

Formal logic 3

**Tutorial letter 202 (second semester)
for
COS3761**

Solutions to Assignment 2



School of Computing

SOLUTION TO ASSIGNMENT 2

QUESTION 1

If you encountered major difficulties with this question, we suggest that you work through the relevant chapters of the prescribed book of Formal Logic 2 again. Also keep in mind that in most cases

- the implication connective \rightarrow needs to be involved when \forall occurs in a sentence, and that
- the conjunction connective \wedge needs to be involved when \exists occurs in a sentence.

Question 1.1

$$\forall x (A(x) \rightarrow \neg R(x))$$

The $\forall x$ operator is needed because the sentence applies to all artists.

An equivalent sentence is

$$\neg \exists x (A(x) \wedge R(x))$$

Question 1.2

$$\forall x ((P(x) \wedge \neg M(b, x)) \rightarrow L(h, x))$$

Question 1.3

$$\exists x (P(x) \wedge L(b, x) \wedge (M(v, x) \vee M(h, x)))$$

Question 1.4

$$\forall x ((P(x) \wedge L(v, x)) \rightarrow M(v, x))$$

We also accepted $\forall x (L(v, x) \rightarrow M(v, x))$

Note how “only if” is translated.

Question 1.5

$$\forall x (P(x) \rightarrow (L(b, x) \leftrightarrow L(v, x)))$$

We also accepted $\forall x (L(b, x) \leftrightarrow L(v, x))$

Note that $\forall x$ (and not $\exists x$) should be used because the statement applies to all paintings.

Question 1.6

$$(A(b) \vee A(h)) \wedge \neg(A(b) \wedge A(h))$$

or

$$(A(b) \vee A(h)) \wedge (\neg A(b) \vee \neg A(h))$$

or

$$(A(b) \wedge \neg A(h)) \vee (\neg A(b) \wedge A(h))$$

QUESTION 2**Question 2.1**

All paintings are painted by some artist.

Question 2.2

There is a rich artist who did not paint any painting.

Question 2.3

Vincent likes all paintings that he has painted.

Question 2.4

Every painting is liked by somebody.

Question 2.5

Somebody likes all paintings he or she did not paint him- or herself.

QUESTION 3

Terms and formulas are defined in your textbook on pages 99 and 100, respectively.

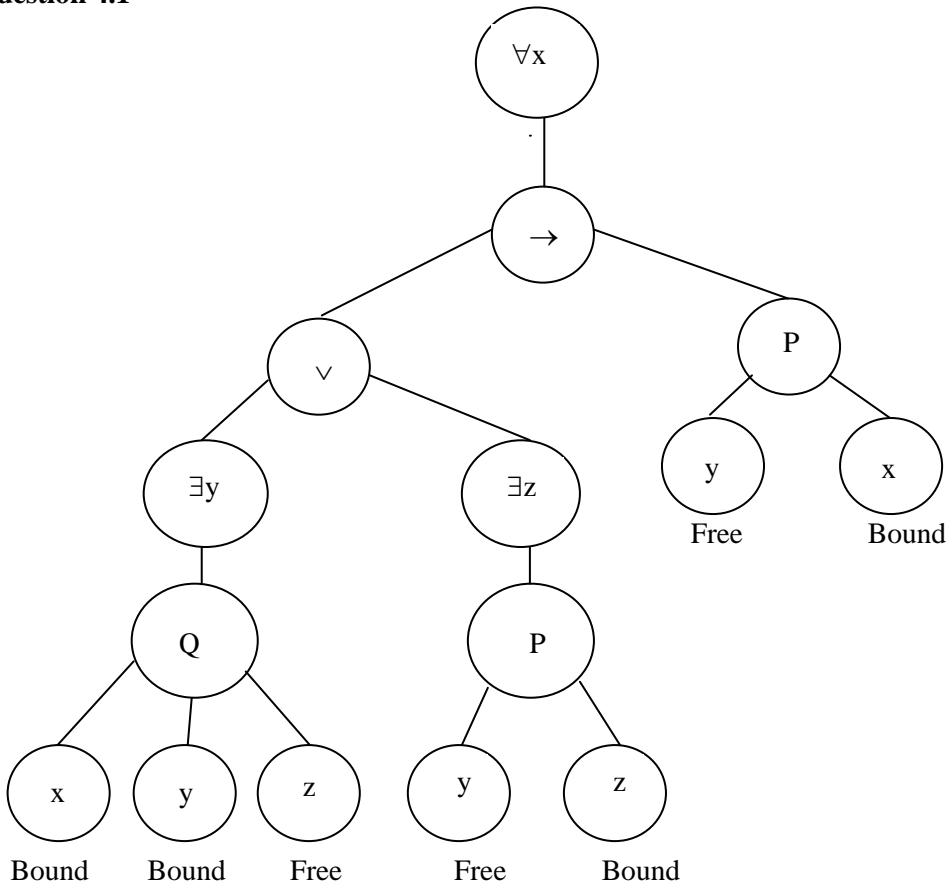
- 3.1 Not a term or a wff. The predicate Q should have two arguments.
- 3.2 Not a term or a wff. The quantifier \exists should be followed by a variable, not a constant.
- 3.3 Not a term or a wff. The term $f(c)$ can only be an argument of a predicate in a wff.
- 3.4 Not a term or a wff. A function (f in this case) cannot have a predicate symbol (Q in this case) as argument.
- 3.5 Not a term or a wff. A predicate symbol (P in this case) cannot have a predicate symbol (Q in this case) as argument.
- 3.6 A term.
- 3.7 A wff.

