UNISA | university of south africa

DEPARTMENT OF DECISION SCIENCES

LINEAR MATHEMATICAL PROGRAMMING

DSC2605

MEMORANDUM

May/June 2017

QUESTION 1 (6/6) **[6]**

Consider the following matrices:

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 4 & 1 \\ 2 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -5 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 0 \\ -4 & 0 \\ -2 & 6 \end{bmatrix}.$$

Perform each of the following calculations where possible, or state why the calculation cannot be performed.

(a) $A^T B - 2C$. (4)

(b) $AB^T + 4C$. (2)

Solutions

(a) $$A^T B - 2C = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 4 & -2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ -8 & 0 \\ -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ -8 & 1 \\ -5 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ -8 & 0 \\ -4 & 12 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(b) Calculations cannot be performed because the matrices A and B^T are not compatible for multiplication. ie A is a 3x3 matrix and B^T is a 2x3 matrix).

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QUESTION 2 (15/15) **[15]**

Let the matrix M be given by

$$M = \begin{bmatrix} 1-k & 0 & -1 \\ 0 & 1+k & 0 \\ 1 & 0 & -(1+k) \end{bmatrix}.$$

(a) Find the determinant $|M|$ of the matrix M. (4)

(b) Find all values of the parameter k for which the inverse of the matrix M does exist. (2)

(c) Find the inverse M^{-1} of the matrix M in terms of the parameter k by using the Gauss-Jordan elimination method. (9)

(a) By expanding along the second row, we have

$$|M| = (1+K)\begin{vmatrix} 1-K & -1 \\ 1 & -(1+K) \end{vmatrix}$$ ✓

$$= (1+K)\left[-(1-K)(1+K)+1\right]$$ ✓ 4/4

$$= (1+K)\,K^2.$$ ✓

(b) The matrix inverse M^{-1} of the matrix M exists if and only if $|M| \neq 0$, that is

$$(1+K)K^2 \neq 0$$ ✓

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$\therefore\ K \neq 0$ or $K \neq -1$. ✓ ✓

Therefore the M^{-1} exists for all $K \in \mathbb{R} \setminus \{0,-1\}$.

(C) The augmented matrix $[M|I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1-K & 0 & -1 & 1 & 0 & 0 \\ 0 & 1+K & 0 & 0 & 1 & 0 \\ 1 & 0 & -(1+K) & 0 & 0 & 1 \end{array}\right]$$ ✓

We exchange R_1 and R_3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -(1+K) & 0 & 0 & 1 \\ 0 & 1+K & 0 & 0 & 1 & 0 \\ 1-K & 0 & -1 & 1 & 0 & 0 \end{array}\right]$$ ✓

We apply the 1-0 method ($R_1' = R_1$, $R_2' = R_2$ and $R_3' = R_3 - (1-K)R_1$) to the 1^{st} column and get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -(1+K) & 0 & 0 & 1 \\ 0 & 1+K & 0 & 0 & 1 & 0 \\ 0 & 0 & -K^2 & 1 & 0 & -(1-K) \end{array}\right]$$ ✓

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Next pivot is in the 2^{nd} row, 2^{nd} column and we apply the 1-0 method

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -(1+K) & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{1+K} & 0 \\ 0 & 0 & -K^2 & 1 & 0 & -(1-K) \end{array}\right]$$ ✓

Next pivot is in 3^{rd} row, 3^{rd} column. We apply the 1-0 method and obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1+K}{K^2} & 0 & 1+\frac{(1-K)^2}{K^2} \\ 0 & 1 & 0 & 0 & \frac{1}{1+K} & 0 \\ 0 & 0 & 1 & -\frac{1}{K^2} & 0 & \frac{1-K}{K^2} \end{array}\right]$$ ✓ ✓

The inverse of the matrix M is

$$M^{-1} = \begin{bmatrix} -\frac{1+k}{k^2} & 0 & 1+\frac{(1-k)^2}{k^2} \\ 0 & \frac{1}{1+k} & 0 \\ -\frac{1}{k^2} & 0 & \frac{1-k}{k^2} \end{bmatrix}$$ ✓

