Tutorial Letter 202/1/2018

Linear Mathematical Programming

DSC2605

Semester 1

Department of Decision Sciences

This tutorial letter contains solutions to Assignment 2

Bar code

Define tomorrow.

UNISA university of south africa

Question 1

Discussion on myUnisa. No solution is provided. [7]

Question 2

(a) Decision variables
Let

- X_A be the number of model A carriages,
- X_B be the number of model B carriages,
- X_{Cc} be the number of model C carriages used for coal,
- X_{Cl} be the number of model B carriages used for lumber.

(b) LP model

- Objective function: Minimise $z = 6X_A + 4X_B + 8\,(X_{Cc} + X_{Cl})$
- Functional constraints:
  1. Fleet capacity of coal: $25X_A + 18X_{Cc} \geq 3\,400$
  2. Fleet capacity of lumber: $8X_B + 10X_{Cl} \geq 1\,800$ or $4X_B + 5X_{CI} \geq 900.$
  3. At least a quarter of the carriages must be able to transport coal, that means

$$X_A + X_{Cc} \geq \frac{1}{4}\,(X_A + X_B + X_{Cc} + X_{Cl})\,,$$

  or equivalently

$$3X_A - X_B + 3X_{Cc} - X_{Cl} \geq 0.$$

The LP model is then given by:

$$\begin{array}{lrcrcrcrl}
\text{Minimize } z = & 6X_A & + & 4X_B & + & 8X_{Cc} & + & 8X_{Cl} & \\
\text{Subject to} & & & & & & & & \\
 & 25X_A & & & + & 18X_{Cc} & & & \geq 3\,400 \\
 & & & 4X_B & & & + & 5X_{Cl} & \geq 900 \\
 & 3X_A & - & X_B & + & 3X_{Cc} & - & X_{Cl} & \geq 0. \\
\text{and} & \multicolumn{8}{l}{X_A,\ X_B,\ X_{Cc},\ X_{Cl} \geq 0.}
\end{array}$$

(c) Lingo codes and solution report

```
MODEL:

    ! Assignment 2, Question 2c);

[OBJECTIVE_FUNCTION] MIN = 6*Xa + 4*Xb + 8*Xcc + 8*Xcl;

              [COAL] 25*Xa + 18*Xcc >= 3400;
            [LUMBER] 4*Xb + 5*Xcl >= 900;
    [COAL_TRANSPORT] 3*Xa  - Xb + 3*Xcc  - Xcl>= 0;
```

Solution report

```
Global optimal solution found.
Objective value:                              1716.000
Infeasibilities:                              0.000000
Total solver iterations:                             2
Elapsed runtime seconds:                          0.05

Model Class:                                        LP

Total variables:                      4
Nonlinear variables:                  0
Integer variables:                    0

Total constraints:                    4
Nonlinear constraints:                0

Total nonzeros:                      12
Nonlinear nonzeros:                   0
```

Variable	Value	Reduced Cost
XA	136.0000	0.000000
XB	225.0000	0.000000
XCC	0.000000	3.680000
XCL	0.000000	3.000000

Row	Slack or Surplus	Dual Price
OBJECTIVE_FUNCTION	1716.000	-1.000000
COAL	0.000000	-0.2400000
LUMBER	0.000000	-1.000000
COAL_TRANSPORT	183.0000	0.000000

(d) Optimal solution:

$$z = 1716 \text{ and,}$$

$$x_A = 136,\ X_B = 225, \quad X_{Cc} = X_{Cl} = 0.$$

Question 3

(a) **LP model**

- Objective function

Revenue $= 40\,(X_{11} + X_{21} + X_{31}) + 50\,(X_{12} + X_{22} + X_{32})$
Cost $= 6\,(X_{11} + X_{12}) + 8\,(X_{21} + X_{22}) + 10\,(X_{31} + X_{32})$
It follows that

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Cost} \\ &= 34X_{11} + 44X_{12} + 32X_{21} + 42X_{22} + 30X_{31} + 40X_{32}\end{aligned}$$

- Functional constraints

(1) Availability of Oil 1, 2 and 3.

$$\begin{aligned} X_{11} + X_{12} &\leq 8\,000 \quad \text{(Availability of Oil 1)} \\ X_{21} + X_{22} &\leq 12\,000 \quad \text{(Availability of Oil 2)} \\ X_{31} + X_{32} &\leq 16\,000 \quad \text{(Availability of Oil 3)} \end{aligned}$$

(2) Constraints on demands of blends.