QUESTION 4

We have

$$\forall x [(\exists y Q(x, y, z) \vee \exists z P(y, z)) \rightarrow P(y, x)]$$

where P is a predicate symbol with two arguments and Q is a predicate symbol with three arguments.

Question 4.1**4.2.1** $\varphi[f(z) / z]$

No problem is created here. Note that there is only one free occurrence of z in φ . The substituted formula is $\forall x [(\exists y Q(x, y, f(z)) \vee \exists z P(y, z)) \rightarrow P(y, x)]$

4.2.2 $\varphi[f(z) / y]$

There are two free occurrences of y in φ . No problem occurs if we replace the second free occurrence of y by $f(z)$, but if we replace the first (free) y in φ by $f(z)$, we are in the scope of $\exists z$ and the argument z of f will become bound by that. We should first rename all the bound occurrences of z in φ before doing the substitution. Suppose we rename z by u . Then the first step would be

$$\forall x [(\exists y Q(x, y, z) \vee \exists u P(y, u)) \rightarrow P(y, x)]$$

and the substitution step would be

$$\forall x [(\exists y Q(x, y, z) \vee \exists u P(f(z), u)) \rightarrow P(f(z), x)]$$

4.2.3 $\varphi[f(x) / y]$

There are two free occurrences of y in φ . Both occur within the scope of $\forall x$. Their substitution by $f(x)$ will therefore create a problem because the argument x of f will become bound by $\forall x$. We should first rename all the bound occurrences of x in φ before doing the substitution. Suppose we rename x by u . Then the first step would be

$$\forall u [(\exists y Q(u, y, z) \vee \exists z P(y, z)) \rightarrow P(y, u)]$$

and the substitution step would be

$$\forall u [(\exists y Q(u, y, z) \vee \exists z P(f(x), z)) \rightarrow P(f(x), u)]$$

QUESTION 5

We give three examples of a model M where both formulas are true. You should check that both sentences are true in each of the given models.

First example:

A : The set of all people in South Africa

S^M : We interpret S as “has a university degree”.

Q^M : We interpret Q as “older than ten years”.

Second example:

$A = \{a, b, c, d, e\}$

$S^M = \{a, b, c\}$

$Q^M = \{a, b, c, d, e\}$

Third example:

A : the set of positive integers

S^M : We interpret S as “is divisible by 10”.

Q^M : We interpret Q as “is divisible by 2”.

QUESTION 6

We give two examples of each. Make sure that you understand why the formula is true or false in every given model.

Model M where the sentence is true:

A : the set of natural numbers

R^M : We interpret $R(x, y)$ as “ x is greater than y ”.

Another model M where the sentence is true:

A : all the pupils in a certain school

R^M : We interpret $R(x, y)$ as “ x and y are in the same class”.

Model M where the sentence is false:

A : the set of natural numbers

R^M : We interpret $R(x, y)$ as “ x is equal to two times y ”.

Another model M where the sentence is false:

A : all the inhabitants of a certain town

R^M : We interpret $R(x, y)$ as “ x is the biological mother of y ”.

QUESTION 7

We want to find out whether the sentence

$$\forall x \exists y (R(x, y) \wedge R(y, y))$$

is true in the model M :

$$A = \{a, b, c, d\}$$

$$R^M = \{(a, a), (b, a), (c, a), (d, b), (b, b)\}$$

We have to investigate every element x of A in order to see whether it is true that there exists an element y such that the tuples (x, y) and (y, y) are both in the set R^M :

Element a : Yes. The tuple (a, a) is in the set, making the sentence true.

Element b : Yes. The tuples (b, a) and (a, a) are in the set, making the sentence true.

