}, \qquad C = -\frac{19}{10} \text{ and } \qquad E = \frac{1}{5}$$

Thus the required particular solution is:

$$x = \frac{1}{4}\left(3e^{t} + 7e^{-t}\right) - \frac{1}{10}\left(19 \cos t - 2 \sin t\right) + \frac{2}{5}e^{2t}$$

$$y = -\frac{1}{12}\left(3e^{t} + 7e^{-t}\right) + \frac{1}{10}\left(19 \cos t - 2 \sin t\right) - \frac{1}{15}e^{2t}$$

3.4 POST-TEST: MODULE 3 (LEARNING UNIT 3)

(*Solutions on myUnisa under additional resources*)

Time: 90 minutes

Solve the following simultaneous differential equations:

1. $Dx - (D+1)y = -e^{t}$ (10)

   $x + (D-1)y = e^{2t}$

2. $(D-1)x + (D+3)y = e^{-t} - 1$ (12)

   $(D+2)x + (D+1)y = e^{2t} + t$

3. $\dfrac{dx}{dt} + 2x - 3y = t$ (11)

   $\dfrac{dy}{dx} - 3x + 2y = e^{2t}$

4. $(D-1)x + (D+2)y = 1 + e^{t}$ (21)

   $(D+2)y + (D+1)z = 2 + e^{t}$

   $(D-1)x + (D+1)z = 3 + e^{t}$

[54]

You should now be able to solve simultaneous differential equations using D-operator methods by elimination and by determinants and Cramer's rule.

We can now move to the next learning unit and solve practical problems resulting in differential equations by using D-operator methods.

MODULE 3

DIFFERENTIAL OPERATORS

LEARNING UNIT 4

PRACTICAL APPLICATIONS

OUTCOMES

At the end of this learning unit you should be able to solve problems using D-operator methods relating to

- electric circuits
- vibrating systems
- beams

Refer to Tutorial letter 101 for the reference to the pages you must study from your prescribed book.

CONTENTS **PAGE**

4.1 APPLICATION TO ELECTRIC CIRCUIT PROBLEMS

Differential equations are indispensable in nearly every branch of engineering. In electrical engineering, for instance, second-order linear differential equations are used to describe the flow of electricity in a circuit. A simple electric circuit can consist of the following elements in series with a switch:

- a battery or a generator supplying an electromotive force E (volts)
- a resistor having resistance R ohms (Ω)
- an inductor having inductance L (henrys)
- a capacitor having capacitance C (farads)

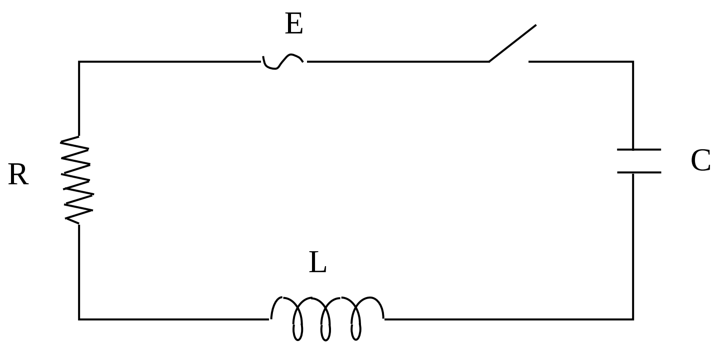


Figure 4.1

When the switch is closed, a charge Q (coulombs) will flow to the capacitor plates. The time rate of flow of charge, that is $\frac{dQ}{dt} = I$, is called the current and is measured in amperes when t is in seconds. Kirchhoff's second law states that the algebraic sum of the voltage drops around a closed loop is equal to zero, so for the circuit shown above we have

$$L\frac{d^2O}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E \qquad (4.1)$$

where

$L\frac{d^2Q}{dt^2} = L\frac{dI}{dt}$	is the voltage drop across the inductor
$R\frac{dQ}{dt} = RI$	the voltage drop across the resistor
$\frac{1}{C}Q$	the voltage drop across the capacitor
E	the voltage drop across the generator

Solving this equation (4.1) will give us the charge $Q(t)$ after time t, from which we can determine the current $I = \frac{dQ}{dt}$.

