Department of Decision Sciences

LINEAR MATHEMATICAL PROGRAMMING

Only study guide for

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Learning Unit 1

Introduction

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Learning objectives

After completing this study unit you should be able to

- explain what "operations research" is
- describe how the scientific approach is applied to Operations Research/Quantitative Management
- explain what a mathematical model is
- explain what "mathematical programming" is
- explain what "linear programming" is
- give the properties and assumptions of linear programming
- give examples of operations research techniques and their areas of application.

Sections from prescribed book

1.1. Introduction to Modeling - *p1*

1.2. The Seven-Step Model-Building Process- *p5*

1.1 Historical background

In the period between the two great wars, military technology developed by leaps and bounds without military leaders having the chance to apply all the new weapons together or to experiment with new weapon systems. When the German air attacks on England started during World War II, some of the British military managers and leaders recognised this fact and realised that they were confronted with problems for which there were no parallels in history.

In order to maximise their war effort, the British government organised teams of scientific and engineering personnel to assist field commanders in solving perplexing strategic and tactical problems. They found that technically trained men could solve problems outside their normal professional competency. They asked biologists to examine problems in electronics, physicists to think in terms of the movement of people rather than the movement of molecules, mathematicians to apply probability theory to improve soldiers' chances of survival, and chemists to study equilibria in systems other than in chemicals. Teams made up of specialists from these different disciplines studied problems ranging from the evaluation of cost and effectiveness of complete military systems (such as the defence system of a country) to the best placement of depth charges in anti-submarine warfare.

The success of the British operational research teams led the United States to institute a similar effort in 1942 (small-scale projects dated back to 1937). The initial project involved the deployment of merchant marine convoys to minimise losses from enemy submarines.

These mathematical and scientific approaches to military operations were then called *operations research.*

The success of operations research techniques during the Second World War led to the techniques being extended and applied to other fields after the war.

Today the term "operations research" means a scientific approach to decision making, which seeks to determine how best to design and operate a system, usually under conditions requiring the allocation of scarce resources.

Operations research is now applied so extensively in many areas that a separate discipline has been established. The diversity of this discipline has led to its being known not only as *operations research*, but also by terms such as *management science*, *decision science*, *decision analysis* and *quantitative management.*

1.2 The scientific approach

Descartes, a French philosopher and mathematician of the 17th century, emphasised the use of reason as the chief tool of enquiry. The scientific approach, also known as the scientific method, is a formalised reasoning process. It consists of the following steps:

(a) The problem for analysis is defined, and the conditions for observation are determined.

(b) Observations are made under different conditions to determine the behaviour of the system containing the problem.

(c) Based on the observations, a hypothesis that describes how the factors involved are thought to interact, or what the best solution to the problem is, is conceived.

(d) An experiment is designed to test the hypothesis.

(e) The experiment is carried out, and measurements are obtained and recorded.

(f) The results of the experiment are analysed, and the hypothesis is either accepted or rejected.

The six steps of the scientific method can be applied to decision making in general and to Operations Research/Quantitative Management in particular, and can be adapted for this purpose as follows:

- identify the problem
- collect the relevant data

- construct a mathematical model to represent the problem
- select a solution method
- derive a solution to the model
- test the model and evaluate the solution
- implement and maintain the solution.

There can be interaction between these steps.

1.3 What is a model?

A model is a representation of a real entity, and may be constructed in order to gain some understanding of, or insight into, that entity. A model should be realistic enough to incorporate the important characteristics of the real-life system it represents, but not so complex as to hide those characteristics.

Three types of model are generally distinguished:

- Iconic models are physical representations of real objects, designed to resemble those objects in appearance. For example, when a tall building is planned, engineers may construct a small-scale model of that building, and of its surrounding area and environment, in order to conduct stress tests in a wind tunnel.
- Analogue models are also physical models, but represent the entities under study by analogue rather than by replica. An example is a graph showing the movements over time of stock market prices; this type of model provides a pictorial representation of numerical data.
- Mathematical models, or symbolic models, are more abstract representations than iconic or analogue models. They attempt to provide, for example, through an equation or system of equations, a description of a real-life system.

Operations research makes extensive use of mathematical models. To be used successfully, a mathematical model must meet the following criteria:

(a) The model should be as simple and understandable as possible.

(b) The model should be reasonable. Its structures should constrain answers to a reasonable range of values and make it difficult for unrealistic answers to result from inputs.

(c) The model should be easy to maintain.

(d) The model should be adaptive. The parameters and structures of the model should be easy to change as new insights and information evolve.

(e) The model should be complete on important issues.

1.4 What does a mathematical model look like?

Consider a business that manufactures and sells a product. The product costs R5 to manufacture and sells for R20. A model that computes the total profit that will accrue from the items sold is given by the following equation:

$$Z = 20x - 5x.$$

Here x represents the number of units of the product that are manufactured and sold, and

Z represents the total profit in rand that will result from the sale of the product.

The symbols, x and Z, are referred to as variables. The term *variable* is used because no set numerical value has been specified for these items. The number of units manufactured and sold and the resulting profit can be any amount (within limits) – they can vary.

These two variables can be further distinguished as follows:

- The variable Z is known as a *dependent* variable because its value is dependent on the number of units manufactured and sold.
- The variable x is an *independent* variable, since the number of units manufactured and sold is not dependent upon anything else (in this model). It is also called the *decision variable*, because its value is usually determined by a conscious decision made by someone with the authority to do so.

The numbers 20 and 5 in the equation are referred to as *parameters.* Parameters are constant values that are generally coefficients of the variables in an equation. Parameters may change in the longer term but usually remain constant for the duration of solving a specific problem, for example, the price of the product may change over time but is fixed for the time being.

The equation as a whole is known as a *functional relationship.* This term is derived from the fact that profit, Z, is a *function* of the number of units manufactured and sold, x.

Since only one functional relationship exists in this example, it is also the model. This model does not represent a real problem – it merely states a functional relationship in a mathematical form – and we expand our example to create a problem situation.

Let us assume that the product is made from steel and that 500 grams of steel is needed to make one unit of the product, and that a total of 100 kilograms of steel is available for production.

If one unit of the product uses 500 grams of steel, then the total number of products manufactured, x, uses $500 \times x$ grams of steel, or $0,5x$ kilograms of steel.

The steel used to manufacture the products may obviously not exceed the steel available for production. A mathematical inequality representing this relationship between steel used and steel available can now be developed and is as follows:

$$0,5x \leq 100 \quad \text{(steel utilisation in kilograms).}$$

The "less than or equal to" sign, $\leq$, is used as *no more than* 100 kilograms may be used.

A negative number of products can obviously not be manufactured and this can be expressed mathematically as $x \geq 0$. The "greater than or equal to" sign, $\geq$, is used as *at least* zero units of the product must be manufactured.

The model now consists of three relationships:

$$\begin{aligned} Z &= 20x - 5x \\ 0,5x &\leq 100 \\ x &\geq 0. \end{aligned}$$

To add some flavour to the model, we now assume that a manager, say the production manager, must decide on the number of units of the product to manufacture. What objective do you think he will have in mind when having to decide on this? Surely to achieve as much profit as possible. The ideal is an infinite profit, but this cannot be reached in practice because of the limited availability of steel. Based on this observation we can now make a distinction between the relationships in the model.

The equation $Z = 20x - 5x$ represents profit and is called the *objective function* of the model. The inequality $0,5x \leq 100$ represents the steel utilisation and is called the resource *constraint*. The inequality $x \geq 0$ specifies the numerical values that the variable x may assume. It is also a constraint and is called the *sign restriction*.

To emphasise the distinction between the objective function and the constraints, the model is written as follows:

$$\begin{array}{ll} \text{Maximise} & Z = 20x - 5x \\ \text{subject to} & \\ & 0,5x \leq 100 \\ \text{and} & x \geq 0. \end{array}$$

This is an example of a very simple model.

1.5 Linear algebra

Linear algebra is the part of algebra that deals with the theory of linear equations and linear transformation in which the specific properties of vector spaces are studied (including matrices).

A matrix (plural matrices) is a rectangular array of numbers. We will do a number of elementary, entrywise operations such as matrix addition and matrix multiplication. The latter operation connects matrices to linear functions, i.e., functions of the form $f(x) = c \times x$, where c is a constant. This map corresponds to a matrix with one row and column, with entry c.

In general matrices are used to keep track of the coefficients of linear equations and to record other data that depend on multiple parameters.

This concept was also one of the historical roots of matrices. Matrices have a long history of application in solving linear equations. The Chinese text from between 300 BC and AD 200, The Nine Chapters on the Mathematical Art (Jiu Zhang Suan Shu), is the first example of the use of matrix methods to solve simultaneous equations.[Wiki]

In the case of square matrices, matrices with equal number of columns and rows, more refined calculations notably the determinant and inverse matrices are done. Both these govern solution properties of the system of linear equations represented by the matrix.

Matrices find many applications.

1.6 Mathematical programming and Linear programming

In the world of quantitative analysis, the word “programming” refers to the *modelling and solving* of practical problems. “Modelling” refers to the construction of a model to represent a problem situation. Our interest is in mathematical models.

“Mathematical programming” therefore means the construction of mathematical models of real-life problem situations and their solutions. All mathematical programming models consist of an objective function that has to be maximised or minimised, and a set of constraints.

“Linear programming” is one category of mathematical programming. Linear programming models are distinguished by the fact that the objective function and the constraints are linear. This means that each term in an equation or inequality is either a number or a number multiplied by a symbol. There are no squares, cubes, cross-products or other funny things present. Linear programming is often denoted by LP and in this module we will follow this convention.

From the above we see that the word “programming” has a special meaning and should not be confused with computer programming. Computer programming has, however, played an important role in the advancement and use of operations research techniques. Most real-life problems are too complex to solve by hand or even with a calculator, and require the use of computer packages.

1.7 Properties and assumptions of linear programming

Linear programming has been applied extensively in the past to military, industrial, financial, marketing, accounting and agricultural problems. Today LP continues to be used in many different fields and even though the applications are so diverse, all LP problems have four properties in common.

Linear programming problems have the following *properties*:

(a) All problems seek to optimise (maximise or minimise) some quantity (usually profit or cost). This property is referred to as the *objective* of the problem. This objective must be clearly stated and mathematically defined in the form of an objective function.

(b) There are restrictions, called *constraints*, on the problem. These constraints limit the degree to which the objective of the problem can be pursued.

(c) There are *alternative courses of action* to choose from. Say, for example, a company manufactures three different products. Then management must decide how to allocate its limited production resources between the three products.

(d) The objective and constraints of a problem are expressed as equations and inequalities and each of these is *linear*.

Linear programming is based on the following *assumptions*:

(a) *Certainty.* This means that all the parameters (numerical values) in the objective function and constraints are known with certainty and do not change during the period being studied.

(b) *Proportionality.* Proportionality exists in the objective function and constraints. Say, for example, that the production of one unit of a product uses three hours of a resource. Then making 10 units of the product will use 10×3 hours of the resource.

(c) *Additivity.* This means that the total of all activities equals the sum of the individual activities. Say, for example, that the objective is to maximise the profit resulting from the sale of two products, and the profit contribution of the first product is R5 and the profit contribution of the second product is R7. Then the total profit resulting from the manufacture of one unit of each product will be R5 + R7 = R12.

(d) *Divisibility.* This means that the actual values of the decision variables need not be in whole numbers (integers). They may take on fractional values, that is, they are divisible. For example, the production of 10,4 computers per day is quite acceptable since the production process is continuous. The rest of the eleventh computer can be completed on the following day. If integer answers are required for practical purposes, rounding off the values to the nearest integers may yield reasonable results.

Sometimes a fractional solution is not acceptable and an integer solution must be forced. In this case, the model must specify that an integer solution is required and this is called integer programming. Integer programming will not be discussed in this module.

1.8 Other operations research techniques

LP is just part of the vast number of techniques available for problem solving and decision making that form part of the tool kit provided by operations research. A few others are inventory control techniques, network techniques (network flow, CPM/PERT), probabilistic techniques (game theory, markov analysis, simulation, forecasting, etc.), nonlinear techniques and other linear techniques such as integer and goal programming.

A schematic representation of some of the techniques of operations research and their areas of application is given in Figure 1.1.