$$\begin{aligned} X_{11} + X_{21} + X_{31} &\geq 12\,000 \quad \text{(Demand of Blend 1)} \\ X_{12} + X_{22} + X_{32} &\geq 12\,000 \quad \text{(Demand of Blend 2)} \end{aligned}$$

(3) Rate of content of each oil for Blend 1

$$\begin{aligned} \frac{X_{11}}{X_{11} + X_{21} + X_{31}} &\leq 0.4 \quad \text{or} \quad 3X_{11} - 2X_{21} - 2X_{31} \leq 0. \\ \frac{X_{21}}{X_{11} + X_{21} + X_{31}} &\geq 0.3 \quad \text{or} \quad 3X_{11} - 7X_{21} + 3X_{31} \leq 0. \\ \frac{X_{31}}{X_{11} + X_{21} + X_{31}} &\leq 0.3 \quad \text{or} \quad 3X_{11} + 3X_{21} - 7X_{31} \geq 0. \end{aligned}$$

(4) Rate of content of each oil for Blend 2

$$\begin{aligned} \frac{X_{12}}{X_{12} + X_{22} + X_{32}} &\geq 0.4 \quad \text{or} \quad 3X_{12} - 2X_{22} - 2X_{32} \geq 0. \\ \frac{X_{22}}{X_{12} + X_{22} + X_{32}} &\leq 0.5 \quad \text{or} \quad X_{12} - X_{22} + X_{32} \geq 0. \\ \frac{X_{32}}{X_{12} + X_{21} + X_{22}} &\geq 0.3 \quad \text{or} \quad 3X_{12} + 3X_{22} - 7X_{32} \leq 0. \end{aligned}$$

– LP model

$$\begin{aligned}
\text{Maximise z} = \quad & 34X_{11} + 44X_{12} + 32X_{21} + 42X_{22} + 30X_{31} + 40X_{32} \\
\text{subject to} \quad & \\
& X_{11} + X_{12} \leq 8\,000 \\
& X_{21} + X_{22} \leq 12\,000 \\
& X_{31} + X_{32} \leq 16\,000 \\
& X_{11} + X_{21} + X_{31} \geq 12\,000 \\
& X_{12} + X_{22} + X_{32} \geq 12\,000 \\
& 3X_{11} - 2X_{21} - 2X_{31} \leq 0 \\
& 3X_{11} - 7X_{21} + 3X_{31} \leq 0 \\
& 3X_{11} + 3X_{21} - 7X_{31} \geq 0 \\
& 3X_{12} - 2X_{22} - 2X_{32} \geq 0 \\
& X_{12} - X_{22} + X_{32} \geq 0 \\
& 3X_{12} + 3X_{22} - 7X_{32} \leq 0 \\
\text{and} \quad & X_{ij} \geq 0, \quad (i = 1, 2, 3 \text{ and } j = 1, 2).
\end{aligned}$$

(b) **LINGO codes and solution report**

```
    MODEL:

! Assignment 2, Question 3b);

[OBJECTIVE_FUNCTION] MAX= 34*X11 + 44*X12 + 32*X21 + 42*X22 + 30*X31 + 40*X32;

! Functional Constraints;

[AVAILABILITY_OIL_1] X11 + X12 <= 8000;
[AVAILABILITY_OIL_2] X21 + X22 <= 12000;
[AVAILABILITY_OIL_3] X31 + X32 <= 16000;

[DEMAND_BLEND_1] X11 + X21 + X31 >= 12000;
[DEMAND_BLEND_2] X12 + X22 + X32 >= 12000;

[CONTENT_OF_OIL_1_FOR_BLEND_1] 3*X11 - 2*X21  - 2*X31 <= 0;
[CONTENT_OF_OIL_2_FOR_BLEND_1] 3*X11 - 7*X21  + 3*X31 <= 0;
[CONTENT_OF_OIL_3_FOR_BLEND_1] 3*X11 + 3*X21  - 7*X31 >= 0;

[CONTENT_OF_OIL_1_FOR_BLEND_2] 3*X12 - 2*X22  - 2*X32 >= 0;
[CONTENT_OF_OIL_2_FOR_BLEND_2] X12 - X22  + X32 >= 0;
[CONTENT_OF_OIL_3_FOR_BLEND_2] 3*X12 + 3*X22  - 7*X32 >= 0;
```