Element c : Yes. The tuples (c, a) and (a, a) are in the set, making the sentence true.

Element d : Yes. The tuples (d, b) and (b, b) are in the set, making the sentence true.

This means that the given model *does* satisfy the sentence.

QUESTION 8

We have to show that the validity of the given sequents cannot be proved by finding for each of them a model where all formulas to the left of \vdash evaluate to T but the formula to the right of \vdash evaluates to F. We give two examples in each case. Make sure that you understand why these models show that the sequents cannot be proved.

Question 8.1

Example 1: Let the universe be the integers and interpret $S(x, y)$ as “ y is equal to 10 times x ”.

Example 2: Let the universe be all people in South Africa and interpret $S(x, y)$ as “ y is the mother of x ”.

Question 8.2

Example 1: Let the universe be the integers and interpret $R(x)$ as “ x is divisible by 5” and $Q(x)$ as “ x is divisible by 6”.

Example 2: Let the universe be all people in South Africa and interpret $R(x)$ as “ x plays netball” and $Q(x)$ as “ x plays cricket”.

QUESTION 9

If you encountered major difficulties with this question, we suggest that you work through the relevant chapters of the prescribed book of Formal Logic 2 again. You could also use the Fitch software to help you. Note that some of the rules are named differently (the \neg elimination rule was called the \perp introduction rule in Formal logic 2) and that premises should be explicitly indicated here, but the application of the rules is exactly the same, of course.

Question 9.1

In line 2 we open a subproof with the assumption of the negation of the goal – the negation of $\neg(\exists x (P(x) \wedge Q(x)))$ is $\exists x (P(x) \wedge Q(x))$ – and then derive the contradiction as the last sentence of the subproof (line 9). The \neg introduction rule is then cited in line 10. You have come across this technique in Formal logic 2 already, namely assume the negation of the right hand side and derive \perp .

Many students have difficulty with the application of the $\exists x$ elimination rule. In many cases when the premise (or, as in this case in line 2, an assumption) has the form $\exists x \phi$, we need to use the $\exists x$ elimination rule. This means that we need to introduce a fresh variable and then assume ϕ with x substituted by the fresh variable. This should be the first line of a (new) subproof. Below this is done in line 3. The $\exists x$ elimination rule is then cited when the specific subproof is exited – here in line 9.

The \neg elimination rule (here cited in line 8) was called the \perp introduction rule in Formal logic 2.

1	$\forall x (P(x) \rightarrow \neg Q(x))$	premise	
2	$\exists x (P(x) \wedge Q(x))$	assumption	
3	x_0 $P(x_0) \wedge Q(x_0)$	assumption	[x_0/x]
4	$P(x_0) \rightarrow \neg Q(x_0)$	$\forall x e$ 1	
5	$P(x_0)$	$\wedge e_1$ 3	
6	$\neg Q(x_0)$	$\rightarrow e$ 4, 5	
7	$Q(x_0)$	$\wedge e_2$ 3	
8	\perp	$\neg e$ 6, 7	
9	\perp	$\exists x e$ 2, 3 – 8	
10	$\neg(\exists x (P(x) \wedge Q(x)))$	$\neg i$ 2 – 9	

Question 9.2

In this question both the $\forall x$ elimination rule and the $\forall x$ introduction rule are used. Students normally find the $\forall x e$ rule easy to use – here it is cited in line 3. Note how the $\forall x i$ rule is used: x_0 should be chosen at the start of a (new) subproof and the rule should then be cited outside the subproof – here lines 2 and 7, respectively.

The goal involves \rightarrow . Thus we start a subproof with the assumption of the left hand side (line 4) and end on the right hand side (line 5) and then cite the $\rightarrow i$ rule outside the subproof (line 6).

1	$\forall x (P(x) \wedge Q(x))$	premise	
2	x_0		
3	$P(x_0) \wedge Q(x_0)$	$\forall x e$ 1	
4	$P(x_0)$	assumption	
5	$Q(x_0)$	$\wedge e_2$ 3	
6	$P(x_0) \rightarrow Q(x_0)$	$\rightarrow i$ 4 – 5	
7	$\forall x (P(x) \rightarrow Q(x))$	$\forall x i$ 2 – 6	

Question 9.3

Because the goal involves the implication symbol \rightarrow in this case, we assume the left hand side of the goal at the start of a subproof (line 2), derive the right hand side as the last line of the subproof (line 9) and then use the \rightarrow introduction rule (line 10).