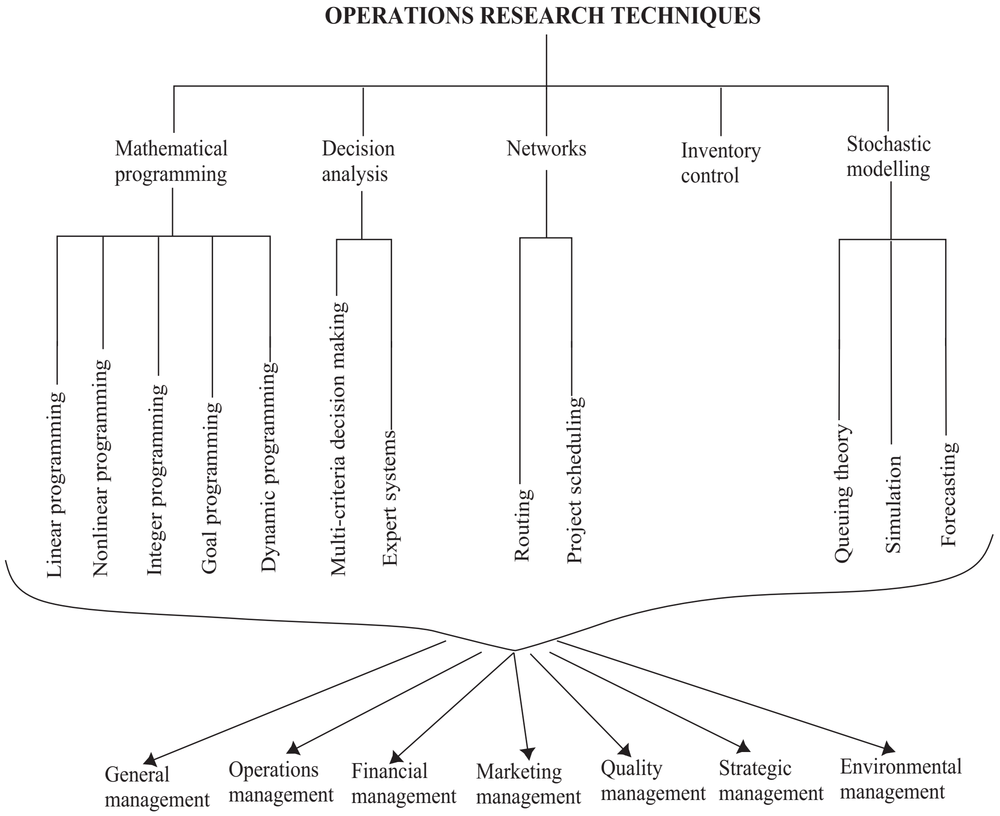

Figure 1.1: A schematic representation

Part 1

Basic Linear Algebra

Learning Unit 2

System of Linear Equations

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Learning objectives

After completing this study unit you should be able to

- do equation operations
- solve a system of linear equation by the elimination method

2.1 System of Linear Equations: Definitions

We know from elementary geometry that a straight line in the xy-plane can be represented algebraically by an equation of the form

$$a_1x + a_2y = b, \qquad E_1$$

where a_1, a_2 and b are constants. Similarly, in three dimensions a plane in a rectangular xyz-coordinate system can be represented by

$$a_1x + a_2y + a_3z = b, \qquad E_2$$

where a_1, a_2, a_3 and b are also constants. Equations E_1 and E_2 are typical examples of a linear equation. We now give the general definition of a linear equation:

Definition 2.1 Linear Equation.

A linear equation in the variables x_1, $x_2, \ldots, x_n$ *is an equation of the form*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where a_1, a_2, $\ldots$, a_n *and* b *are real or complex numbers. If* $b = 0$, *the equation is called a "****homogenous linear equation****".*

For example $2x+4y-z+19w = 11$ is a linear equation in the variables x, y, z and w,, $6x-2y+z = 0$ is a homogenous linear equation in the variables x, y and z, but $2x+9y-5z^2 = 4$ is not a linear equation (because of the appearance of z^2).

Definition 2.2 System of linear equations.

Suppose that m *and* n *are positive integers. A system of* m *linear equations in the variables* $x_1,\ x_2, \ldots, x_n$ *is a set of equations of the form*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots\ \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots\ \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots\ \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

where $a_{ij}\,(i = 1, \ldots, m;\ j = 1, \ldots,\ n)$ *and* $b_i\,(i = 1,\ \ldots,\ m)$ *are real or complex numbers. This system of linear equations is said to consist of* m *equations with* n *unknowns (or variables). We will denote the* i^{th} *equation by* E_i.

For example,

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & - & 5x_3 & + & 4x_4 & - & 2x_5 & = & 4 \\ -4x_1 & + & x_2 & & & - & 5x_4 & & & = & -1 \\ 3x_1 & - & 2x_2 & & & + & 4x_4 & - & 2x_5 & = & 2 \end{array}$$

is a system of three linear equations in five unknowns. The “unknowns” or “variables” of the system are x_1, x_2, x_3, x_4 and x_5.

Definition 2.3 Solution of a linear system.

A solution of the linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

is an ordered n*-tuple of numbers* $(s_1,\ s_2, \ldots, s_n)$ *such that the equations in the system are satisfied when the substitution* $x_1 = s_1,\ x_2 = s_2, \ldots, x_n = s_n$ *is made.*

Example 2.1

The 5-tuple $(0,\ -1,\ -\frac{12}{5},\ 0,\ 3)$ is a solution of the linear system

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & - & 5x_3 & + & 4x_4 & - & 2x_5 & = & 4 \\ -4x_1 & + & x_2 & & & - & 5x_4 & & & = & -1 \\ 3x_1 & - & 2x_2 & & & + & 4x_4 & - & 2x_5 & = & -4 \end{array}$$

because after making the substitution

$$\begin{array}{rcr} x_1 & = & 0 \\ x_2 & = & -1 \\ x_3 & = & -\frac{12}{5} \\ x_4 & = & 0 \\ x_5 & = & 3 \end{array}$$

we get

$$\begin{array}{rcrcrcrcrcr} 2(0) & + & 2(-1) & - & 5\left(-\frac{12}{5}\right) & + & 4(0) & - & 2(3) & = & 4 \\ -4(0) & + & (-1) & & & - & 5(0) & & & = & -1 \\ 3(0) & - & 2(-1) & & & + & 4(0) & - & 2(3) & = & -4. \end{array}$$

Similarly, the 5-tuple $(1,\ 3,\ \frac{3}{5},\ 0,\ \frac{1}{2})$ is also a solution of the system, but the 5-tuple $(1,\ 3,\ \frac{3}{5},\ 0,\ 1)$ is not a solution of the system simply because

$$2(1) + 2(3) - 5(\frac{3}{5}) + 4(0) - 2(1) = 3 \neq 4$$

and in the third equation

$$3(1) - 2(3) + 4(0) - 2(1) = -5 \neq -4.$$

Definition 2.4 Solution set.
The solution set of a system of linear equations is the set of all solutions of the system. A linear system is said to be consistent if it has at least one solution; otherwise it is inconsistent.

2.2 Solving systems of linear equations

In this section, we investigate the existence of different types of solution sets of a system of linear equations. Before we do that, let's give some additional important definitions:

Definition 2.5 Equivalent Systems. *Two linear systems are said to be equivalent to each other if they have the same solution set.*

For example, the following linear systems are equivalent

$$\left.\begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 5 \\ x_1 & - & 2x_2 & = & 1 \end{array}\right\}$$

$$\left.\begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 5 \\ -2x_1 & + & 4x_2 & = & -2 \end{array}\right\}$$

$$\left.\begin{array}{rcrcr} 2x_1 & + & 3x_2 & = & 5 \\ & & 7x_2 & = & 3 \end{array}\right\}$$

2.2.1 Equation Operations

Given a system of linear equations, the following three operations will transform the system into an equivalent system, and each operation is known as an equation operation.

Note: E_i denotes Equation i and $E_i^{(k)}$ denotes Equation i after k transformations.

Operation 1. Swap the location of two equations in the list of equations (eg. $E_i^{(1)} = E_j$ and $E_j^{(1)} = E_i$).

For example, the system

$$\begin{array}{rcrcrcrl} & & x_2 & + & 2x_3 & = & 3 & \qquad E_1 \\ x_1 & - & 2x_2 & + & x_3 & = & 1 & \qquad E_2 \\ 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 & \qquad E_3 \end{array}$$

is equivalent to

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & - & 2x_2 & + & x_3 & = & 1 \\ & & x_2 & + & 2x_3 & = & 3. \end{array}$$

We have swaped the location of the first and third equations, i.e. $E_1^{(1)} = E_3$ and $E_3^{(1)} = E_1$.

Operation 2. Multiply each term of an equation by a nonzero number (eg. $E_i^{(1)} = cE_i$, $c \neq 0$).

For example, the system

$$\begin{array}{rcrcrcrl} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 & \quad E_1 \\ x_1 & - & 2x_2 & + & x_3 & = & 1 & \quad E_2 \\ & & x_2 & + & 2x_3 & = & 3 & \quad E_3 \end{array}$$

is equivalent to

$$\begin{array}{rcrcrcrl} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 & \quad E_1^{(1)} \\ -4x_1 & + & 8x_2 & - & 4x_3 & = & -4 & \quad E_2^{(1)} \\ & - & 8x_2 & - & 16x_3 & = & -24 & \quad E_3^{(1)} \end{array}$$

Here we have multiplied the second equation E_2 by -4 and the third equation E_3 by -8.

Operation 3. Take an equation and add a nonzero multiple of another equation to it (eg. $E_i^{(1)} = E_i + cEj,\ i \neq j,\ c \neq 0$).

For example, the system

$$\begin{array}{rcrcrcrl} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 & \quad E_1 \\ x_1 & - & 2x_2 & + & x_3 & = & 1 & \quad E_2 \\ & & x_2 & + & 2x_3 & = & 3 & \quad E_3 \end{array}$$

is equivalent to

$$\begin{array}{rcrcrcrl} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 & \quad E_1^{(1)} \\ & - & \frac{5}{2}x_2 & + & \frac{1}{4}x_3 & = & \frac{1}{2} & \quad E_2^{(1)} \\ & & x_2 & + & 2x_3 & = & 3 & \quad E_3^{(1)} \end{array}$$

Here we have transformed the second equation E_2 by performing the operation $E_2^{(1)} = E_2 - \frac{1}{4}E_1$.

We note that the original system and the transformed system are equivalent, that means they have the same solution set. Equation operations are necessary tools to complete our strategy for solving systems of equations. We will use these operations to move from one system to another, all the while keeping the solution set the same.

2.2.2 Types of solution sets

System of linear equations with only one solution: Singleton solution set

Example 2.2

(a) We solve the following system

$$\begin{array}{rcrcll} x_1 & + & x_2 & = & 2 & \qquad E_1 \\ 2x_1 & + & 4x_2 & = & 4 & \qquad E_2 \end{array}$$

We keep the first equation E_1 as it is and we eliminate the variable x_1 in the second equation E_2. This means the new equivalent linear system has equations $E_1^{(1)}$ and $E_2^{(1)}$ such that

$$\begin{array}{rcl} E_1^{(1)} & = & E_1 \\ E_2^{(1)} & = & E_2 - 2E_1 \end{array}$$

that is

$$\begin{array}{rcrcll} x_1 & + & x_2 & = & 2 & \qquad E_1^{(1)} \\ & & 2x_2 & = & 0 & \qquad E_2^{(1)} \end{array}$$

which is equivalent to

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & 2 \\ & & x_2 & = & 0 \end{array}$$

By substitution of $x_2 = 0$ in the equation $x_1+x_2 = 2$, we get $x_1 = 2$. The solution is then given by

$$x_1 = 2 \text{ and } x_2 = 0.$$

(b) Let's solve the following system of three equations in three unknowns

$$\begin{array}{rcrcrcll} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & \quad E_1 \\ x_1 & + & 3x_2 & + & 3x_3 & = & 5 & \quad E_2 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & \quad E_3 \end{array}$$

- **Step 1.** We keep the first equation E_1, we transform the second equation E_2 by performing the operation $E_2^{(1)} = E_2 - E_1$ and we keep the third equation as it is. We get the following equivalent system:

$$\begin{array}{rcrcrcllcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & \quad E_1^{(1)} & = & E_1 \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 1 & \quad E_2^{(1)} & = & E_2 \; - \; E_1 \\ 2x_1 & + & 6x_2 & + & 5x_3 & = & 6 & \quad E_3^{(1)} & = & E_3 \end{array}$$

- **Step 2.** We keep the first equation $E_1^{(1)}$, we also keep the second equation $E_2^{(1)}$ and we transform the third by using the operation $E_3^{(2)} = E_3^{(1)} - 2E_1^{(1)}$. We get the following equivalent system

$$\begin{array}{rcrcrcrlcll} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & E_1^{(2)} & = & E_1^{(1)} & \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 1 & E_2^{(2)} & = & E_2^{(1)} & \\ 0x_1 & + & 2x_2 & + & 1x_3 & = & -2 & E_3^{(2)} & = & E_3^{(1)} & - \ 2E_1^{(1)} \end{array}$$

  It is also possible to combine the transformations used in steps 1 and 2 in just one step.

- **Step 3.** We keep the first and second equations and we transform again the third equation by using the operation $E_3^{(3)} = E_3^{(2)} - 2E_2^{(2)}$. We get:

$$\begin{array}{rcrcrcrlcl} x_1 & + & 2x_2 & + & 2x_3 & = & 4 & E_1^{(3)} & = & E_1^{(2)} \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 1 & E_2^{(3)} & = & E_2^{(2)} \\ 0x_1 & + & 0x_2 & - & 1x_3 & = & -4 & E_3^{(3)} & = & E_3^{(2)} - 2E_2^{(2)} \end{array}$$

- **Step 4.** Keep the first and second equations as they are and transform once more the 3rd equation by the operation $E_3^{(4)} = -E_3^{(3)}$. We get

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ 0x_1 & + & 1x_2 & + & 1x_3 & = & 1 \\ 0x_1 & + & 0x_2 & + & 1x_3 & = & 4 \end{array}$$

  which can be written more clearly as

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 4 \\ & & x_2 & + & x_3 & = & 1 \\ & & & & x_3 & = & 4. \end{array}$$

  By substitution, we have

$$\begin{array}{rcl} x_3 & = & 4 \\ x_2 + 4 & = & 1 \Rightarrow x_2 = -3 \\ x_1 + 2(-3) + 2(4) & = & 4 \Rightarrow x_1 = 2. \end{array}$$

  The solution is then given by

$$x_1 = 2;\ x_2 = -3 \text{ and } x_3 = 4$$

or we can write $(x_1,\ x_2,\ x_3) = (2,\ -3,\ 4)$ which is the unique solution to the original system of equations. The solution set is $\{(2,\ -3,\ 4)\}$, which is a singleton set.