Solution report

```
Global optimal solution found.
Objective value:                              1078857.
Infeasibilities:                              0.000000
Total solver iterations:                             5
Elapsed runtime seconds:                          0.06

Model Class:                                        LP

Total variables:                      6
Nonlinear variables:                  0
Integer variables:                    0

Total constraints:                   12
Nonlinear constraints:                0

Total nonzeros:                      36
Nonlinear nonzeros:                   0


                          Variable           Value        Reduced Cost
                               X11        1371.429            0.000000
                               X12        6628.571            0.000000
                               X21        7028.571            0.000000
                               X22        4971.429            0.000000
                               X31        3600.000            0.000000
                               X32        4971.429            0.000000

                               Row    Slack or Surplus      Dual Price
                OBJECTIVE_FUNCTION        1078857.            1.000000
               AVAILABILITY_OIL_1         0.000000            61.14286
               AVAILABILITY_OIL_2         0.000000            59.14286
               AVAILABILITY_OIL_3         7428.571            0.000000
                   DEMAND_BLEND_1         0.000000           -10.00000
                   DEMAND_BLEND_2         4571.429            0.000000
     CONTENT_OF_OIL_1_FOR_BLEND_1         17142.86            0.000000
     CONTENT_OF_OIL_2_FOR_BLEND_1         34285.71            0.000000
     CONTENT_OF_OIL_3_FOR_BLEND_1         0.000000           -5.714286
     CONTENT_OF_OIL_1_FOR_BLEND_2         0.000000            0.000000
     CONTENT_OF_OIL_2_FOR_BLEND_2         6628.571            0.000000
     CONTENT_OF_OIL_3_FOR_BLEND_2         0.000000           -5.714286
```

(c) Optimal solution

$$z = 1\,078\,857,$$
$$X_{11} = 1371.429 \quad X_{12} = 6628.571,$$
$$X_{21} = 7028.571 \quad X_{22} = 4971.429,$$
$$X_{31} = 3600.000 \quad X_{32} = 4971.429.$$

Question 4

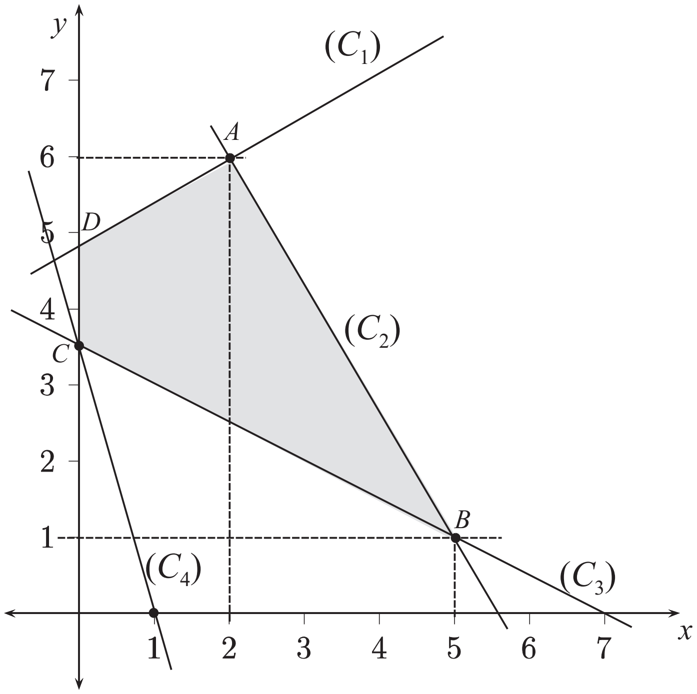
Minimise $z = x + 2y$

Subject to

$$\begin{array}{lcccl} -2x & + & 3y & \leq & 14 \quad (C_1) \\ 5x & + & 3y & \leq & 28 \quad (C_2) \\ x & + & 2y & \geq & 7 \quad (C_3) \\ 7x & + & 2y & \geq & 7 \quad (C_4) \end{array}$$

and $x,\ y \geq 0$.

(a) Graphical representation of the LP model



(b) Corner points and values of objective function.

Corner-points	**Value of** $z = x + 2y$
$A = (2,\ 6)$	$2 + 2(6) = 14$
$B = (5,\ 1)$	$5 + 2(1) = 7$
$C = (0,\ 7/2)$	$0 + 2(7/2) = 7$
$D = (0,\ 14/3)$	$0 + 2(14/3) = 28/3$

(c) The minimum value of the objective function $z = x + 2y$ occurs at corner-points $B = (5, 1)$ and $C = (0, 7/2)$. Therefore, the LP model has multiple optimal solutions given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 7/2 \end{bmatrix},$$

$$= \frac{\alpha}{2} \begin{bmatrix} 10 \\ -5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \quad \alpha \in [0,\, 1].$$

(d)
- Redundant constraint: $7x + 2y \geq 7,\ x \geq 0$.
- Binding constraints: $5x + 3y \leq 28,\ \ x + 2y \geq 7$ and $y \geq 0$.
- Nonbinding constraint: $-2x + 3y \leq 14$.