Equation (4.1) is a second-order linear differential equation with constant coefficients which can be solved using the D-operator methods studied in learning units 2 and 3. Note, however, that in this case Q and t are the dependent and independent variables, respectively, instead of y and x to which we have been accustomed.

The operator $\frac{d}{dt}$, that is the time rate of change of a function, frequently crops up in practical applications and is often denoted by a dot over the variable.

For example

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dy}{dt} = \dot{y}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\dot{x} = \ddot{x}$$

We found it convenient to use the letter D for the differential operator $\frac{d}{dx}$. Since D has no numerical value of its own and merely indicates the process of differentiation of the function to which it is attached, we can also use D in place of $\frac{d}{dt}$ so that $\frac{dx}{dt}$ can be written Dx.

EXAMPLE 1

An inductor of 1 henry, a resistor of 6 Ω and a capacitor of 0.1 F are connected in series with an electromotive force of 20sin2t volts. At $t = 0$ charges on the capacitor and the current in the circuit are zero. Find the charge and current at any time $t > 0$.

SOLUTION

Let Q be the instantaneous charge at time t. Then by Kirchhoff`s laws:

$$\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + \frac{1}{0.1}Q = 20\sin 2t$$

or $$\frac{d^2Q}{dt^2}+6\frac{dQ}{dt}+10Q=20\sin 2t$$

On using the operator $D=\frac{d}{dt}$, this equation becomes:

$$\left(D^2+6D+10\right)Q=20\sin 2t$$

To find the complementary function (C.F.), we solve

$$\left(D^2+6D+10\right)Q=0$$

which has the roots $D=\frac{-6\pm\sqrt{36-40}}{2}$

$$=-3\pm\frac{\sqrt{4i^2}}{2}$$

$$=-3\pm i$$

The C.F. is thus $y=e^{-3t}\left(c_1\cos t+c_2\sin t\right)$

The given equation can be written as

$$Q=\frac{1}{D^2+6D+10}20\sin 2t$$

from which we evaluate the particular integral (P.I.):

$$Q=\frac{1}{D^2+6D+10}20\sin 2t$$

$$=20\ \frac{1}{6D+6}\sin 2t$$

$$=\frac{10}{3}\ \ \frac{1}{D+1}\cdot\frac{(D-1)}{(D-1)}\sin 2t$$

$$=\frac{10}{3}\frac{(D-1)}{D^2-1}\sin 2t$$

$$=\frac{10}{3}\cdot\frac{1}{-5}(D-1)\sin 2t$$

$$=-\frac{2}{3}(2\ \ \cos 2t-\sin 2t)$$

$$=\frac{2}{3}\sin 2t-\frac{4}{3}\cos 2t$$

The general solution is

$$Q=c_1e^{-3t}\cos t+c_2e^{-3t}\sin t-\frac{4}{3}\cos 2t+\frac{2}{3}\sin 2t \qquad (4.2)$$

Since initial conditions are given, we can find the values of the constants c_1 and c_2, and thus a particular solution. Remember that c_1 and c_2 cannot be found from the C.F., but only from the general solution.

At $t = 0$, $Q = 0$ and $I = \dfrac{dQ}{dt} = 0$.