Example 2.3

System of linear equations with infinitely many solutions: Infinite solution set

(a) Given the following system

$$\begin{array}{rcrcrl} 4x_1 & + & 2x_2 & = & 3 & \quad E_1 \\ 12x_1 & + & 6x_2 & = & 9 & \quad E_2 \end{array}$$

let's find its solution:

Step 1. We multiply E_1 by $\frac{1}{4}$. We get the following equivalent system

$$\begin{array}{rcrcrlcl} x_1 & + & \frac{1}{2}x_2 & = & \frac{3}{4} & \quad E_1^{(1)} & = & \frac{1}{4}E_1 \\ 12x_1 & + & 6x_2 & = & 9 & \quad E_2^{(1)} & = & E_2. \end{array}$$

Step 2. We keep the first equation as it is and perform the operation $E_2^{(1)} - 12E_1^{(1)}$ on the second equation. We get the following system

$$\begin{array}{rcll} x_1 + \dfrac{1}{2}x_2 & = & \dfrac{3}{4} & \qquad E_1^{(2)} = E_1^{(1)} \\ 0x_1 + 0x_2 & = & 0 & \qquad E_2^{(2)} = E_2^{(1)} - 12E_1^{(1)} \end{array}$$

which can be written more clearly as

$$\begin{array}{rcl} x_1 + \dfrac{1}{2}x_2 & = & \dfrac{3}{4} \\ 0 & = & 0. \end{array}$$

What does the equation $0 = 0$ mean? We can choose any values for x_1 or x_2 and this equation will always be true. Let us choose $x_2 = t$, where t is any real number. Then we obtain

$$x_1 + \frac{1}{2}t = \frac{3}{4}.$$

Hence $x_1 = \frac{3}{4} - \frac{1}{2}t$.

The solution is then given by

$$x_1 = \frac{3}{4} - \frac{1}{2}t \quad ; \quad x_2 = t, \quad \text{where } t \in \mathbb{R}.$$

We can see that for every chosen value of t there corresponds a solution. For example, if we choose $t = 0$, then we have

$$\begin{aligned} x_1 &= \frac{3}{4} \\ x_2 &= 0. \end{aligned}$$

If $t = \frac{1}{2}$; then

$$\begin{aligned} x_1 &= \frac{1}{2} \\ x_2 &= \frac{1}{2}, \end{aligned}$$

and so forth. The entire solution set is written as

$$\begin{aligned} S &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} - \frac{1}{2}t \\ t \end{bmatrix} ;\ t \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ 0 \end{bmatrix} ;\ t \in \mathbb{R} \right\}. \end{aligned}$$

In this example, we see that the solution set S is an infinite set.

(b) Solve the following linear system

$$\begin{array}{ccccccccl} & - & 2x_2 & + & 3x_3 & = & 0 & & E_1 \\ x_1 & + & x_2 & - & 2x_3 & = & 4 & & E_2 \\ 2x_1 & & & - & x_3 & = & 8 & & E_3 \end{array}$$

Step 1. We swap the locations of equations E_1 and E_2 and get

$$\begin{array}{ccccccclcl} x_1 & + & x_2 & - & 2x_3 & = & 4 & E_1^{(1)} & = & E_2 \\ & - & 2x_2 & + & 3x_3 & = & 0 & E_2^{(1)} & = & E_1 \\ 2x_1 & & & - & x_3 & = & 8 & E_3^{(1)} & = & E_3 \end{array}$$

Step 2. We keep the second equation $E_2^{(1)}$ and transform the third equation $E_3^{(1)}$ by performing the operation $E_3^{(2)} = E_3^{(1)} - 2E_1^{(1)}$. We get

$$\begin{array}{ccccccclcl} x_1 & + & x_2 & - & 2x_3 & = & 4 & E_1^{(2)} & = & E_1^{(1)} \\ & - & 2x_2 & + & 3x_3 & = & 0 & E_2^{(2)} & = & E_2^{(1)} \\ 0x_1 & - & 2x_2 & + & 3x_3 & = & 0 & E_3^{(2)} & = & E_3^{(1)} - 2E_1^{(1)} \end{array}$$

Step 3. We divide the second equation by -2 and get the system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 2x_3 & = & 4 \qquad E_1^{(3)} = E_1^{(2)} \\ 0x_1 & + & x_2 & - & \frac{3}{2}x_3 & = & 0 \qquad E_2^{(3)} = -\frac{1}{2}E_2^{(2)} \\ 0x_1 & - & 2x_2 & + & 3x_3 & = & 0 \qquad E_3^{(3)} = E_3^{(2)} \end{array}$$

Step 4. We keep the first and second equations and transform the third equation by performing the operation $E_3^{(4)} = E_3^{(3)} + 2E_2^{(3)}$. We get

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 2x_3 & = & 4 \qquad E_1^{(4)} = E_1^{(3)} \\ 0x_1 & + & x_2 & - & \frac{3}{2}x_3 & = & 0 \qquad E_2^{(4)} = E_2^{(3)} \\ 0x_1 & - & 0x_2 & + & 0x_3 & = & 0 \qquad E_3^{(4)} = E_3^{(3)} + 2E_2^{(3)} \end{array}$$

This can be clearly written as

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 2x_3 & = & 4 \\ 0x_1 & + & x_2 & - & \frac{3}{2}x_3 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

As the equation $0 = 0$ is always true, we can choose $x_3 = t$, for all $t \in \mathbb{R}$. Then we have

$$\begin{aligned} x_3 &= t, \\ x_2 - \frac{3}{2}t = 0 \Rightarrow x_2 &= \frac{3}{2}t, \\ x_1 + \frac{3}{2}t - 2t = 4 \Rightarrow x_1 &= 4 + \frac{1}{2}t. \end{aligned}$$

The general solution is then given by

$$\begin{aligned} x_1 &= 4 + \frac{1}{2}t, \\ x_2 &= \frac{3}{2}t, \\ x_3 &= t, \ t \in \mathbb{R}. \end{aligned}$$

The set of solutions can be written as

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + \frac{1}{2}t \\ \frac{3}{2}t \\ t \end{bmatrix} ; \ t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} ; \ t \in \mathbb{R} \right\}.$$

If we want to avoid fractions in the above solution, we could choose $x_3 = 2s$. Then $x_2 = 3s$ and $x_1 = 4 + s$. In this case, the general solution is

$$\begin{aligned} x_1 &= 4 + s \\ x_2 &= 3s \\ x_3 &= 2s, \ s \in \mathbb{R} \end{aligned}$$

and the solution set is

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+s \\ 3s \\ 2s \end{bmatrix} ; \ s \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} ; \ s \in \mathbb{R} \right\}.$$

The solution set of each of the linear systems considered in the above example consists of infinitely many solutions. Therefore we have an infinite solution set.

Example 2.4

System of linear equations without a solution: Empty solution set

(a) We solve the following system

$$\begin{aligned} x_1 + 2x_2 &= 3 \qquad E_1 \\ 2x_1 + 4x_2 &= 4 \qquad E_2 \end{aligned}$$

We do the same as in the previous examples. We keep the first equation and we eliminate the variable x_1 in the second equation by using the operation $E_2^{(1)} = E_2 - 2E_1$. We get:

$$\begin{aligned} x_1 + 2x_2 &= 3 \qquad E_1^{(1)} = E_1 \\ 0x_1 + 0x_2 &= -2 \qquad E_2^{(1)} = E_2 - 2E_1 \end{aligned}$$

We see that whatever values we give to x_1 and x_2, the equation $E_2^{(1)}$ can never be satisfied. Thus the system does not have a solution.

(b) Once again, let's solve the following system

$$\begin{array}{rcrcrl} 2x_1 & + & x_2 & = & 3 & \quad E_1 \\ 2x_1 & + & x_2 & = & 0 & \quad E_2 \\ x_1 & + & 2x_2 & = & 4 & \quad E_3 \end{array}$$

By using row operations (please check!) we find the equivalent system

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 3 \\ 0x_1 & + & 0x_2 & = & 3 \\ x_1 & + & 2x_2 & = & 4 \end{array}$$

In this system, the second equation cannot be satisfied for any value of x_1 and x_2 and therefore there *is no solution* for this system. The solution set of each of the linear systems considered in the above example has no elements and is called the empty set, denoted by ϕ or $\{\ \ \}$.

From the examples studied in this section we see that a system of linear equations has either exactly one solution, infinitely many solutions or no solution. Thus the solution set of a linear system is either a singleton set, or an infinite set, or an empty set.

2.3 Evaluation Exercises

Find all solutions to the following systems of linear equations:

(a)

$$\begin{array}{rcrcr} 3x & + & 2y & = & 4 \\ 6x & + & 4y & = & 8 \end{array}$$

(b)

$$\begin{array}{rcrcr} 3x & + & 2y & = & 4 \\ 6x & + & 4y & = & -2 \end{array}$$

(c)

$$\begin{array}{rcrcr} 3x & + & 2y & = & 1 \\ x & - & y & = & 2 \\ 4x & + & 2y & = & 2 \end{array}$$

(d)

$$\begin{aligned} x &+ 2y &= 8 \\ x &- y &= 2 \\ x &+ y &= 4 \end{aligned}$$

(e)

$$\begin{array}{rcrcrcr} x &+& y &-& z &=& -1 \\ x &-& y &-& z &=& -1 \\ && && z &=& 2 \end{array}$$

(f)

$$\begin{array}{rcrcrcr} x &+& y &-& z &=& -5 \\ x &-& y &-& z &=& -3 \\ x &+& y &-& z &=& 0 \end{array}$$

(g)

$$\begin{array}{rcrcrcr} 2x_1 &+& x_2 &+& 3x_3 &=& 12 \\ x_1 &+& 2x_2 &+& 2x_3 &=& 3 \\ 3x_1 &+& x_2 &-& x_3 &=& 11 \end{array}$$

Learning Unit 3

Matrices and vectors

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Learning objectives

After completing this study unit you should be able to

- understand what matrices and vectors are
- define different types of matrices
- do matrix operations
- calculate the determinant of a matrix
- transform a matrix into its reduced row echelon form
- find the inverse of a matrix.

Sections from prescribed book

2.1. Matrices and Vectors - *p11*

2.2. Matrices and System of Linear Equations - *p20*

2.5. Inverse of a Matrix - *p36*

2.6. Determinants - *p42*

3.1 Definitions

Roughly speaking, a system of linear equations can be well represented with tables. For example, the system given by

$$\begin{array}{rcrcrcr} 4x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & - & 2x_2 & + & x_3 & = & 1 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$$

can be written as

$$\begin{bmatrix} 4 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

or more compactly as,

$$\mathbf{Ax} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 3 \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

The table **A** is called a matrix and the tables **x** and **b** are called vectors. In addition, many types of numerical data are best displayed in a table. For example, suppose that a company owns two bookstores (Bookstore 1 and Bookstore 2), each of which sells newspapers, magazines and books. Assume that the sales, in rands, of the two bookstores are represented by the following table:

	Bookstore 1	**Bookstore 2**
Newspapers	4 000	5 000
Magazines	11 000	13 000
Books	38 000	42 000

This shows that Bookstore 1 sold R4 000 worth of newspapers, R11 000 worth of magazines and R38 000 worth of books. Similarly Bookstore 2 sold R5 000 worth of newspapers, R13 000 worth of magazines and R42 000 worth of books.

The above information can be simply represented as

$$\begin{bmatrix} 4\,000 & 5\,000 \\ 11\,000 & 13\,000 \\ 38\,000 & 42\,000 \end{bmatrix}.$$

Such a table of numbers is also called a “matrix”. Hereunder we give some fundamental concepts of matrices.

Definition 3.1 Matrix.

A matrix is a collection of numbers ordered by rows (horizontally) and by columns (vertically).
If a matrix $\boldsymbol{A}$ *has* m *rows and* n *columns, we say that the size of the matrix* $\boldsymbol{A}$ *is* m *by* n*, written as* $m \times n$*. The matrix* $\boldsymbol{A}$ *can then be represented by*

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

where a_{ij}; $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$. *This is usually denoted simply as* $\boldsymbol{A} = [a_{ij}]$. *We say that* a_{ij} *is the entry (or element) in row* i *and column* j. *The* i*th row of* $\boldsymbol{A}$ *is*

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}] \qquad (1 \leq i \leq m);$$

and the j*th column of* $\boldsymbol{A}$ *is*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \leq j \leq n).$$

If $m = n$, *we say that* $\boldsymbol{A}$ *is a* ***square matrix*** *of order* n *and the numbers* $a_{11},\ a_{22},\ \ldots,\ a_{nn}$ *form the* ***principal diagonal*** *of* $\boldsymbol{A}$.

If all the elements of an $m \times n$ *matrix* $\boldsymbol{A}$ *are zero, we say that* $\boldsymbol{A}$ *is the "**zero matrix**" or "**null matrix**" generally denoted by* $\boldsymbol{O}_{m,n}$. *If it is unimportant to emphasize the size, we denote the null matrix by* $\boldsymbol{O}$.

Example 3.1

(a) The matrix **X** defined by

$$\mathbf{X} = \begin{bmatrix} 5 & 8 & 2 \\ -1 & 0 & 7 \end{bmatrix}$$

is a 2×3 matrix. The entries of this matrix are

$$x_{11} = 5\ ;\quad x_{12} = 8\ ;\quad x_{13} = 2$$

$$x_{21} = -1\ ;\quad x_{22} = 0\ ;\quad x_{23} = 7.$$

(b) The matrix **M** defined by

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

is a square matrix of order 3. The entries of the matrix **M** are

$$m_{11} = 1\ ;\quad m_{12} = 1\ ;\quad m_{13} = 0$$

$$m_{21} = 2\ ;\quad m_{22} = 0\ ;\quad m_{23} = 1$$

$$m_{31} = 3\ ;\quad m_{32} = -1\ ;\quad m_{33} = 2.$$

The entries $m_{11} = 1$; $m_{22} = 0$ and $m_{33} = 2$ form the principal diagonal.