(e) The general equation of isocost lines are given by all lines parallel the straight line $\overline{AC}$. That is the equation given by

$$x + 2y = b, \quad b \in \mathbb{R}$$

Question 5

$$\begin{array}{llcccccccc} \text{Maximise z} = & x_1 & + & 2x_2 & + & x_3 & & \\ \text{Subject to} & & & & & & & \\ & x_1 & + & 2x_2 & + & x_3 & = & 4 \\ & 3x_1 & + & x_2 & - & x_3 & = & 6 \\ \text{and} & x,\ x_2,\ x_3 \geq 0. & & & & & & \end{array}$$

The LP model in augmented form is given by

$$\begin{array}{llcccccccccc} \text{Maximise z} = & x_1 & + & 2x_2 & + & x_3 & - & Ma_1 & - & Ma_2 & & \\ \text{Subject to} & & & & & & & & & & & \\ & x_1 & + & 2x_2 & + & x_3 & + & a_1 & & & = & 4 \\ & 3x_1 & + & x_2 & - & x_3 & + & a_2 & & & = & 6 \\ \text{and} & x,\ x_2,\ x_3,\ a_1,\ a_2 \geq 0. & & & & & & & & & & \end{array}$$

Initial simplex tableau

	z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
R_1	0	1	2	1	1	0	4	a_1	$4/1$
R_2	0	(3)	1	-1	0	1	6	a_2	$6/3 = 2$
R_0	1	-1	-2	-1	M	M	0	z	
R_0'	1	$-1-4M$	$-2-3M$	-1	0	0	$-10M$	z	

Where R_0' is the new objective row obtained by performing the following elementary row operation:

$$R_0' = R_0 - (R_1 + R_2)\, M.$$

- The entering variable is x_1 because is the most negative number.
- $\min\theta = \min\{4,\, 2\} = 2$. It follows that the leaving variable is a_2.
- The pivot is found in x_1's column and is 3. Therefore, we apply the one-zero method to x_1 's column by making one the pivot and all other elements in the pivot column to zeroes. We then obtain the next simplex tableau given by

Simplex tableau 1

z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
0	0	5/3	4/3	1	−1/3	2	a_1	6/5
0	1	1/3	−1/3	0	1/3	2	x_1	6/1
1	0	$-5/3(1+M)$	$-4/3(1+M)$	0	$\frac{1}{3}$	$2-2M$	z	

- The entering variable is x_2.
- $\theta = \min\{6/5,\, 6\} = 6/5$. The leaving variable is a_1.
- We find the pivot in the x_2's column and is given by 5/3. We apply the one-zero method and obtain the next simple tableau:

Simplex tableau 2

z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
0	0	1	4/5	3/5	−1/5	6/5	x_2	3/2
0	1	0	−3/5	−1/5	3/5	8/5	x_1	–
1	0	0	0	$\frac{5}{3}(1+M)$	M	4	z	

From this tableau, we have reached an optimal solution given by

$$z = 4,\; x_1 = 8/5,\; x_2 = 6/5 \text{ and } x_3 = 0.$$

Since the nonbasic variable x_3 has a zero coefficient in the objective row R_0, this indicates that an alternative solution exists. We reincorporate nonbasic variable x_3 into the basis. It follows that:

- The entering variable is x_3.
- $\min\theta = \min\{3/2,\, -\} = 3/2$. This implies that the leaving variables is x_2.
- We find the pivot in the x_3's column and is given by 4/5. We apply the one-zero method and obtain the following simple tableau:

Simplex tableau 2'

z	x_1	x_2	x_3	a_1	a_2	rhs	BV
0	0	5/4	1	3/4	−1/4	3/2	x_3
0	1	3/4	0	1/4	9/20	5/2	x_1
1	0	0	0	$\frac{5}{3}(1+M)$	M	4	z

Thus the alternative solution is given by

$$z = 4,\ \ x_1 = 2/5,\ \ x_2 = 0 \text{ and } x_3 = 3/2.$$

The general optimal solution is given by $z = 4$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 8/5 \\ 6/5 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 5/2 \\ 0 \\ 3/2 \end{bmatrix}, \quad \alpha \in [0,\ 1]$$

$$= \frac{\alpha}{10} \begin{bmatrix} -9 \\ 12 \\ -15 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}, \quad \alpha \in [0,\ 1].$$