We insert the given values for t and Q into (4.2) to obtain:

$$0 = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 - \frac{4}{3}\cos 0 + \frac{2}{3}\sin 0$$

$$0 = c_1(1)(1) + c_2(1)(0) - \frac{4}{3}(1) + \frac{2}{3}(0)$$

$$0 = c_1 - \frac{4}{3}$$

$$c_1 = \frac{4}{3}$$

Differentiation of (4.2) yields:

$$\frac{dQ}{dt} = -3c_1 e^{-3t}\cos t - c_1 e^{-3t}\sin t - 3c_2 e^{-3t}\sin t + c_2 e^{-3t}\cos t + \frac{8}{3}\sin 2t + \frac{4}{3}\cos 2t$$

We now substitute for t, $\dfrac{dQ}{dt}$ and c_1, so that

$$0 = -3\cdot\frac{4}{3} + c_2 + \frac{4}{3}$$

$$\therefore\ c_2 = \frac{8}{3}$$

$$\therefore\ Q(t) = \frac{4}{3}e^{-3t}\cos t + \frac{8}{3}e^{-3t}\sin t - \frac{4}{3}\cos 2t + \frac{2}{3}\sin 2t$$

and

$$I = \frac{dQ}{dt} = -\frac{4}{3}e^{-3t}\cos t - \frac{28}{3}e^{-3t}\sin t + \frac{8}{3}\sin 2t + \frac{4}{3}\cos 2t$$

As t increases, e^{-3t} tends to zero very rapidly so that for large t the terms containing e^{-3t} are negligible. These are called **transient terms** or the transient part of the solution, while the remaining terms, usually of the form $A\cos at + B\sin at$, are known as the **steady-state terms** or steady-state solution.

4.2 APPLICATION TO VIBRATING SYSTEM PROBLEMS

In this section we shall study the motion of a mass m which is fastened to the lower end of a vertical coiled spring, the upper end of which is rigidly secured.

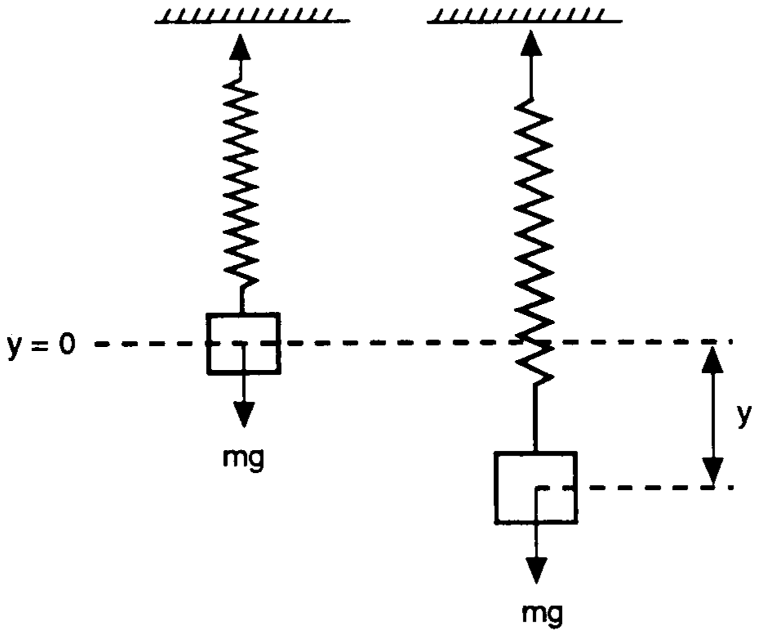


Figure 4.2

We assume that the tension in the spring is proportional to the extension (i.e. we assume that Hooke's law holds), so that if a tension P produces an extension s, we have $P = ks$, where k is known as the **spring constant**.

When a mass m is attached to the spring, then $P = mg$ so that for equilibrium

$$mg = ks$$

and $$k = \frac{mg}{s} \qquad (g = 9.81 \text{ m/s}^2)$$

Suppose now that the mass is given a small vertical displacement and then released. It will undergo vibrations along a vertical line. Let y be the distance of the mass from the equilibrium position at any instant.

By Newton's second law the equation of motion is $m\frac{d^2y}{dt^2} = -ky$ (4.3)

where $-ky$ is the restoring force exerted by the spring on the mass. The acceleration $\frac{d^2y}{dt^2}$ is upwards and hence positive, while the displacement y is negative because it is downwards.