Inside the proof the $\forall x$ elimination rule, the $\forall x$ introduction rule and the $\exists y$ elimination rule are used. Make sure that you understand the application of all the rules: whether they should be outside or inside a subproof starting with a fresh variable, which previous lines should be cited, etc. We use two nested subproofs in order to introduce two free variables so that we may first use the $\forall x$ and $\forall y$ elimination rules inside the subproofs and then the $\forall x$ introduction rule and $\exists y$ elimination outside the two subproofs, respectively.

1	$\forall x \forall y (Q(y) \rightarrow F(x))$	premise	
2	$\exists y Q(y)$	assumption	
3	$y_0 \quad Q(y_0)$	assumption	[y_0/y]
4	x_0		
5	$\forall y (Q(y) \rightarrow F(x_0))$	$\forall x e 1$	
6	$Q(y_0) \rightarrow F(x_0)$	$\forall y e 5$	
7	$F(x_0)$	$\rightarrow e 3, 6$	
8	$\forall x F(x)$	$\forall x i 4 - 7$	
9	$\forall x F(x)$	$\exists y e 2, 3 - 8$	
10	$\exists y Q(y) \rightarrow \forall x F(x)$	$\rightarrow i 2 - 9$	

Question 9.4

One of the premises involves the \vee connective. After the $\forall x e$ rule has been applied to that premise, we are left with a sentence of the form $A \vee B$ (line 5). It is an indication that we should possibly use the \vee elimination rule at some stage in the proof. Remember that this rule requires two subproofs, one (here lines 9 – 10) starting with the assumption of A and the other (here lines 11 – 12) starting with the assumption of B . The two subproofs should end on the same sentence and then that sentence can be derived outside the subproofs using the $\vee e$ rule (here line 13).

Note how the $\forall x$ and $\exists x$ introduction and elimination rules are applied. This is the first example in this assignment of the $\exists x i$ rule (line 15): it has to be cited *inside* the subproof containing the fresh variable.

1	$\forall x (P(x) \vee Q(x))$	premise	
2	$\exists x \neg Q(x)$	premise	
3	$\forall x (R(x) \rightarrow \neg P(x))$	premise	
4	$x_0 \quad \neg Q(x_0)$	assumption	[x_0/x]
5	$P(x_0) \vee Q(x_0)$	$\forall x e 1$	
6	$R(x_0) \rightarrow \neg P(x_0)$	$\forall x e 3$	
7	$R(x_0)$	assumption	
8	$\neg P(x_0)$	$\rightarrow e 6, 7$	
9	$P(x_0)$	assumption	
10	\perp	$\neg e 8, 9$	
11	$Q(x_0)$	assumption	
12	\perp	$\neg e 4, 11$	
13	\perp	$\vee e 5, 9 - 10, 11 - 12$	
14	$\neg R(x_0)$	$\neg i 7 - 13$	
15	$\exists x \neg R(x)$	$\exists x i 14$	
16	$\exists x \neg R(x)$	$\exists x e 2, 4 - 15$	

Question 9.5

You have previously come across all the rules and techniques used in this proof.

- Because the goal starts with $\forall x$, we introduce a fresh variable (line 3) so that the $\forall x$ introduction rule may be used (line 16).
- Because of the \rightarrow connective in the goal, we start a subproof in line 4 with the assumption of the left hand side of \rightarrow (but with x replaced by the dummy variable).
- Note the two sub-subproofs in lines 7 - 8 and 9 - 13 so that the \vee elimination rule can be used in line 14.
- Also note the application of the \perp e rule in line 13: we may derive any sentence after \perp has been derived, provided that we are still inside the same subproof.

1	$\forall x (P(x) \rightarrow (Q(x) \vee R(x)))$	premise
2	$\neg \exists x (P(x) \wedge R(x))$	premise
3	x_0	
4	$P(x_0)$	assumption
5	$P(x_0) \rightarrow (Q(x_0) \vee R(x_0))$	$\forall x$ e 1
6	$Q(x_0) \vee R(x_0)$	\rightarrow e 4, 5
7	$Q(x_0)$	assumption
8	$Q(x_0)$	copy 7
9	$R(x_0)$	assumption
10	$P(x_0) \wedge R(x_0)$	\wedge i 4, 9
11	$\exists x (P(x) \wedge R(x))$	$\exists x$ i 10
12	\perp	\neg e 2, 11
13	$Q(x_0)$	\perp e 12
14	$Q(x_0)$	\vee e 6, 7 - 8, 9 - 13
15	$P(x_0) \rightarrow Q(x_0)$	\rightarrow i 4 - 14
16	$\forall x (P(x) \rightarrow Q(x))$	$\forall x$ i 3 - 15