(c) The following are null matrices: $\mathbf{O}_{1;1} = \begin{bmatrix} 0 \end{bmatrix}$, $\mathbf{O}_{2;2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{O}_{2;3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{O}_{3;3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Definition 3.2 Matrix Equality.

The $m \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal, written $\boldsymbol{A} = \boldsymbol{B}$, if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We may simply say that two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same size and the same corresponding entries or elements.

Definition 3.3 Transpose of a Matrix.

Given an $m \times n$ matrix $\boldsymbol{A} = [a_{ij}]$ its transpose is the $n \times m$ matrix denoted by $\boldsymbol{A}^T = [b_{ij}]$ such that $b_{ij} = a_{ji}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, the transpose of an $m \times n$ matrix **A** is obtained by interchanging the rows and columns of **A**, that is, the first row of **A** becomes the first column of $\mathbf{A}^T$, the second row becomes the second column, etc.

Example 3.2

Let

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 3 & 7 & 2 & -3 \\ -1 & 4 & 2 & 8 \\ 0 & 3 & -2 & 5 \end{bmatrix}.$$

Then the transpose of matrices **A**, **B** and **C** are given respectively by

$$\mathbf{A}^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \ \mathbf{B}^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \text{ and } \mathbf{C}^T = \begin{bmatrix} 3 & -1 & 0 \\ 7 & 4 & 3 \\ 2 & 2 & -2 \\ -3 & 8 & 5 \end{bmatrix}.$$

Definition 3.4 Symmetric Matrix.

The matrix $\boldsymbol{A} = [a_{ij}]$ *is symmetric if* $\boldsymbol{A} = \boldsymbol{A}^T$. *In other words, a symmetric matrix* $\boldsymbol{A}$ *is a square matrix in which* $a_{ij} = a_{ji}$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq n$.

Example 3.3

(a) The following matrices are symmetric:

$$\mathbf{E} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} 9 & 1 & 5 \\ 1 & 6 & 2 \\ 5 & 2 & 7 \end{bmatrix};$$

simply because $\mathbf{E} = \mathbf{E}^T$ and $\mathbf{F} = \mathbf{F}^T$.

(b) The matrix

$$\mathbf{H} = \begin{bmatrix} 9 & 1 & 5 \\ 2 & 6 & 2 \\ 5 & 1 & 7 \end{bmatrix}$$

is not symmetric since $\mathbf{H} \neq \mathbf{H}^T$.

Definition 3.5 Diagonal Matrix.

A diagonal matrix is a symmetric matrix where all the off diagonal elements are zero.

Example 3.4

Matrices

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal while the matrix

$$\mathbf{C} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

is not a diagonal matrix since $c_{23} \neq 0$.

Definition 3.6 Identity Matrix.

An identity matrix is a diagonal matrix with ones on the principal diagonal and zeros elsewhere. The identity matrix of order n is written as

$$\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Example 3.5

The following are identity matrices of order 1, 2, 3 and 4 respectively:

$$\mathbf{I}_1 = \begin{bmatrix} 1 \end{bmatrix}, \ \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Definition 3.7 Row and Column vectors.
emphA matrix that has exactly one row is called a row vector and a matrix that has exactly one column is called a column vector. Most of the time, row and column vectors are just simply called vectors. The elements of a vector are called "**entries**" or "**components**". When all the components are zero, we get the so-called "**zero vector**".

Note: In some textbooks, a vector is represented by a lower case Latin letter from the end of the alphabet with an arrow on top of the letter such as $\overrightarrow{u}$, $\overrightarrow{v}$ or $\overrightarrow{w}$.

Example 3.6

(a) The matrix

$$\mathbf{u} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

is a column vector of dimension 3 while the matrix

$$\mathbf{v} = \begin{bmatrix} 6 & 0 & 4 & 7 \end{bmatrix}$$

is a row vector of dimension 4.

(b) The zero column vector of dimension 2 is given by

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Definition 3.8 Augmented Matrix.

Let $\boldsymbol{A} = [a_{ij}]$ *be an* $m \times n$ *matrix and* $\boldsymbol{B} = [b_{ij}]$ *be an* $m \times p$ *matrix. The augmented matrix* $[\boldsymbol{A}|\boldsymbol{B}]$ *is the matrix obtained by adjoining the matrix* $\boldsymbol{B}$ *to the right side of the matrix* $\boldsymbol{A}$*. In other words*

$$[\mathbf{A}|\mathbf{B}] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1p} \\ a_{21} & a_{22} & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_{m1} & b_{m2} & \dots & b_{mp} \end{array}\right].$$

For example, if

$$\mathbf{A} = \begin{bmatrix} -6 & 1 \\ 0 & 7 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 2 & 3 & -5 \\ -1 & 4 & 0 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$$

then

$$[\mathbf{A}|\mathbf{B}] = \left[\begin{array}{cc|ccc} -6 & 1 & 2 & 3 & -5 \\ 0 & 7 & -1 & 4 & 0 \end{array}\right],$$

$$[\mathbf{B}|\mathbf{A}] = \left[\begin{array}{ccc|cc} 2 & 3 & -5 & -6 & 1 \\ -1 & 4 & 0 & 0 & 7 \end{array}\right],$$

$$[\mathbf{A}|\mathbf{C}] = \left[\begin{array}{cc|c} -6 & 1 & 9 \\ 0 & 7 & 10 \end{array}\right]$$

and

$$[\mathbf{B}|\mathbf{I}_2] = \left[\begin{array}{ccc|cc} 2 & 3 & -5 & 1 & 0 \\ -1 & 4 & 0 & 0 & 1 \end{array}\right].$$

3.2 Matrix Operations

3.2.1 Addition and subtraction of two matrices

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two matrices with the same size $m \times n$. Then the sum $\mathbf{A} + \mathbf{B}$ is the $m \times n$ matrix $\mathbf{C} = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij} \qquad (1 \leq i \leq m, \ 1 \leq j \leq n).$$

Therefore, the sum $\mathbf{C}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ is obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Subtraction works in the same way, except that entries are subtracted instead of added.

Example 3.7

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -1 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 0 & 2 & -4 \\ 1 & 3 & 1 \end{bmatrix}.$$

Then, the *sum* of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}1+0 & -2+2 & 4+(-4)\\ 2+1 & -1+3 & 3+1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 3 & 2 & 4\end{bmatrix}$$

and the *difference* between **A** and **B** is

$$\mathbf{A}-\mathbf{B}=\begin{bmatrix}1-0 & -2-2 & 4-(-4)\\ 2-1 & -1-3 & 3-1\end{bmatrix}=\begin{bmatrix}1 & -4 & 8\\ 1 & -4 & 2\end{bmatrix}.$$

We notice that a matrix $\mathbf{A}=[a_{ij}]$ subtracted from itself gives the zero matrix, that is $\mathbf{A}-\mathbf{A}=\mathbf{O}$.

3.2.2 The scalar multiple of a matrix

Given any matrix **A** and any number (or scalar) c, the matrix c**A** is the matrix obtained by multiplying each entry of **A** by the scalar c.

Example 3.8

If the matrix **A** is given by

$$\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

then

$$2\mathbf{A}=2\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}2\cdot 1 & 2\cdot 2\\ 2\cdot 3 & 2\cdot 4\end{bmatrix}=\begin{bmatrix}2 & 4\\ 6 & 8\end{bmatrix}$$

and

$$-\mathbf{A}=-1\cdot\mathbf{A}=-1\cdot\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}-1\cdot 1 & -1\cdot 2\\ -1\cdot 3 & -1\cdot 4\end{bmatrix}=\begin{bmatrix}-1 & -2\\ -3 & -4\end{bmatrix}.$$

The matrix $-\mathbf{A}$ is referred to as the negative or opposite matrix of the matrix **A**.

Property 3.1 Matrix addition and scalar multiplication.

Let $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *be* $m \times n$ *matrices,* $\mathbf{O}_{mn}$ *be the* $m \times n$ *zero (null) matrix and* p, q *two scalars. Then*

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

(c) $\mathbf{A} + \mathbf{O} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$

(e) $(p\,q)\mathbf{A} = p(q\mathbf{A})$

(f) $p(\mathbf{A} + \mathbf{B}) = p\mathbf{A} + p\mathbf{B}$

(g) $(p + q)\mathbf{A} = p\mathbf{A} + q\mathbf{A}$

We note that the above properties are also applicable for vectors as a vector is a matrix with only one column or row.

3.2.3 Matrix multiplication

In the previous sections, we learnt how to add matrices and how to multiply matrices by scalars. In this section we mix all these ideas together and produce an operation known as "matrix multiplication". We begin with the definition of the product of a vector by a matrix.

Definition 3.9 Matrix-vector product.

Assume that $\boldsymbol{A}$ *is an* $m \times n$ *matrix with columns* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ *and* $\mathbf{u}$ *is a column vector of dimension* n *defined by*

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

The matrix-vector product of $\boldsymbol{A}$ *with* $\mathbf{u}$ *is given by*

$$\mathbf{Au} = [\mathbf{A}_1\, \mathbf{A}_2\, \ldots\, \mathbf{A}_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

or

$$\boldsymbol{Au} = u_1\, \mathbf{A}_1 + u_2\, \mathbf{A}_2 + \ldots + u_n\, \mathbf{A}_n.$$

Example 3.9

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ -3 & 2 & 0 & 1 & -2 \\ 1 & 6 & -3 & -1 & 5 \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}.$$

According to Definition 3.9, we have

$$\mathbf{A}_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \ \mathbf{A}_4 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{A}_5 = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix},$$

and

$$u_1 = 2,\ u_2 = 1,\ u_3 = -2,\ u_4 = 3 \text{ and } u_5 = -1.$$

We then have

$$\mathbf{Au} = 2 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4-4+9-4 \\ -6+2+0+3+2 \\ 2+6+6-3-5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 6 \end{bmatrix}.$$

Definition 3.10 Matrix Multiplication.

Suppose that $\boldsymbol{A}$ *is an* $m \times n$ *matrix and* $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3, \ldots,$ $\mathbf{B}_p$ *are the columns of an* $n \times p$ *matrix* $\boldsymbol{B}$. *Then the matrix product of* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is the* $m \times p$ *matrix, where column* i *is the matrix-vector product* $\mathbf{AB}_i$.

Explicitly,

$$\begin{aligned} \mathbf{AB} &= \mathbf{A}[\mathbf{B}_1 \ \ \mathbf{B}_2 \ldots \mathbf{B}_p] \\ &= [\mathbf{AB}_1 \ \ \mathbf{AB}_2 \ldots \mathbf{AB}_p]. \end{aligned}$$

Example 3.10

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -1 & 3 & 0 \end{bmatrix}.$$

Then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 6 \\ 1 \end{bmatrix} & \mathbf{A}\begin{bmatrix} 6 \\ 4 \\ 1 \\ 4 \\ -1 \end{bmatrix} & \mathbf{A}\begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \\ 3 \end{bmatrix} & \mathbf{A}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 28 & 23 & 20 & 10 \\ 20 & -10 & -3 & -1 \\ -18 & -40 & 12 & -3 \end{bmatrix}.$$

This method of matrix multiplication may be difficult for you to handle.

The theorem below gives an alternative method called matrix multiplication, entry-by-entry (or in some text books element-by-element).

Definition 3.11 Matrix product.

If $\mathbf{A} = [a_{ij}]$ *is an* $m \times n$ *matrix and* $\boldsymbol{B} = [b_{ij}]$ *is an* $n \times p$ *matrix, then the matrix product* $\mathbf{AB} = \mathbf{C} = [\mathbf{c}_{ij}]$ *is the* $m \times p$ *matrix with entries defined by*

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}\, b_{kj} \end{aligned}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq p$.

Note: The product **AB** of two matrices **A** and **B** is defined only if the number of columns of **A** is the same as that number of rows of **B**.

Example 3.11

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{AB} = \begin{bmatrix} (1)(2) + (2)(0) & (1)(1) + (2)(1) \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix},$$

while **BA** is undefined since the number of columns of **B** is not the same as the number of rows of **A**.

Example 3.12

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \qquad \text{while} \qquad \mathbf{BA} = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Example 3.13

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 70 & 60 & 20 \\ 40 & 50 & 60 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}.$$

The product of **AB** is:

$$\mathbf{AB} = \begin{bmatrix} (70)(2)+(60)(3)+(20)(4) & (70)(5)+(60)(6)+(20)(7) \\ (40)(2)+(50)(3)+(60)(4) & (40)(5)+(50)(6)+(60)(7) \end{bmatrix} = \begin{bmatrix} 400 & 850 \\ 470 & 920 \end{bmatrix}.$$

Property 3.2 Matrix multiplication properties.

Suppose that $\boldsymbol{A}$ *is an* $m \times n$ *matrix, and* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are* $n \times p$ *matrices and* $\boldsymbol{D}$ *is a* $p \times s$ *matrix. Then*

1. $\mathbf{AO}_{n\times p} = \mathbf{O}_{m\times p}$
2. $\mathbf{O}_{p\times m}\mathbf{A} = \mathbf{O}_{p\times n}$
3. $\mathbf{AI}_n = \mathbf{A}$
4. $\mathbf{I}_m\mathbf{A} = \mathbf{A}$
5. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
6. $(\mathbf{B}+\mathbf{C})\mathbf{D} = \mathbf{BD}+\mathbf{CD}$
7. $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$ *(Here* α *is a scalar).*
8. $\mathbf{A}(\mathbf{BD}) = (\mathbf{AB})\mathbf{D}$ *(Matrix multiplication is associative).*
9. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

3.3 Determinant of a Matrix

A determinant is a function that takes a square matrix as an input and produces a scalar as an output. This function is recursive, that is, the determinant of a large matrix is defined in terms of the determinants of smaller matrices. This works exactly the same as the factorial function, the only one difference is the factorial is applied to integer numbers (or integer scalars) and the determinant is applied to matrices.