EXAMPLE 2

A spring is stretched 98.1 mm by a body of mass 4 kg. Let a body of mass 16 kg be attached to the spring and released 250 mm below the point of equilibrium with an initial velocity of 1 m/s directed downwards. Describe the motion of the body (g = 9.81 m/s^2).

SOLUTION

From the given information we can find the spring constant k.

We have m = 4 kg and s = 98.1 mm = 0.0981 m so that:

$$\begin{aligned} k &= \frac{4 \times 9.81}{0.0981} \\ &= 400 \end{aligned}$$

We now substitute in equation (4.3) to get:

$$16\frac{d^2y}{dt^2} = -400y$$

or

$$\frac{d^2y}{dt^2} + 25y = 0$$

This second-order linear differential equation is then solved in the usual way.

The D-operator equation $(D^2 + 25)y = 0$ has the roots $\pm 5i$, so that the general solution is

$$y = c_1 \cos 5t + c_2 \sin 5t \text{ .}$$

The initial conditions enable us to evaluate c_1 and c_2.

At $t = 0$, $y = -250 \text{ mm} = -\frac{1}{4} \text{ m}$. We work in m since the mass is in kg and g = 9.81 m/s^2.

$$-\frac{1}{4} = c_1 \cos 0 + c_2 \sin 0$$

$$-\frac{1}{4} = c_1(1) + c_2(0)$$

$$\therefore\ c_1 = -\frac{1}{4}$$

Differentiation of the solution yields $\frac{dy}{dt} = -5c_1 \sin 5t + 5c_2 \cos 5t$.

We are given that at $t = 0$, $\frac{dt}{dt} = -1$. (The initial velocity is directed downwards at 1 m/s.)

$$\frac{dy}{dt} = -5c_1 \sin 5t + 5c_2 \cos 5t$$

$$-1 = -5\left(-\frac{1}{4}\right)\sin 0 + 5c_2 \cos 0$$

$$\therefore -1 = 5 \quad c_2$$

$$c_2 = -\frac{1}{5}$$

The general solution of the system is given by $y = -\frac{1}{4}\cos 5t - \frac{1}{5}\sin 5t$.

The amplitude of the motion is $a = \sqrt{c_1^2 + c_1^2}$

$$= \sqrt{\frac{1}{16} + \frac{1}{25}}$$

$$= \frac{\sqrt{41}}{20}$$

and its period is $T = 2\pi\sqrt{\frac{m}{k}}$

$$= 2\pi\sqrt{\frac{16}{400}}$$

$$= \frac{2\pi}{5}$$

So far the only force which we have taken into account is the restoring force which is proportional to the displacement. When this is the only force acting, the vibrations are said to be free. However, it often happens that there is also a resistance known as a damping force which is proportional to the velocity $\frac{dy}{dt}$ and oppositely directed. When this force is present, equation (4.3) becomes

$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}$$

or $$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad (4.4)$$

where c is called the damping constant. This equation gives the motion of a system performing damped vibrations.

EXAMPLE 3

A system vibrates according to the equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+4y=0$$

where y is the displacement and t the time. Determine y in terms of t given that when $t = 0$, $y = 0$ and $\frac{dy}{dt}=2\sqrt{3}$. Discuss the motion of the system as $t \to \infty$.

SOLUTION

The auxiliary equation

$$m^2+2m+4=0$$

has the roots

$$\begin{aligned} m &= \frac{-2\pm\sqrt{4-16}}{2} \\ &= -1\pm\frac{\sqrt{12i^2}}{2} \\ &= -1\pm\sqrt{3}i \end{aligned}$$

so that the general solution is $y=e^{-t}\left(c_1\cos\sqrt{3}\,t+c_2\sin\sqrt{3}\,t\right)$ (4.5)

The initial conditions tell us that when $t = 0$, $y = 0$ and $\frac{dy}{dt}=2\sqrt{3}$, so we substitute for t and y in (4.5) to find that $c_1=0$.