QUESTION 3 (10/10) **[10]**

Solve the following system of linear equations using the Gauss-Jordan elimination method and write down the solution set clearly.

$$\begin{array}{rcrcrcrcr} 2x & + & y & - & 4z & - & 2w & = & 1 \\ -x & - & 2y & + & z & - & w & = & -2 \end{array}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 2 & 1 & -4 & -2 & 1 \\ -1 & -2 & 1 & -1 & -2 \end{array}\right]$$ ✓

We exchange R_1 and R_2

$$\left[\begin{array}{cccc|c} -1 & -2 & 1 & -1 & -2 \\ 2 & 1 & -4 & -2 & 1 \end{array}\right]$$

We apply i-o method

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -3 & -2 & -4 & -3 \end{array}\right]$$ ✓

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -2 & -1 & \frac{1}{2} \\ 0 & -\frac{3}{2} & -1 & -2 & -\frac{3}{2} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{3} & -\frac{5}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{4}{3} & 1 \end{array}\right]$$ ✓

The next pivot is found in the 2^{nd} row, 2^{nd} column and apply the 1-0 method

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{3} & -\frac{5}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{4}{3} & 1 \end{array}\right]$$

x and y are leading variables and, z and w are free variables The general solution is

$x = \frac{7}{3}s + \frac{5}{3}t$, $y = 1 - \frac{2}{3}s - \frac{4}{3}t$,

$z = s$, $w = t$; $s, t \in \mathbb{R}$.

The solution set is

$$S = \left\{ \begin{bmatrix} \frac{7}{3}s + \frac{5}{3}t \\ 1 - \frac{2}{3}s - \frac{4}{3}t \\ s \\ t \end{bmatrix}, s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{7}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} \frac{5}{3} \\ -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} t, s, t \in \mathbb{R} \right\}.$$

QUESTION 4 (10/10) **[10]**

The Juice Company sells bags of oranges and cartons of orange juice. The company grades oranges on a scale from 1 (poor) to 10 (excellent). At present, the company only has on hand 50 000 kg of grade 9 oranges and 60 000 kg of grade 6 oranges. The average quality of oranges sold in bags must be at most 8, and the average quality of the oranges used to produce orange juice must be at least 6.
Each kg of oranges that is used for juice yields a revenue of R30 and incurs a variable cost (consisting of labour costs, variable overhead costs, inventory costs etc.) of R20. Each kg of oranges sold in bags yields a revenue of R10 and incurs a variable cost of R5.

(a) Define all decision variables of the problem clearly. (2)

(b) Formulate this problem as a linear programming model that maximises the profit of the company. (8)

(a) x_{ij} = number of kg of grade i oranges ($i=1$ for grade 6 and $i=2$ for grade 9) used in product j ($j=1$ for bags and $j=2$ for juice). 2/2

(b) The LP model is then given by

Maximise $z = 10(X_{12}+X_{22}) + 5(X_{11}+X_{21})$
subject to

$$\frac{6X_{11}+9X_{21}}{X_{11}+X_{21}} \leq 8$$

$$\frac{6X_{12}+9X_{22}}{X_{12}+X_{22}} \geq 6$$ 8/8

$$X_{11}+X_{12} \leq 60\,000$$

$$X_{21}+X_{22} \leq 50\,000$$

and $X_{ij} \geq 0$.

QUESTION 5 (11/11) **[11]**

A student would like to design a breakfast of corn flakes and milk that is as economical as possible. On the basis of what he eats during his other meals, he decides that breakfast should supply him with at least 9 grams of protein, at least a third of the recommended daily allowance (RDA) of vitamin D, and at least a quarter of the RDA of calcium. He finds the following nutritional information on the milk and corn flakes containers:

	Milk ($\frac{1}{2}$ cup)	**Corn flakes** (1 Portion = 30 g)
Cost	R1	75c
Protein	4 grams	2 grams
Vitamin D	$\frac{1}{8}$ of RDA	$\frac{1}{10}$ of RDA
Calcium	$\frac{1}{6}$ of RDA	None

In order not to make his mixture too soggy or too dry, the student decides to limit himself to mixtures that contain 1 to 3 portions of corn flakes per cup of milk. Assume that the student wants to determine the quantities of milk and corn flakes that should be used to minimise the cost of his breakfast.