3.3.1 Definitions and examples

Definition 3.12 Submatrix.

Assume that $\boldsymbol{A}$ is an $m \times n$ matrix. Then the submatrix $\mathbf{A}(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by removing row i and column j.

Example 3.14

For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 4 & -2 & 0 & 1 \\ 3 & 5 & 2 & 1 \end{bmatrix}$$

we have the submatrices

$$\mathbf{A}(2|3) = \begin{bmatrix} 1 & -2 & \cancel{3} & 9 \\ \cancel{4} & \cancel{-2} & \cancel{0} & \cancel{1} \\ 3 & 5 & \cancel{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 9 \\ 3 & 5 & 1 \end{bmatrix},$$

$$\mathbf{A}(3|1) = \begin{bmatrix} \cancel{1} & -2 & 3 & 9 \\ \cancel{4} & -2 & 0 & 1 \\ \cancel{3} & \cancel{5} & \cancel{2} & \cancel{1} \end{bmatrix} = \begin{bmatrix} -2 & 3 & 9 \\ -2 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{A}(1|1) = \begin{bmatrix} \cancel{1} & \cancel{-2} & \cancel{3} & \cancel{9} \\ \cancel{4} & -2 & 0 & 1 \\ \cancel{3} & 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 5 & 2 & 1 \end{bmatrix}.$$

Before we define the determinant of an $n \times n$ matrix $\mathbf{A}$ we require the following definition.

Definition 3.13 Minor. *The minor M_{ij} of a square matrix $\mathbf{A}$ is the determinant of the submatrix $\mathbf{A}(i|j)$, that is*

$$M_{ij} = |\mathbf{A}(i|j)|.$$

Definition 3.14 Determinant of a Matrix. *Assume that $\mathbf{A}$ is a square matrix. Then its determinant, $det(\mathbf{A}) = |\mathbf{A}|$ is a scalar defined recursively by:*

1. If $\mathbf{A}$ is a 1×1 matrix, that is $\mathbf{A} = [\alpha]$, where α is a real number then the determinant of $\mathbf{A}$, $|\mathbf{A}| = \alpha$.
2. If $\mathbf{A} = [a_{ij}]$ is a square matrix of size n with $n \geq 2$, then

$$|\mathbf{A}| = a_{11}M_{11} - a_{12}\,M_{12} + a_{13}\,M_{13} - a_{14}M_{14} + \cdots + (-1)^{n+1}\,a_{1n}\,M_{1n}$$

or more generally

$$\begin{aligned} |\mathbf{A}| &= (-1)^{i+1}a_{i1}M_{i1} + (-1)^{i+2}a_{i2}M_{i2} + \cdots + (-1)^{i+j}a_{ij}M_{ij} + \cdots + (-1)^{i+n}a_{in}M_{in} \\ &= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}M_{ij}. \end{aligned}$$

*This is called the cofactor expansion of $|\mathbf{A}|$ along row i. The term "**cofactor**" will be defined in Section 3.4.2*

Example 3.15

(a) The determinant of

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $|\mathbf{A}| = ad - bc$.

This is because by applying Definition 3.14, we have

$$\begin{aligned} |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= a|d| - b|c| \\ &= ad - bc. \end{aligned}$$

(b) Let

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}.$$

To find $|\mathbf{B}|$, let's choose a row from which we can compute minors. Let's say row 1. Then

$$M_{11} = |\mathbf{B}(1|1)| = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{-1} \\ \cancel{5} & 3 & 4 \\ \cancel{-2} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3,$$

$$M_{12} = |\mathbf{B}(1|2)| = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{-1} \\ 5 & \cancel{3} & 4 \\ -2 & \cancel{0} & 1 \end{vmatrix} = \begin{vmatrix} 5 & 4 \\ -2 & 1 \end{vmatrix} = 13$$

and

$$M_{13} = |\mathbf{B}(1|3)| = \begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{-1} \\ 5 & 3 & \cancel{4} \\ -2 & 0 & \cancel{1} \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ -2 & 0 \end{vmatrix} = 6.$$

Then it follows from Definition 3.14 that

$$\begin{aligned} |\mathbf{B}| &= b_{11}\, M_{11} - b_{12}\, M_{12} + b_{13}\, M_{13} \\ &= 1 \cdot 3 - 2 \cdot 13 + (-1) \cdot 6 \\ &= 3 - 26 - 6 = -29. \end{aligned}$$

We note that the choice of the row from which we want to compute minors does not matter. You can choose any row but you need to pay more attention to the signs. If we choose the second row, we have:

$$M_{21} = |\mathbf{B}(2|1)| = \begin{vmatrix} \cancel{1} & 2 & -1 \\ \cancel{5} & \cancel{3} & \cancel{4} \\ \cancel{-2} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2,$$

$$M_{22} = |\mathbf{B}(2|2)| = \begin{vmatrix} 1 & \cancel{2} & -1 \\ \cancel{5} & \cancel{3} & \cancel{4} \\ -2 & \cancel{0} & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = -1$$

and

$$M_{23} = |\mathbf{B}(2|3)| = \begin{vmatrix} 1 & 2 & \cancel{-1} \\ \cancel{5} & \cancel{3} & \cancel{4} \\ -2 & 0 & \cancel{1} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4$$

and hence

$$\begin{aligned} |\mathbf{B}| &= -b_{21}M_{21} + b_{22}\,M_{22} - b_{23}\,M_{23} \\ &= -5\cdot 2 + 3\cdot(-1) - 4(4) \\ &= -10 - 3 - 16 \\ &= -29. \end{aligned}$$

Although we can evaluate a determinant by expanding along any row it will be less work to expand along a row that has the most zeros. In this example it would make sense to expand along the third row since

$$\begin{aligned} |\mathbf{B}| &= b_{31}M_{31} - b_{32}M_{32} + b_{33}M_{33} \\ &= -2\cdot M_{31} - 0\cdot M_{32} + 1\cdot M_{33}. \end{aligned}$$

We then only need to determine two minors, namely M_{31} and M_{33} since $0\cdot M_{32} = 0$.

(c) To calculate the determinant of the 4×4 matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

we calculate, for example, the minors M_{11}, M_{12}, M_{13} and M_{14}:

$$M_{11} = \begin{vmatrix} 0 & 2 & -2 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 0\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 2,$$

$$M_{12} = \begin{vmatrix} -1 & 2 & -2 \\ 3 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -1,$$

$$M_{13} = \begin{vmatrix} -1 & 0 & -2 \\ 3 & -1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$$

and

$$M_{14} = \begin{vmatrix} -1 & 0 & 2 \\ 3 & -1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 3.$$

Therefore, the determinant of $\mathbf{C}$ is

$$|\mathbf{C}| = 1(2) - 2(-1) + (-1)(-2) - 1(3) = 3.$$

Example 3.16

Find all minors of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Solution

$$M_{11} = |\mathbf{A}(1|1)| = \begin{vmatrix} \cancel{1} & \cancel{-1} & \cancel{2} \\ \cancel{2} & -1 & 1 \\ \cancel{1} & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$$

$$M_{12} = |\mathbf{A}(1|2)| = \begin{vmatrix} \cancel{1} & \cancel{-1} & \cancel{2} \\ 2 & \cancel{-1} & 1 \\ 1 & \cancel{1} & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$M_{13} = |\mathbf{A}(1|3)| = \begin{vmatrix} \cancel{1} & \cancel{-1} & \cancel{2} \\ 2 & -1 & \cancel{1} \\ 1 & 1 & \cancel{-1} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3$$

$$M_{21} = |\mathbf{A}(2|1)| = \begin{vmatrix} \cancel{1} & -1 & 2 \\ \cancel{2} & \cancel{-1} & \cancel{1} \\ \cancel{1} & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$$

$$M_{22} = |\mathbf{A}(2|2)| = \begin{vmatrix} 1 & \cancel{-1} & 2 \\ \cancel{2} & \cancel{-1} & \cancel{1} \\ 1 & \cancel{1} & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -3$$

$$M_{23} = |\mathbf{A}(2|3)| = \begin{vmatrix} 1 & -1 & \cancel{2} \\ \cancel{2} & \cancel{-1} & \cancel{1} \\ 1 & 1 & \cancel{-1} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$M_{31} = |\mathbf{A}(3|1)| = \begin{vmatrix} \cancel{1} & -1 & 2 \\ \cancel{2} & -1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{-1} \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} = 1$$

$$M_{32} = |\mathbf{A}(3|2)| = \begin{vmatrix} 1 & \cancel{-1} & 2 \\ 2 & \cancel{-1} & 1 \\ \cancel{1} & \cancel{1} & \cancel{-1} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$M_{33} = |\mathbf{A}(3|3)| = \begin{vmatrix} 1 & -1 & \cancel{2} \\ 2 & -1 & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{-1} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$$

Note: In Example 3.16, the determinant $|\mathbf{A}|$ may either be found using M_{11}, M_{12} and M_{13}, or M_{21}, M_{22} and M_{23}, or M_{31}, M_{32} and M_{33}.

3.3.2 Properties of determinants

In the previous subsection, we learnt how to compute the determinant of a matrix. We give some properties that will help you to compute certain determinants.

1. If **A** is any square matrix, then

$$|\mathbf{A}^T| = |\mathbf{A}|.$$

It follows from this property that a determinant can also be determined by using the minors of a column instead of a row. In this case

$$\begin{aligned}|\mathbf{A}| &= (-1)^{1+j}a_{1j}M_{1j} + (-1)^{2+j}a_{2j}M_{2j} + \cdots + (-1)^{i+j}a_{ij}M_{ij} + \cdots + (-1)^{m+j}a_{mj}M_{mj} \\ &= \sum_{i=1}^{m}(-1)^{i+j}a_{ij}M_{ij}.\end{aligned}$$

This is called the cofactor expansion of $|\mathbf{A}|$ along the column j.

2. If **A** is a square matrix with a row where every entry is zero, or a column where every entry is zero, then $|\mathbf{A}| = 0$.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 4 & 7 \\ 2 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

has a determinant equal to zero.

3. Assume that **A** is a square matrix. Let **B** be the square matrix obtained from **A** by interchanging the location of two rows or two columns. Then $|\mathbf{B}| = -|\mathbf{A}|$.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}$$

has the determinant $|\mathbf{A}| = -29$. Then the matrix **B** defined by

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 5 & 3 & 4 \end{bmatrix}$$

will have the determinant $|\mathbf{B}| = -|\mathbf{A}| = 29$ and the matrix $\mathbf{C}$ defined by

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \\ 5 & 4 & 3 \end{bmatrix}$$

will have the determinant $|\mathbf{C}| = -|\mathbf{B}| = -(-|\mathbf{A}|) = -29$.

4. Assume that $\mathbf{A}$ is a square matrix. Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $|\mathbf{B}| = \alpha|\mathbf{A}|$.

For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}$$

with $|\mathbf{A}| = -29$, then for the matrix $\mathbf{B}$ defined by

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ 4 & 0 & -2 \end{bmatrix}$$

the determinant $|\mathbf{B}| = -2|\mathbf{A}| = 58$ and the determinant of the matrix $\mathbf{C}$ defined by

$$\mathbf{C} = \begin{bmatrix} 1 & 10 & -1 \\ 5 & 15 & 4 \\ 4 & 0 & -2 \end{bmatrix}$$

is $|\mathbf{C}| = 5 \times |\mathbf{B}| = 5 \times (-2|\mathbf{A}|) = 290$.

5. If $\mathbf{A}$ is a square matrix with two equal rows or two equal columns, then $|\mathbf{A}| = 0$. For example, the determinant of the matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ 5 & 3 & 4 \end{bmatrix}$$

is $|\mathbf{A}| = 0$, and the determinant of the matrix $\mathbf{B}$ defined by

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 3 & 3 \\ -2 & 0 & 0 \end{bmatrix}$$

is also zero.

6. For every identity matrix $\mathbf{I}_n$, $|\mathbf{I}_n| = 1$.

7. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size. Then

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|.$$

3.4 Inverse of a Matrix

The inverse of a matrix is an important concept in linear algebra, and can be used to solve certain linear systems. In this section we restrict our attention to square matrices and formulate the notion corresponding to the reciprocal of a nonzero number.

3.4.1 Definition and properties

Definition 3.15
An $n \times n$ matrix $\boldsymbol{A}$ is invertible (or nonsingular) if there exists an $n \times n$ matrix denoted by $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \boldsymbol{I}_n.$$

The matrix $\mathbf{A}^{-1}$ is called the inverse of $\boldsymbol{A}$. If the matrix $\mathbf{A}^{-1}$ does not exist, then $\boldsymbol{A}$ is called ***noninvertible*** *(or* ***singular****).*

Example 3.17

The matrix inverse of

$$\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$$

is given by

$$\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}$$

since

$$\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

More generally, for a 2 × 2 matrix given by

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

with an additional condition:

$$ad - bc \neq 0.$$

If $ad - bc = 0$, then **A** is noninvertible (or singular).

Property 3.3 Properties of the Inverse.

(a) If an $n \times n$ matrix $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is also invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ invertible matrices, then $\mathbf{AB}$ is also invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

More generally, if $\mathbf{A}_1,\ \mathbf{A}_2,\ \ldots,\mathbf{A}_r$ are $r\ n \times n$ invertible matrices, then the product $\mathbf{A}_1\ \mathbf{A}_2 \ldots \mathbf{A}_r$ is also invertible and

$$(\mathbf{A}_1\,\mathbf{A}_2 \ldots \mathbf{A}_r)^{-1} = \mathbf{A}_r^{-1} \ldots \mathbf{A}_1^{-1}.$$

(c) If $\mathbf{A}$ is an $n \times n$ invertible matrix, then

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T.$$

(d) If $\mathbf{A}$ is an $n \times n$ invertible matrix and α a scalar, then

(d1) $(\alpha\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}$

(d2) $|(\mathbf{A})^{-1}| = (|(\mathbf{A})|)^{-1} = \frac{1}{|\mathbf{A}|}$.