Differentiation of equation (4.5) gives

$$\frac{dy}{dt}=-c_1e^{-t}\cos\sqrt{3}t-c_1\sqrt{3}e^{-t}\sin\sqrt{3}\,t-c_2e^{-t}\sin\sqrt{3}\,t+c_2\sqrt{3}e^{-t}\cos\sqrt{3}\,t$$

and we substitute for t, c_1 and $\frac{dy}{dt}$ in this to obtain

$$\begin{aligned} 2\sqrt{3} &= c_2\sqrt{3} \\ \therefore\ c_2 &= 2 \end{aligned}$$

So $y=2e^{-t}\sin\sqrt{3}t$ and as $t \to \infty$, $y \to 0$.

Frequently, in addition to the forces already discussed, there may also be a variable external force $f(t)$ acting vertically on the mass. In this case it is said to undergo forced vibrations.

The motion of a body undergoing damped forced vibrations is described by the differential equation:

$$m\frac{d^2y}{dt^2}+c\frac{dt}{dt}+ky= f(t)$$

The most common form of $f(t)$ in practical application is

$f(t)=c_1\cos at\ +\ c_2\sin at$

EXAMPLE 4

The equation of motion of a body performing damped forced vibrations is

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=\cos t$$

Solve this equation given that y = 0.1 and $\frac{dy}{dt}=0$ when t = 0.

SOLUTION

The C.F. is found from the auxiliary equation:

$$m^2+5m+6=0$$

$$(m+3)(m+2)=0$$

Thus $y=c_1e^{-2t}+c_2e^{-3t}$

To find the P.I. we assume the general form

$$\begin{aligned}y&=\frac{1}{D^2+5D+6}\ \cos t\\&=\frac{1}{5D+5}\ \cos\ t\\&=\frac{1}{5}\ \frac{1}{D+1}\cdot\frac{(D-1)}{D-1}\ \cos t\\&=\frac{1}{5}\ \frac{(D-1)}{D^2-1}\ \cos t\\&=-\frac{1}{10}\ (D-1)\ \cos t\\&=-\frac{1}{10}\ (-\sin t-\cos t)\\&=\frac{1}{10}\ (\sin t+\cos t)\end{aligned}$$

and the general solution is $y=c_1e^{-2t}+c_2e^{-3t}+\frac{1}{10}\cos t+\frac{1}{10}\sin t$.

We are now able to find a particular solution from the given initial conditions.

When $t = 0$, $y = 0.1$: $y = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{10}\cos t + \frac{1}{10}\sin t$

$$0.1 = c_1e^0 + c_2e^0 + 0.1\cos 0 + 0.1\sin 0$$
$$0.1 = c_1 + c_2 + 0{,}1$$
$$\therefore\ c_1 = -c_2$$

Substituting for $t = 0$ and $\frac{dy}{dt} = 0$:

$$\frac{dy}{dt} = -2c_1e^{-2t} - 3c_2e^{-3t} - 0.1\sin t + 0.1\cos t$$
$$0 = -2c_1 - 3c_2 + 0{,}1$$
$$\text{but } c_1 = -c_2$$
$$\therefore\ c_1 = -0.1 \text{ and } c_2 = 0.1$$

The required particular solution is therefore

$$y = 0.1(e^{-3t} - e^{-2t} + \cos t + \sin t)$$

4.3 APPLICATION TO BEAM PROBLEMS

The force of gravity causes a beam which is supported at both ends to sag in the middle and a beam rigidly fixed at one end to sag only at the free end. In each case the beam is said to be deflected and the shape into which its axis is bent is called the curve of deflection or elastic curve. An important problem in mechanical engineering is to find the equation of this curve.

Let x be a point on a transversely loaded beam. The bending moment at x is given by

$$M(x) = \frac{EI}{R} \qquad (4.6)$$

where E is the modulus of elasticity for the beam,

I the moment of inertia of a small cross-section at x and

R the radius of curvature of the curve of deflection at x.