(a) Define all decision variables of the problem clearly. (2)

(b) Formulate this problem as a linear programming model that minimises the cost of the student's breakfast. (9)

(a) Let x_1 be the quantity of milk used (measured in $\frac{1}{2}$-cup unit), ✓

x_2 be the quantity of corn flakes (measured in 30 gr portions). ✓ 2/2

(b) The LP model is

Minimise $z = x_1 + \frac{3}{4}x_2$ ✓

subject to

$$4x_1 + 2x_2 \geq 9$$ ✓

$$\frac{1}{8}x_1 + \frac{1}{10}x_2 \geq \frac{1}{3}$$ ✓ 9/9

$$\frac{1}{6}x_1 \geq \frac{1}{4}$$ ✓

$$x_1 - 2x_2 \leq 0$$ ✓

$$3x_1 - 2x_2 \geq 0$$ ✓

and $x_1, x_2 \geq 0$. ✓

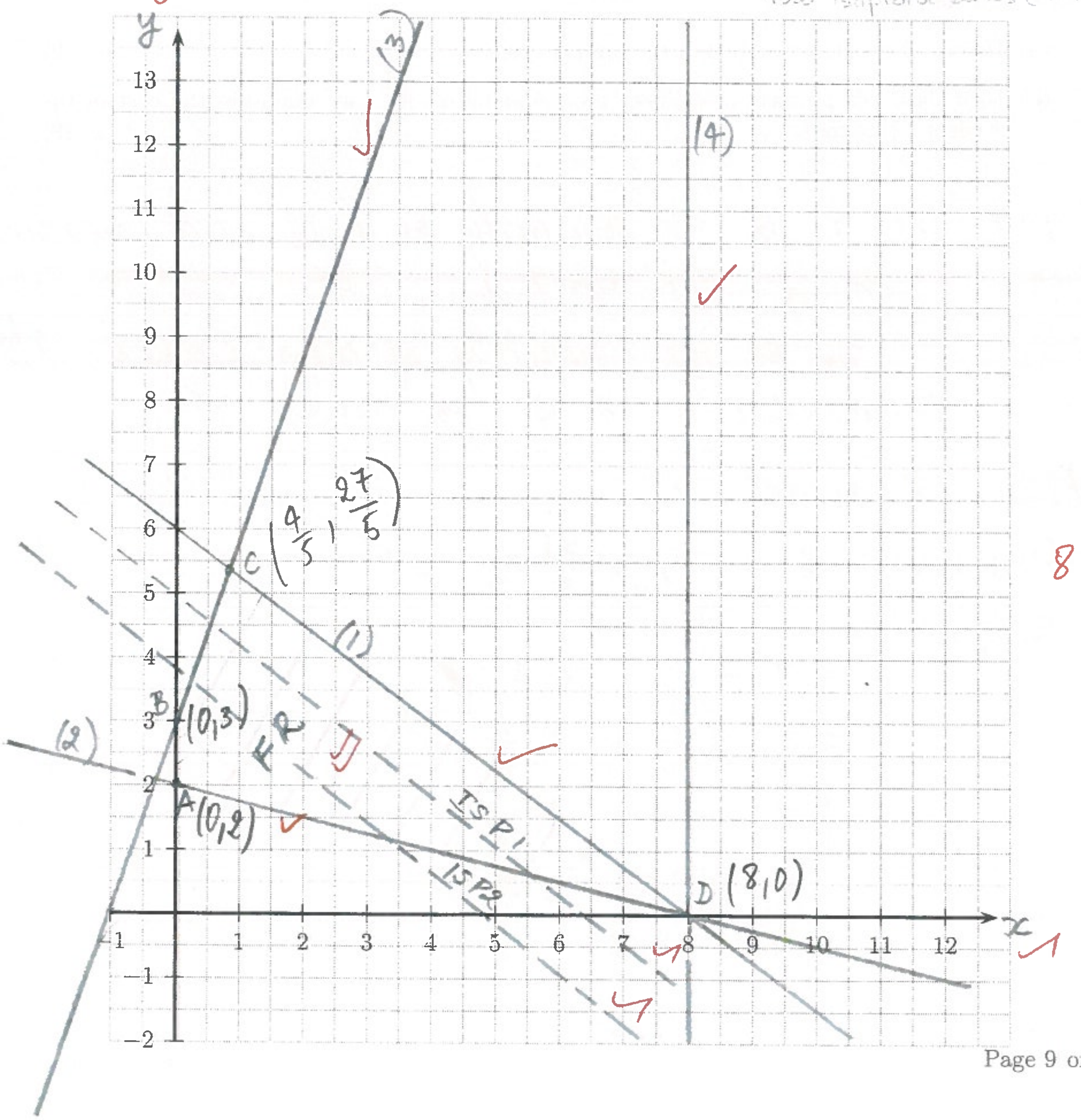
QUESTION 6 (18/18) [18]