3.4.2 Methods for finding the inverse of a matrix

There are several methods used for finding the inverse of a matrix, including Newton's method, Cayley-Hamilton method, the Eigen decomposition, Cholesky decomposition, the cofactor method and the Gauss-Jordan elimination method. In this module we will be limited to the two last methods and we will focus more on the Gauss-Jordan elimination method as it is a key in understanding upcoming methods such as the Simplex method and its variants that will be used in the second part.

Cofactor (or determinant) method

In this method two important terms are used: the determinant and the cofactor matrix. As we are familiar with the determinant of a matrix, let's then define the cofactor matrix.

Definition 3.16 Cofactor Matrix.

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The (ij) cofactor C_{ij} of $\boldsymbol{A}$ is defined as

$$C_{ij} = (-1)^{i+j}\, M_{ij}.$$

The matrix, cof($\boldsymbol{A}$), containing the cofactors of $\boldsymbol{A}$ is called the cofactor matrix. *The transpose of the cofactor matrix is called* ***the adjoint matrix****.*

Example 3.18

(a) In Example 3.16 we found all minors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

namely

$$\begin{array}{rclrclrcl} M_{11} & = & 0 & M_{12} & = & -3 & M_{13} & = & 3 \\ M_{21} & = & -1 & M_{22} & = & -3 & M_{23} & = & 2 \\ M_{31} & = & 1 & M_{32} & = & -3 & M_{33} & = & 1. \end{array}$$

Then the entries of cof($\mathbf{A}$) are

$$\begin{array}{lrclrc} C_{11} = (-1)^{1+1} \cdot M_{11} = 0, & C_{12} = (-1)^{1+2} \cdot M_{12} & = & 3, & C_{13} = (-1)^{1+3} \cdot M_{13} & = \;\; 3 \\ C_{21} = (-1)^{2+1} \cdot M_{21} = 1, & C_{22} = (-1)^{2+2} \cdot M_{22} & = & -3, & C_{23} = (-1)^{2+3} \cdot M_{23} & = -2 \\ C_{31} = (-1)^{3+1} \cdot M_{31} = 1; & C_{32} = (-1)^{3+2} \cdot M_{32} & = & 3, & C_{33} = (-1)^{3+3} \cdot M_{33} & = \;\; 1. \end{array}$$

It follows that

$$\text{cof}(\mathbf{A}) = \begin{bmatrix} 0 & 3 & 3 \\ 1 & -3 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

and the adjoint matrix of $\mathbf{A}$ is

$$\text{cof}(\mathbf{A})^T = \begin{bmatrix} 0 & 1 & 1 \\ 3 & -3 & 3 \\ 3 & -2 & 1 \end{bmatrix}.$$

(b) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Find cof($\mathbf{A}$) and the adjoint matrix.

Solution

$$\begin{array}{lllllllllllllll} C_{11} & = & M_{11} & = & -2; & C_{12} & = & -M_{12} & = & -2; & C_{13} & = & M_{13} & = & 6 \\ C_{21} & = & M_{21} & = & -(-4) = 4; & C_{22} & = & M_{22} & = & -2; & C_{23} & = & -M_{23} & = & 0 \\ C_{31} & = & M_{31} & = & 2; & C_{32} & = & -M_{32} & = & -(-8) = 8; & C_{33} & = & M_{33} & = & -6 \end{array}$$

It follows that

$$\text{cof}(\mathbf{A}) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix},$$

and the adjoint matrix of $\mathbf{A}$ is

$$\text{cof}(\mathbf{A})^T = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}.$$

Now we can find the inverse of a matrix using the cofactor (determinant) method by applying the following formula.

⌟

Theorem 3.1 *Given an* $n \times n$ *matrix* $\boldsymbol{A}$*, the inverse of* $\boldsymbol{A}$ *is given by*

$$\mathbf{A}^{-1} = \frac{cof(\mathbf{A})^T}{|\mathbf{A}|}.$$

If $|\mathbf{A}| = 0$*, that is* $\frac{1}{|\mathbf{A}|} \to \frac{1}{0}$*, then we say that the matrix* $\boldsymbol{A}$ *has no inverse or* $\boldsymbol{A}$ *is singular.*

From the theorem above, we can give the following procedure for finding the inverse of a matrix $\mathbf{A}$ by the cofactor method:

Step 1: Evaluate $|\mathbf{A}|$. If $|\mathbf{A}| = 0$, then there is no inverse. (In this case, do not continue with the remaining steps.)

Step 2: Find cof($\mathbf{A}$)

Step 3: Find the adjoint matrix, that is the transpose of cof($\mathbf{A}$) denoted by cof$(\mathbf{A})^T$.

Step 4: Find the inverse $\mathbf{A}^{-1}$ using the formula

$$\mathbf{A}^{-1} = \frac{\text{cof}(\mathbf{A})^T}{|\mathbf{A}|}.$$

Example 3.19

Find the inverse of the matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.$$

Solution

Step 1: $|\mathbf{D}| = -13$, hence the inverse $\mathbf{D}^{-1}$ exists.

Step 2: The cofactors of $\mathbf{D}$ are

$$\begin{array}{rclrclrcl} C_{11} & = & -5; & C_{12} & = & -1; & C_{13} & = & 4 \\ C_{21} & = & -6; & C_{22} & = & 4; & C_{23} & = & -3 \\ C_{31} & = & 4; & C_{32} & = & -7; & C_{33} & = & 2. \end{array}$$

Hence,

$$\text{cof }(\mathbf{D}) = \begin{bmatrix} -5 & -1 & 4 \\ -6 & 4 & -3 \\ 4 & -7 & 2 \end{bmatrix}.$$

Step 3: Adjoint of $\mathbf{D}$

$$\text{cof }(\mathbf{D})^T = \begin{bmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{bmatrix}.$$

Step 4: Inverse of $\mathbf{D}$

$$\mathbf{D}^{-1} = -\frac{1}{13}\begin{bmatrix} -5 & -6 & 4 \\ -1 & 4 & -7 \\ 4 & -3 & 2 \end{bmatrix}.$$

Gauss – Jordan Elimination method

Before we start dealing with this method, there are two very important concepts that we need to introduce: The elementary row operations (EROs) and the reduced row echelon form of a matrix.

Elementary Row Operations (EROs)

The EROs are similar to Equation Operations as introduced in Learning unit 2, (see Subsection 2.2.1), the only difference is in equation operations, we will apply operations on equations while in EROs we apply operations on rows of a given matrix. So if you understand how to do Equation Operations, then EROs will be just a repetition!

In what follows R_i represents the i^{th} row of a matrix and $R_i^{(k)}$ denotes the i^{th} row after k transformations.

Definition 3.17 Elementary Row Operations (EROs).

An ERO is an operation performed on a row that transforms a given matrix $\mathbf{A}$ *into a new one, say* $\mathbf{A}_1$. *These operations are of three types:*

ERO1: *Interchange two rows of a matrix*

For example, let

$$\mathbf{A} = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 \\ -4 & -2 & 5 \\ 2 & -4 & 1 \end{bmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

After interchanging rows R_2 and R_3, we get a new matrix $\mathbf{A}_1$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 1 \\ -4 & -2 & 5 \end{bmatrix} \begin{array}{lcl} R_1^{(1)} & = & R_1 \\ R_2^{(1)} & = & R_3 \\ R_3^{(1)} & = & R_2 \end{array}$$

In the matrix $\mathbf{A}$, we have interchanged R_2 and R_3 to obtain $\mathbf{A}_1$ and that is why $R_2^{(1)} = R_3$ and $R_3^1 = R_2$.

(Here R_1, R_2 and R_3 denote respectively the first, second and third row of the original matrix $\mathbf{A}$, and $R_1^{(1)}$, $R_2^{(1)}$ and $R_3^{(1)}$ for the rows of $\mathbf{A}_1$).

ERO2: *Multiply a row by a nonzero number.*

For example, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -4 & -2 & 5 \\ 2 & -4 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3. \end{array}$$

If we perform the operation $R_1^{(1)} = 4R_1$, we get

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 8 & 0 \\ -4 & -2 & 5 \\ 2 & -4 & 1 \end{bmatrix} \begin{array}{lcl} R_1^{(1)} & = & 4R_1 \\ R_2^{(1)} & = & R_2 \\ R_3^{(1)} & = & R_3. \end{array}$$

ERO3: *Take a row and add a nonzero multiple of another row to it.*

For example, let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -4 & -2 & 5 \\ 2 & -4 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

If we perform the following elementary row operations

$$\begin{array}{lcl} R_2^{(1)} & = & R_2 + 4R_1 \\ R_3^{(1)} & = & R_3 - 2R_1 \end{array}$$

respectively on the second and third row of $\mathbf{A}$, we get

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 6 & 5 \\ 0 & -8 & 1 \end{bmatrix} \begin{array}{l} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \end{array}$$

Definition 3.18 Elementary matrix.

An elementary matrix is a matrix obtained from the identity matrix by performing a single elementary row operation (ERO) on it.

Example 3.20

- **Elementary matrix of type I:** Interchange two rows of an identity matrix. For example,

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

  is the elementary matrix obtained by interchanging the first and third rows (ERO1) of the identity matrix $\mathbf{I}_3$.

- **Elementary matrix of type II:** Multiply a row by a nonzero number. For example,

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  is the elementary matrix obtained by performing $R_2^{(1)} = -5R_2$ (ERO2) on $\mathbf{I}_3$.

- **Elementary matrix of type III:** Take a row of an identity matrix and add a nonzero multiple of another row to it. For example,

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

  is the elementary matrix obtained by performing $R_3^{(1)} = R_3 + 3R_1$ (ERO3) on $\mathbf{I}_3$.

Echelon form of a matrix

We now discuss the row echelon form and the reduced row echelon form of a matrix.

Definition 3.19 Row Echelon Form.

A matrix is in row echelon form when it satisfies the following conditions:

1. *The first nonzero entry (called the leading entry) in each row is a 1. This is called the leading* 1.
2. *Each leading entry is in a column to the right of the leading entry in the previous row.*
3. *If there is a row where every element (or entry) is zero, then this row lies below any other row that contains a nonzero entry.*

Example 3.21

The following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

while the following matrices are NOT in row echelon form

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Definition 3.20 Reduced Row Echelon Form.

A matrix is in reduced row echelon form when it satisfies the following conditions:

1. *The matrix satisfies the conditions for a matrix in echelon form.*
2. *The leading 1 in each row is the only nonzero entry in its column.*

Example 3.22

The following matrices are in reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

whereas the following matrices are not:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Procedure for obtaining a row reduced echelon form

Any matrix can be transformed into a matrix in echelon form by using a series of elementary row operations. Below is a procedure of how to transform a matrix into a reduced row echelon form.

Step 1. Locate the left most column that does not consist only of zeros.

Step 2. If the first entry of the column found in step 1 is zero then swap the first row with a row that has a nonzero entry a in the same column. The nonzero entry a which is now in the first row and the column found in step 1 is called a pivot. The row (column) in which the pivot occurs is called the pivot row (column).

Step 3. Divide the first row by a, to make the pivot 1 (i.e. to get a leading **1** in the first row) and transform all entries (except the pivot in the pivot column to zeros by performing appropriate EROs.
We shall refer to the method in this step as the "*one-zero method*".
This completes the first row.

Step 4. Repeat steps 1-3 on the matrix consisting of the remaining rows until there are no more pivots to be processed.

Theorem 3.2 *Let $\boldsymbol{A}$ be an $m \times n$ matrix. Suppose that $\boldsymbol{B}$ and $\boldsymbol{C}$ are reduced row echelon forms of the matrix $\boldsymbol{A}$, then $\boldsymbol{B} = \boldsymbol{C}$. This does* not *necessary hold if $\boldsymbol{B}$ and $\boldsymbol{C}$ are row echelon forms of $\boldsymbol{A}$.*

This theorem implies that the reduced row-echelon form of a matrix is unique but a row echelon form is *NOT* necessary unique.

Example 3.23

Transform each of the following matrices into the matrix in reduced row echelon form.