Since the curvature is very small, $\frac{1}{R} = \frac{d^2y}{dx^2}$ and equation (4.6) becomes:

$$M = EI\frac{d^2y}{dx^2} \qquad (4.7)$$

This is called the bending equation for a transversely loaded beam. The exact form of M depends on the manner in which the beam is supported. When we know this, we are able to substitute for M in equation (4.7) and hence solve the equation to find the curve of deflection.

EXAMPLE 5

For a simply supported beam, that is one supported at both ends, the bending moment is given by $M = \frac{kx}{2}(x-b)$

(4.8)

where b is the length of the beam and

k its uniform load per unit length.

Find the equation of the curve of deflection and its maximum deflection.

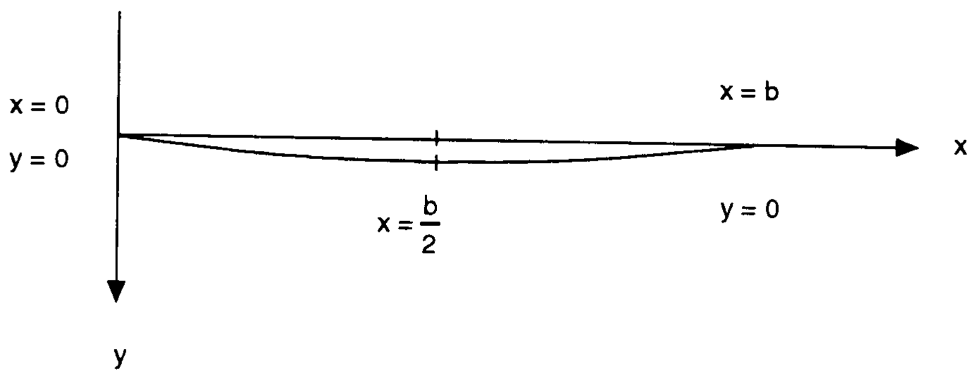


Figure 4.3

SOLUTION

Combining equations (4.7) and (4.8) we get:

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left(\frac{kx^2}{2} - \frac{kbx}{2}\right)$$

We integrate this equation to obtain:

$$\frac{dy}{dx} = \frac{k}{2EI}\left(\frac{x^3}{3} - \frac{bx^2}{2}\right) + c_1 \qquad (4.9)$$

At the midpoint of the beam, that is where $x = \frac{b}{2}$, the curve levels out and the beam is horizontal, so that $\frac{dy}{dx} = 0$. In order to evaluate c_1, we substitute these values in equation (4.9):

$$\begin{aligned} 0 &= \frac{k}{2EI}\left(\frac{b^3}{24} - \frac{b^3}{8}\right) + c_1 \\ &= \frac{k}{2EI}\left(-\frac{b^3}{12}\right) + c_1 \\ c_1 &= \frac{k}{2EI} \cdot \frac{b^3}{12} \end{aligned}$$

Equation (4.9) now becomes:

$$\frac{dy}{dx} = \frac{k}{2EI}\left(\frac{x^3}{3} - \frac{bx^2}{2} + \frac{b^3}{12}\right)$$

Integrating again we get:

$$y = \frac{k}{2EI}\left(\frac{x^4}{12} - \frac{bx^3}{6} + \frac{b^3 x}{12}\right) + c_2$$

At the two ends of the beam, that is at $x = 0$ and $x = b$, we have $y = 0$.
Substitution of these values yields $c_2 = 0$

The equation of the curve of deflection of the given beam therefore is:

$$y = \frac{k}{24EI}\left(x^4 - 2bx^3 + b^3 x\right)$$

The maximum deflection of the beam occurs at the midpoint, that is at $x = \frac{b}{2}$.

At this point:

$$\begin{aligned} y &= \frac{k}{24EI}\left(\frac{b^4}{16} - \frac{b^4}{4} + \frac{b^4}{2}\right) \\ &= \frac{k}{24EI} \cdot \frac{5b^4}{16} \\ &= \frac{5}{384}\frac{kb^4}{EI} \end{aligned}$$

EXAMPLE 6

A beam that has one end built in horizontally and the other end free to move is called a cantilever beam. For this type of beam the bending moment is given by

$$M = \frac{k}{2}(b-x)^2 \qquad (4.10)$$

where b is the length of the beam and
k its uniform load per unit length.