Consider the following LP problem:

$$\begin{array}{llrcl} \text{Maximise} & z = 3x + 4y & & & \\ \text{subject to} & & & & \\ & 3x + 4y & \leq & 24 & (1) \\ & x + 4y & \geq & 8 & (2) \\ & -3x + y & \leq & 3 & (3) \\ & x & \leq & 8 & (4) \\ \text{and} & x, y & \geq & 0. & \end{array}$$

(a) Represent all the constraints of the problem on the graph below. Represent x on the horizontal axis and y on the vertical axis. Label all relevant lines on the graph and indicate the feasible region clearly. Include on the graph two isoprofits ~~lines.~~ in dashed lines (8)

two isoprofit lines (in dashed lines).



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(b) Find all corner-points of the feasible region and evaluate the objective function at each of them. (4)

(c) Deduce the optimal solution. If the LP problem is infeasible or unbounded, give the reason for this and if the LP problem has multiple optimal solutions, find the general optimal solution. (3)

(d) State clearly all redundant, binding and nonbinding constraints in this linear programming problem. (3)

(b) corner-points

$A = (0, 2)$; $z = 8$ ✓

$B = (0, 3)$; $z = 12$ ✓

$C = \left(\frac{4}{5}, \frac{27}{5}\right)$; $z = 24$ ✓ 4/4

$D = (8, 0)$; $z = 24$ ✓

(c) The LP problem has multiple optimal solutions and the general optimal solution is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} \frac{4}{5} \\ \frac{27}{5} \end{bmatrix} (1-\alpha) , \quad \alpha \in [0,1].$$ 3/3

With $z = 24$. ✓

(d)

- Redundant constraint : $x \le 8$ (4) ✓
- Binding constraints : $x + 4y \geq 8$ (1), $3x + 4y \le 24$ (2), ✓ 3/3
- Nonbinding constraints : $-3x + y \le 3$ (3) ✓

Or

QUESTION 7 **[18]**

Consider the following LP model:

$$\begin{array}{lrcl}
\text{Maximise} & r = 2x - 2y + z & & \\
\text{subject to} & & & \\
& x + 2y - z & \leq & 5 \\
& 2x + y + 3z & \geq & 2 \\
\text{and} & x, y, z & \geq & 0.
\end{array}$$

(a) Write down the augmented form of the LP model and the initial simplex tableau. (5)

- The LP model in augmented form is

Maximise $r = x - 2y + z - Ma$ ✓

subject to

$x + 2y - z + s_1 = 5$ ✓

$2x + y + 3z - s_2 + a = 2$ ✓

and $x, y, z, s_1, s_2, a \geq 0.$ ✓

- Initial simplex tableau

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	r	x	y	z	s_1	s_2	a	rhs	BV
R_1	0	1	2	−1	1	0	0	5	s_1
R_2	0	2	1	3	0	−1	1	2	a
R_0	1	−2	2	−1	0	0	M	0	r
R_0'	1	$-2-2M$ ✓	$2-M$ ✓	$-1-3M$ ✓	0 ✓	M ✓	0	$-2M$ ✓	r

Where the new objective row R_0 is obtained by performing the ERO: $R_0' = R_0 - MR_2$.