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}$$

Solution

$$\mathbf{A} = \begin{bmatrix} \textcircled{2} & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

Step 1: We find the pivot in the first column first row. We apply the one-zero method as follows: We make the pivot 1 by multiplying R_1 by $\frac{1}{2}$ (ERO2). We get

$$\mathbf{A}_1 = \begin{bmatrix} \textcircled{1} & 1 & \frac{1}{2} \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{array}{l} R_1^{(1)} = \frac{1}{2}R_1 \\ R_2^{(1)} = R_2 \\ R_3^{(1)} = R_3 \end{array}$$

In the pivot column we transform all nonzero entries to zeros (except the pivot 1) using the following EROs:

$$\begin{aligned} R_1^{(2)} &= R_1^{(1)} \\ R_2^{(2)} &= R_2^{(1)} - 2R_1^{(1)} \\ R_3^{(2)} &= R_3^{(1)} - R_1^{(1)}. \end{aligned}$$

We get the following matrix

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & \textcircled{-3} & 1 \\ 0 & -2 & \frac{3}{2} \end{bmatrix} \begin{array}{l} R_1^{(2)} \\ R_2^{(2)} \\ R_3^{(2)} \end{array}$$

Step 2: We find the pivot in the second column, 2nd row of $\mathbf{A}_2$. We apply the one-zero method. We first make the pivot 1 by multiplying the 2nd row by $-\frac{1}{3}$ (ERO2). We get

$$\mathbf{A}_3 = \left[\begin{array}{ccc} 1 & 1 & \frac{1}{2} \\ 0 & \textcircled{1} & -\frac{1}{3} \\ 0 & -2 & \frac{3}{2} \end{array}\right] \begin{array}{l} R_1^{(3)} = R_1^{(2)} \\ R_2^{(3)} = -\frac{1}{3}R_2^{(2)} \\ R_3^{(3)} = R_3^{(2)} \end{array}$$

In the new pivot column we transform all nonzero entries (except the pivot) to zeros by performing the following EROs:

$$\begin{aligned} R_1^{(4)} &= R_1^{(3)} - R_2^{(3)} \\ R_2^{(4)} &= R_2^{(3)} \\ R_3^{(4)} &= R_3^{(3)} + 2R_2^{(3)}. \end{aligned}$$

We get the following matrix

$$\mathbf{A}_4 = \left[\begin{array}{ccc} 1 & 0 & \frac{5}{6} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \textcircled{\frac{5}{6}} \end{array}\right] \begin{array}{l} R_1^{(4)} \\ R_2^{(4)} \\ R_3^{(4)} \end{array}$$

Step 3: We find the pivot in the third column, third row. We apply the one-zero method. We first make the pivot 1 by multiplying the third row by $\frac{6}{5}$ (ERO2).

$$\mathbf{A}_5 = \left[\begin{array}{ccc} 1 & 0 & \frac{5}{6} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \textcircled{1} \end{array}\right] \begin{array}{lcl} R_1^{(5)} & = & R_1^{(4)} \\ R_2^{(5)} & = & R_2^{(4)} \\ R_3^{(5)} & = & \frac{6}{5}R_3^{(4)}. \end{array}$$

In the new pivot column we transform all nonzero entries except the pivot to zeros using the following EROs:

$$\begin{aligned} R_1^{(6)} &= R_1^{(5)} - \frac{5}{6}R_3^{(5)} \\ R_2^{(6)} &= R_2^{(5)} + \frac{1}{3}R_3^{(5)} \\ R_3^{(6)} &= R_3^{(5)}. \end{aligned}$$

Finally, we get

$$\mathbf{A}_6 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

There is no other pivot to be processed, therefore the reduced row echelon form of the matrix **A** is $\mathbf{A}_6$ but the row echelon forms of **A** are $\mathbf{A}_5$ and $\mathbf{A}_6$.

$$\mathbf{B} = \begin{array}{cc} \left[\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{array}\right] & \begin{array}{l} R_1 \\ R_2 \\ R_3. \end{array} \end{array}$$

Step 1: We cannot locate a pivot in the first row, first column. Thus we interchange the first and the second row (ERO1). We get

$$\mathbf{B}_1 = \begin{array}{cc} \left[\begin{array}{ccc} \textcircled{1} & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{array}\right] & \begin{array}{lcl} R_1^{(1)} & = & R_2 \\ R_2^{(1)} & = & R_1 \\ R_3^{(1)} & = & R_3. \end{array} \end{array}$$

Step 2: We now locate the pivot in the first row, first column. We then transform all nonzero entries in the pivot column, except the pivot to zero by performing the following EROs

$$\begin{array}{lcl} R_1^{(2)} & = & R_1^{(1)} \\ R_2^{(2)} & = & R_2^{(1)} \\ R_3^{(2)} & = & R_3^{(1)} - 2R_2^{(1)}. \end{array}$$

We get

$$\mathbf{B}_2 = \begin{array}{cc} \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & \textcircled{1} & 2 \\ 0 & 3 & 6 \end{array}\right] & \begin{array}{l} R_1^{(2)} \\ R_2^{(2)} \\ R_3^{(2)} \end{array} \end{array}$$

Step 3: We locate the pivot 1 in the 2nd row, 2nd column and we then apply the one-zero method. We transform all nonzero entries in the pivot column, except the pivot to zeros by using the following EROs

$$\begin{array}{lcl} R_1^{(3)} & = & R_1^{(2)} - 2R_2^{(2)} \\ R_2^{(3)} & = & R_2^{(2)} \\ R_3^{(3)} & = & R_3^{(2)} - 3R_2^{(2)}. \end{array}$$

We get

$$\mathbf{B}_3 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The reduced row echelon form of **B** is $\mathbf{B}_3$.

As we are now familiar with elementary row operations and the reduced row echelon form of a matrix, we can give the procedure for finding the inverse of a matrix using the Gauss-Jordan elimination method. Suppose we are given the $n \times n$ matrix $\mathbf{A}$. To find the inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$, we need to follow the steps below:

Step 1: Write down the augmented matrix of $\mathbf{A}$ and $\mathbf{I}_n$, namely $[\mathbf{A}|\mathbf{I}_n]$ (refer to Definition 3.8)

For example, if the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix},$$

then the augmented matrix of $\mathbf{A}$ and $\mathbf{I}_3$ is

$$[\mathbf{A}|\mathbf{I}_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right].$$

Step 2: Transform the left matrix (that is the matrix $\mathbf{A}$) to the reduced row echelon form using elementary row operations (EROs) on the whole matrix (that is on the augmented matrix $[\mathbf{A}|\mathbf{I}_n]$).

As a result you will get the reduced row echelon form $[\mathbf{I}_n^{'}|\mathbf{A}^{'}]$. If $\mathbf{I}_n^{'} = \mathbf{I}_n$, then the inverse of the matrix $\mathbf{A}$ exists and is given by $\mathbf{A}^{'} = \mathbf{A}^{-1}$.

If $\mathbf{I}_\mathbf{n}^{'} \neq \mathbf{I}_n$, then the inverse of the matrix $\mathbf{A}$ does not exist.

Example 3.24

Compute the inverse of the matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

Solution

The augmented matrix $[\mathbf{A}|\mathbf{I_3}]$ is given by

$$[\mathbf{A}|\mathbf{I}_3] = \left[\begin{array}{rrr|rrr} \boxed{-7} & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right].$$

Step 1: We find the pivot in the first column, first row and we apply the one-zero method to the first column (current Pivot column).

We get the following matrix

$$\left[\begin{array}{rrr|rrr} 1 & \frac{6}{7} & \frac{12}{7} & -\frac{1}{7} & 0 & 0 \\ 0 & \boxed{\frac{5}{7}} & -\frac{11}{7} & \frac{5}{7} & 1 & 0 \\ 0 & -\frac{6}{7} & \frac{16}{7} & \frac{1}{7} & 0 & 1 \end{array}\right]$$

Step 2: We find the pivot in the second column, second row and we apply the one-zero method.

We get the following matrix

$$\left[\begin{array}{rrr|rrr} 1 & 0 & \frac{18}{5} & -1 & -\frac{6}{5} & 0 \\ 0 & 1 & -\frac{11}{5} & 1 & \frac{7}{5} & 0 \\ 0 & 0 & \boxed{\frac{2}{5}} & 1 & \frac{6}{5} & 1 \end{array}\right]$$

Step 3: We find the pivot in the third column, third row and we apply the one-zero method to the third column.

We get the following matrix

$$[\mathbf{I}_3^{'}|\mathbf{A}^{'}] = \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -10 & -12 & -9 \\ 0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2} \end{array}\right]$$

As the left matrix is an identity matrix, that is $\mathbf{I}_3^{'} = \mathbf{I}_3$, the inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$ exists and is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}.$$

Comment: In this example we could have begun by interchanging the first and third rows to get

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 4 & 0 & 0 & 1 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ -7 & -6 & -12 & 1 & 0 & 0 \end{array}\right].$$

The pivot located in the first column, first row is now 1. In this case the calculations will be easier as some fractions will be avoided. Now reduce the above matrix to find $\mathbf{A}^{-1}$ and ensure that you obtain the same answer.

Example 3.25

Compute the inverse of the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution

The augmented matrix $[\mathbf{B}|\mathbf{I}_3]$ is given by

$$[\mathbf{B}|\mathbf{I}_3] = \left[\begin{array}{rrr|rrr} \enclose{circle}{1} & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right].$$

Step 1: We find the pivot in the first row and first column and we apply the one-zero method.

We get the following matrix

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & \enclose{circle}{-3} & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right].$$

Step 2: We find the pivot in the second column, second row and apply the one-zero method.

We get

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & -\frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 2 & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right].$$

There are no more pivots to be processed. The left matrix is **NOT** an identity matrix. Therefore, the inverse matrix $\mathbf{B}^{-1}$ **DOES NOT** exist.

3.5 Evaluation Exercises

1. Doing the computations by hand, find the determinant of each of the matrices below and check the final answer with maxima.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

2. Find the value of k so that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & k \end{bmatrix}$$

has $|\mathbf{A}| = 0$.

3. Find the value of k so that the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 3 & k \end{bmatrix}$$

has $|\mathbf{B}| = 0$.

4. Given the matrix

$$\mathbf{B} = \begin{bmatrix} 2-x & 1 \\ 4 & 2-x \end{bmatrix}$$

find all values of x that are solutions of $|\mathbf{B}| = 0$.

5. Suppose that $\mathbf{A}$ is a square matrix of size n and α a scalar. Prove that

$$|\alpha\mathbf{A}| = \alpha^n|\mathbf{A}|.$$

6. Consider the two matrices $\mathbf{A}$ and $\mathbf{B}$ below, and suppose that you already have $|\mathbf{A}| = -120$. Use a property of determinants to find $|\mathbf{B}|$.

$$\mathbf{A} = \begin{bmatrix} 0 & 8 & 3 & -4 \\ -1 & 2 & -2 & 5 \\ -2 & 8 & 4 & 3 \\ 0 & -4 & 2 & -3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 8 & 3 & -4 \\ 0 & -4 & 2 & -3 \\ -2 & 8 & 4 & 3 \\ -1 & 2 & -2 & 5 \end{bmatrix}$$

7. Find the inverse of each of the following matrices using the cofactor method.

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ -5 & 5 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 2 & -3 \\ -1 & \frac{3}{2} \end{bmatrix}$$

8. Let

$$\mathbf{A} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 3 & 1 \\ 1 & -3 & 4 & 2 \\ 2 & 4 & 2 & 3 \\ -1 & 5 & 5 & 4 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4. \end{array}$$

Find a new matrix $\mathbf{A}_1$ that has rows $R_1^{(1)}$, $R_2^{(1)}$, $R_3^{(1)}$ and $R_4^{(1)}$ such that

$$\begin{aligned} R_1^{(1)} &= R_1 \\ R_2^{(1)} &= R_2 - R_1 \\ R_3^{(1)} &= R_3 - 2R_1 \\ R_4^{(1)} &= R_4 + R_1 \end{aligned}$$

9. Let

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & -3 & 4 \\ 2 & 4 & 2 \end{bmatrix}.$$

If after performing a series of elementary row operations you have a new matrix $\mathbf{M}'$ given by

$$\mathbf{M}' = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}$$

find one set of values for a, b, c, d, e and f.

10. Let the matrix $\mathbf{K}$ be defined by

$$\mathbf{K} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 5 & 2 & -3 \end{bmatrix}.$$

If after performing a series of elementary row operations you have a new matrix $\mathbf{K}'$ given by

$$\mathbf{K}' = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{bmatrix}$$

find one set of values for x, y and z.

11. Determine whether each of the following matrices are in reduced row echelon form, row echelon form, or neither:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

12. (a) Find a series of elementary row operations that have been performed to transform a matrix $\mathbf{A}$ to a new matrix $\mathbf{A}'$ defined respectively by

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 5 \\ 5 & 2 & -4 \\ 3 & 1 & 6 \end{bmatrix} \text{ and } \mathbf{A}' = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{9}{2} & -\frac{33}{2} \\ 0 & \frac{5}{2} & -\frac{3}{2} \end{bmatrix}.$$

(b) Find a series of elementary row operations that have been performed to transform a matrix **B** to a matrix $\mathbf{B}^{'}$ and from $\mathbf{B}^{'}$ to a matrix $\mathbf{B}^{''}$ defined respectively by

$$\mathbf{B} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 2 & -1 & 0 \end{bmatrix},$$

$$\mathbf{B}^{'} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 4 \\ 0 & 2 & -1 \end{bmatrix} \text{ and } \mathbf{B}^{''} = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 4 \end{bmatrix}.$$

13. Find the reduced row echelon form of each of the following matrices:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

14. Find the inverse of each of the following matrices using the Gauss-Jordan Elimination method:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}.$$

Learning Unit 4

Solving Linear Systems: A Gauss-Jordan Elimination approach

Contents

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Learning objectives

After completing this study unit you should be able to

- write a system of linear equations as an augmented matrix
- use the Gauss-Jordan elimination method to solve systems of linear equations
- determine whether a set of vectors is linearly dependent or independent.

Sections from prescribed book

2.3. The Gauss-Jordan Method for Solving Systems of Linear Equations. - *p22*

2.4. Linear Independence and Linear Dependence. - *p32*

4.1 The matrix form of a linear system

Definition 4.1 *Consider the following linear system of m equations in n variables:*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where x_1, $x_2, \ldots, x_n$ are variables (or unknowns), a_{ij} and b_i are constants. Then the linear system above can be represented by the matrix equation

$$\mathbf{Ax} = \boldsymbol{b},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where $\boldsymbol{x}$ *and* $\boldsymbol{b}$ *are vectors defined by*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

which is obtained by adjoining $\boldsymbol{b}$ *to* $\boldsymbol{A}$ *is called the augmented matrix of the linear system.*

Note:

1. The vector $\mathbf{b}$ is referred to as a vector of constants while $\mathbf{x}$ is a vector of variables.
2. The matrix $\mathbf{A}$ is called the coefficient matrix.