Calculate the maximum deflection of the beam.

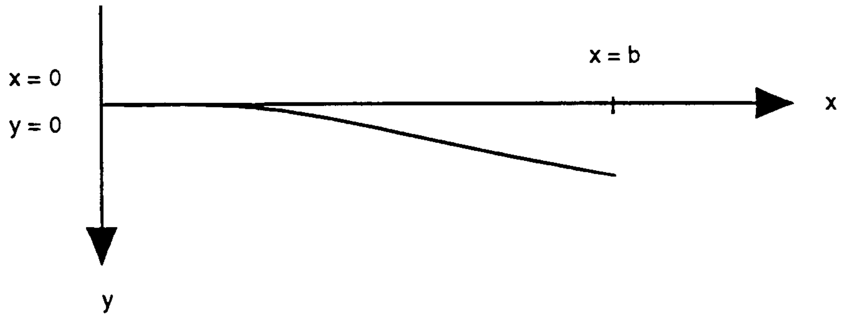


Figure 4.4

SOLUTION

In order to calculate the maximum deflection of the beam, we must first find the equation of the curve of deflection.

We combine equations (4.9) and (4.10) to get:

$$\frac{d^2y}{dx^2} = \frac{k}{2EI}\left(b-x\right)^2$$

$$= \frac{k}{2EI}\left(b^2 - 2bx\right) + x^2$$

Integration yields $$\frac{dy}{dx} = \frac{k}{2EI}\left(b^2x - bx^2 + \frac{x^3}{3}\right) + c_1 \qquad (4.11)$$

At the attached end of the beam, that is at $x = 0$, the beam is horizontal and so $\frac{dy}{dx} = 0$.

On substituting these values in (4.11) we find that $c_1 = 0$.

Integrating again we get $y = \frac{k}{2EI}\left(\frac{b^2x^2}{2} - \frac{bx^3}{3} + \frac{x^4}{12}\right) + c_2$.

At $x = 0$, $y = 0$ and substitution of these gives $c_2 = 0$,

so that the curve of deflection of the beam is given by the equation:

$$y = \frac{k}{24\ EI}\left(x^4 - 4bx^3 + 6b^2x^2\right)$$

The maximum deflection occurs at the free end, that is at $x = b$.

At that point:

$$y = \frac{k}{24\ EI}\left(b^4 - 4b^4 + 6b^4\right)$$

$$= \frac{kb^4}{8\ EI}$$

4.4 POST-TEST: MODULE 3 (LEARNING UNIT 4)

(*Solutions on myUnisa under additional resources*)

Time: 84 minutes

1. An inductor of 1 H, a resistor of 4 Ω and a capacitor of 0.2 F are connected in series with an electromotive force of 8 cos t V. At $t = 0$, the charge on the capacitor and the current in the circuit are zero. Find the current at any time > 0 and discuss the solution as $t \to \infty$. (21)

2. A body of mass 5 kg attached to the end of a spring is released 60 mm above the point of equilibrium with an initial velocity of 480 mm/s directed downwards. Describe the motion of the body and find the amplitude and period of the motion. The differential equation is $m\dfrac{d^2y}{dt^2} = -180y$. (14)

3. A system vibrates according to the equation

$$8\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = \sin t - 2\cos t$$

where y is the displacement and t the time. Determine y in terms of t. (12)

4. A cantilever beam, built in at $x = 0$ and free at $x = b$, carries a concentrated load k_0 at its free end. If the bending moment of the beam is given by

$$M = EI\frac{d^2y}{dx^2}$$

find the deflection of its free end. (9)

[56]

You should now be able to solve problems using D-operator methods relating to electric circuits, vibrating systems and beams.

This is the end of study guide 1. In study guide 2, module 4, learning unit 1, we will start to explore another method of solving differential equations by using Laplace transforms.