(b) Assume that the next tableau after the initial simplex tableau is

r	x	y	z	s_1	s_2	a	rhs	BV	θ
0	$\frac{5}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{17}{3}$	s_1	17/5
0	$\frac{2}{3}$	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	z	1
1	$-\frac{4}{3}$	$\frac{7}{3}$	0	0	$-\frac{1}{3}$	$\frac{1}{3}+M$	$\frac{2}{3}$	r	

where s_1, s_2 and a are respectively the slack, the surplus and the artificial variables.

Continue the iterations by using the Big M method to find the optimal solution to the LP problem. (13)

Note: *If the LP problem is infeasible or unbounded, give the reason for this and if the LP problem has multiple optimal solutions, find the general optimal solution.*

- The entering variable is x.
- Min $\theta = \min\{\frac{17}{5}, 1\}$. The leaving variable is z
- We apply the 1-0 method to x's column and obtain the next simplex tableau

r	x	y	z	s_1	s_2	a	rhs	BV	θ
0	0	$\frac{3}{2}$	$-\frac{5}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	4	s_1	8
0	1	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	x	—
1	0	3	2	0	-1	$1+M$	2	r	

- The entering variable is s_2
- min $\theta = \min\{8, -\} = 8$. The leaving variable is s_1
- We apply the 1-0 method to s_2's column and obtain the following simplex tableau

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r	x	y	z	s_1	s_2	a	Rhs	BV	θ
0	0	3	−5	2	1	−1	0	s_2	–
0	1	2	−1	1	0	0	2	x	–
1	0	6	−3	2	0	M	10	r	

↑

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The entering variable is z but there is no leaving variable. The solution is unbounded.

QUESTION 8 (12/12) **[12]**

Assume that the table below is the final simplex tableau for an LP problem with the non-negative decision variables x_1, x_2 and x_3 and three constraints. The objective function z is maximised and slack variables s_1 and s_2, and artificial variables a_1 and a_2 were added.

z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs	BV
0	0	0	-1	0	1	1	0	k	x_1
0	0	1	p	1	0	0	1	2	x_2
0	1	0	-4	0	0	0	0	3	s_1
1	3	1	q	8	12	$2M$	$3M-1$	6	z

(1) Give conditions on parameters p, q and k for which the problem has

(a) a unique optimal solution. (3)

(b) no solution. (2)

(2) Assume now that $q = 0$, $p = 1$ and $k = 0$ in the above tableau. Find the optimal solution or the general optimal solution if the LP problem has multiple optimal solutions. (7)

(1) (a) $q > 0$, any p, $k \geq 0$. 3/3

(b) The problem will always have at least one optimal solution because $a_1 = a_2 = 0$. 2/

(2) If $q = 0$, $p = 1$ and $k = 0$, then the problem has multiple optimal solutions.
- We reincorporate the nonbasic variable x_3 in the basis. This means that x_3 is the entering variable.
- $\min \theta = \min\{-, 2, -\} = 2$. The leaving variable is x_2.
- We apply the 1-0 method to the x_3's column and obtain the following alternative final simplex tableau.

z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	rhs	BV
0	0	1	0	1	1	1	1	2	x_1
0	0	1	1	1	0	0	1	2	x_3
0	1	4	0	4	0	0	4	11	s_1
1	3	1	0	8	12	2M	3M-1	6	z

The two optimal solutions are respectively given by

$$x' = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad x'' = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

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The general optimal solution is given by

$$x = \alpha \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \alpha \in [0,1].$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}, \quad \alpha \in [0,1].$$