Example 4.1

(a) Consider the following linear system

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & + & x_5 & = & 9 \\ 3x_1 & + & x_2 & & & + & x_4 & - & 3x_5 & = & 0 \\ -2x_1 & + & 7x_2 & - & 5x_3 & + & 2x_4 & + & 2x_5 & = & -3. \end{array}$$

The coefficient matrix, vector of constants and vector of variables are given, respectively, by

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

In a compact form, we write the system as $\mathbf{Ax} = \mathbf{b}$.

(b) The linear system

$$\begin{array}{rcrcrcl} 2x & + & 3y & - & 4z & = & 5 \\ -2x & & & + & z & = & 7 \\ 3x & + & 2y & + & 2z & = & 3 \end{array}$$

can be written in matrix form as

$$\mathbf{Ax} = \mathbf{b},$$

where

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -4 \\ -2 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}.$$

Here **A** is the coefficient matrix and the augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{rrr|r} 2 & 3 & -4 & 5 \\ -2 & 0 & 1 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

(c) The matrix

$$\left[\begin{array}{rrr|r} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

is the augmented matrix of the linear system

$$\begin{array}{rcrcrcl} 2x & - & y & + & 3z & = & 4 \\ 3x & & & + & 2z & = & 5. \end{array}$$

Theorem 4.1 Equal Matrices and Matrix vector products.

Assume that $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* $m \times n$ *matrices such that* $\mathbf{Ax} = \mathbf{Bx}$ *for every* n*-dimensional vector of variables* $\boldsymbol{x}$*. Then* $\mathbf{A} = \mathbf{B}$.

4.2 Solutions of a linear system

We first consider a linear system in which the number of equations m is equal to the number of variables n, and the coefficient matrix $\mathbf{A}$ is invertible. Then, when solving

$$\mathbf{Ax} = \mathbf{b},$$

we can multiply both sides of the equation on the left by $\mathbf{A}^{-1}$ to obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$$

i.e.

$$\mathbf{I}_n\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

i.e.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

and we see that a unique solution exists. However, if $\mathbf{A}$ is a square noninvertible matrix, or an $m \times n$ matrix, where $m \neq n$, then we cannot use the above method and it may be difficult to decide whether the linear system has a solution or possibly infinitely many solutions. Another method that we can use to solve $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{A}$ is invertible is Cramer's rule. This rule states that the solution of

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A}$ is invertible, is given by

$$\mathbf{x} = [x_1,\, x_2, \ldots, x_n]^T,$$

where

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|}, \text{ for all } 1 \leq i \leq n,$$

where $|\mathbf{A}_i|$ is the determinant of the matrix $\mathbf{A}_i$ obtained by replacing the ith column of $\mathbf{A}$ by the vector $\mathbf{b}$.

Note that $\dfrac{|\mathbf{A}_i|}{|\mathbf{A}|}$ is defined since $|\mathbf{A}| \neq 0$ because $\mathbf{A}$ is invertible. It is obvious that this method does not work if $\mathbf{A}$ is singular since then $|\mathbf{A}| = 0$ and division by 0 is undefined.

We illustrate this method by means of an example. Let the coefficient matrix $\mathbf{A}$ and vectors $\mathbf{b}$ and $\mathbf{x}$ be given respectively by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & -1 & 1 \end{bmatrix}; \ \mathbf{b} = \begin{bmatrix} 1 \\ -7 \\ 0 \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 2 & -1 \\ -7 & -3 & 1 \\ 0 & -1 & 1 \end{bmatrix}; \text{ and } |\mathbf{A}_1| = 5,$$

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -7 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \text{ and } |\mathbf{A}_2| = -15,$$

$$\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -7 \\ 1 & -1 & 0 \end{bmatrix}; \text{ and } |\mathbf{A}_3| = -20,$$

and $|\mathbf{A}| = -5$.

Applying Cramer's rule, we have

$$\begin{aligned} x_1 &= \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{5}{(-5)} = -1 \\ x_2 &= \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-15}{(-5)} = 3 \\ x_3 &= \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{-20}{(-5)} = 4. \end{aligned}$$

We also note that Cramer's rule does not work for non square matrices, for example, for

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 3 \\ 0 & 5 & 2 \end{bmatrix},$$

simply because $|\mathbf{A}|$ does not exist. This is a motivation for introducing the Gauss-Jordan elimination method for solving linear systems. The following theorem summarizes the nature of the solutions for such systems.

Theorem 4.2 *Consider the augmented matrix of the system* $\mathbf{Ax} = \mathbf{b}$ *given by*

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

(a) *If the reduced row echelon form of the augmented matrix [*$\mathbf{A}|\mathbf{b}$*] after deleting any zero rows is [*$\mathbf{I_n}|\mathbf{s}$*], where*

$$[\mathbf{I_n}|\mathbf{s}] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & 0 & \cdots & 0 & s_2 \\ 0 & 0 & 1 & \cdots & 0 & s_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_n \end{array}\right]$$

then the linear system has the unique solution

$$x_1 = s_1,\ x_2 = s_2, \ldots, x_n = s_n.$$

(b) *If the reduced row echelon form of the augmented matrix* $[\mathbf{A}|\mathbf{b}]$

  (i) *contains a row of the form [*$0\ 0\ \cdots\ 0|c$*] with* $c \neq 0$*, then the system has no solutions.*

  (ii) *does not contain such a row and there are less nonzero rows than variables, then infinitely many solutions exist.*

Note that the row $[0\ 0 \cdots 0|c]$, $c \neq 0$ corresponds to the equation

$$0x_1 + 0x_2 + \cdots + 0x_n = c,\ c \neq 0.$$

It is clear that there are no values for $x_1,\ x_2, \ldots, x_n$ which satisfy this equation since the left hand side will always be 0.

To illustrate this theorem, we consider a number of examples, but before doing so, we define the concepts '*leading*' and '*free variables*'.

Definition 4.2 *If the* i^{th} *column of an augmented matrix in reduced row echelon form does not contain the **leading 1** of some row, then the* i^{th} *variable* x_i *is called a **free-variable for the system**. Otherwise* x_i *is known as a **leading variable**.*

Example 4.2

Solve the system

$$\begin{aligned} x_1 - x_2 &= 5 \\ 2x_1 + x_2 &= 1. \end{aligned}$$

Solution

The augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{cc|c} \textcircled{1} & -1 & 5 \\ 2 & 1 & 1 \end{array}\right].$$

We find the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$ by performing a series of EROs.

Step 1. We locate the pivot in the 1st column, 1st row and we apply the one-zero method to the first column.

We get the matrix

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & \textcircled{3} & -9 \end{array}\right].$$

Step 2. We locate the next pivot in the 2nd column, 2nd row and we apply the one-zero method to the 2nd column (that is the current pivot column). We get the matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] = [\mathbf{I_2}|\mathbf{s}].$$

Then, according to Theorem 4.2 part (a), the system has the unique solution $x_1 = 2$ and $x_2 = -3$.

Note that both x_1 and x_2 are leading variables since the first and second columns contain leading 1's.

Example 4.3

Solve the linear system

$$\begin{array}{rcrcrcr} & & 2x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & - & 2x_3 & = & -6 \\ x_1 & - & x_2 & + & x_3 & = & 0 \end{array}$$

Solution

The augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ 1 & 1 & -2 & -6 \\ 1 & -1 & 1 & 0 \end{array}\right].$$

We find the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$ by performing a series of EROs.

Step 1. We interchange the first and second rows. We get

$$\left[\begin{array}{ccc|c} \textcircled{1} & 1 & -2 & -6 \\ 0 & 2 & 1 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right].$$

Step 2. We locate the pivot 1 in the first column, first row and apply the one-zero method to the first column.

We get

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -6 \\ 0 & \textcircled{2} & 1 & 2 \\ 0 & -2 & 3 & 6 \end{array}\right].$$

Step 3. We find the pivot in the second row, second column and we apply the one-zero method to the second column. We obtain the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{2} & -7 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & \textcircled{4} & 8 \end{array}\right].$$

Step 4. We find the pivot in the third column, third row and we apply the one-zero method to the third column (which is the current pivot column). We get the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] = [\mathbf{I_3}|\mathbf{s}].$$

According to Theorem 4.2 part (a), the system has the unique solution

$$x_1 = -2,\ x_2 = 0,\ x_3 = 2.$$

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Example 4.4

Solve the system

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 4 \\ -4x_1 & + & 2x_2 & = & -1 \end{array}$$

Solution

The augmented matrix of this system is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{rr|r} \textcircled{2} & -1 & 4 \\ -4 & 2 & -1 \end{array}\right].$$

We find the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$ by performing a series of EROs.

Step 1. We locate the pivot in the first row, first column and we apply the one-zero method. We get

$$\left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 7 \end{array}\right].$$

According to Theorem 4.2, part (b) (i), the system does not have a solution because there is a row of the form $[0\ 0\ |c]$, where $c = 7 \neq 0$.

Example 4.5

Solve the system

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 2x_3 & = & 2 \\ 4x_1 & - & 3x_2 & + & x_3 & = & -5 \\ -x_1 & + & 2x_2 & + & x_3 & = & -2. \end{array}$$

Solution

The augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{rrr|r} 2 & 1 & -1 & 1 \\ 1 & -1 & 2 & 2 \\ 4 & -3 & 1 & -5 \\ -1 & 2 & 1 & -2 \end{array}\right].$$

Next we find the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$ by performing a series of EROs.

Step 1. We interchange R_1 and R_2, and get

$$\left[\begin{array}{ccc|c} \textcircled{1} & -1 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 4 & -3 & 1 & -5 \\ -1 & 2 & 1 & -2 \end{array}\right].$$

We could have made the pivot 1 in the 1st column, 1st row by dividing the first row of $[\mathbf{A}|\mathbf{b}]$ by 2, but this will produce fractions in the row. It is easier to swap R_1 with R_2 as above. We now apply the one-zero method to the first column.

We get the following matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & -5 & -3 \\ 0 & 1 & -7 & -13 \\ 0 & 1 & 3 & 0 \end{array}\right].$$

Step 2. We interchange the second and third rows. We get

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & \textcircled{1} & -7 & -13 \\ 0 & 3 & -5 & -3 \\ 0 & 1 & 3 & 0 \end{array}\right].$$

We locate the pivot in the 2nd column, 2nd row and apply the one-zero method. We obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -11 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & \textcircled{16} & 36 \\ 0 & 0 & 10 & 13 \end{array}\right].$$

Step 3. We find the pivot in the third row, third column and we apply the one-zero method to the third column. We get the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{11}{4} \\ 0 & 0 & 1 & \frac{9}{4} \\ 0 & 0 & 0 & -\frac{19}{2} \end{array}\right].$$

According to Theorem 4.2, part (b) (i), the system does not have a solution because there is a row of the form $[0 \;\; 0 \;\; 0|c]$, where $c = -\frac{19}{2} \neq 0$.

Example 4.6

Solve

$$\begin{array}{rcrcr} x_1 & - & x_2 & = & 5 \\ -2x_1 & + & 2x_2 & = & -10. \end{array}$$

Solution

The augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{rr|r} \textcircled{1} & -1 & 5 \\ -2 & 2 & -10 \end{array}\right].$$

The reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$ can be computed using the following steps.

Step 1. We locate the pivot 1 in the 1st column, 1st row and we apply the one-zero method to the first column. We get

$$\left[\begin{array}{rr|r} 1 & -1 & 5 \\ 0 & 0 & 0 \end{array}\right].$$

According to Theorem 4.2 part (b) (ii), the system has infinitely many solutions. The corresponding system is then

$$x_1 - x_2 = 5.$$

We note that x_1, is the leading variable and x_2 the free variable. We then set the free-variable

$$x_2 = t, \text{ where } t \in \mathbb{R}.$$

Hence,

$$\begin{array}{rrcrcr} & x_1 & = & 5 & + & x_2, \\ \text{i.e.} & x_1 & = & 5 & + & t. \end{array}$$

The general solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5+t \\ t \end{bmatrix}, \text{ where } t \in \mathbb{R}$$

or equivalently

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

The solution set is

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} \ : \ t \in \mathbb{R} \right\}.$$

Example 4.7

Solve the system

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & - & x_4 & = & 1 \\ -2x_1 & - & 3x_2 & + & 4x_3 & + & 5x_4 & = & 2 \\ 2x_1 & + & 4x_2 & & & - & 2x_4 & = & 2 \end{array}$$

Solution

Here the augmented matrix is

$$[\mathbf{A}|\mathbf{b}] = \left[\begin{array}{rrrr|r} \textcircled{1} & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 2 \end{array}\right].$$

Let's find the reduced row echelon form of $[\mathbf{A}|\mathbf{b}]$.

Step 1. We locate the pivot in the 1st column, 1st row and we apply the one-zero method. We get

$$[\mathbf{A}_1|\mathbf{b}_1] = \left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 1 \\ 0 & \textcircled{1} & 4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Step 2. We locate the pivot in the 2nd column, 2nd row and we apply the one-zero method to the 2nd column. We get the matrix

$$\left[\begin{array}{rrrr|r} 1 & 0 & -8 & -7 & -7 \\ 0 & 1 & 4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

By Theorem 4.2, part (b) (ii), the system has infinitely many solution. The corresponding system is

$$\begin{array}{rcrcrcrl} x_1 & & & - & 8x_3 & - & 7x_4 = -7 & \quad E_1 \\ & & x_2 & + & 4x_3 & + & 3x_4 = 4 & \quad E_2 \end{array}$$

In this case, the free variables are x_3 and x_4. We then set

$$\begin{array}{rcl} x_3 & = & s \\ x_4 & = & t, \end{array}$$

where t, $s \in \mathbb{R}$.

From equation (E_1) we have

$$x_1 = -7 + 8x_3 + 7x_4$$

i.e.

$$x_1 = -7 + 8s + 7t.$$

Also from equation (E_2)

$$x_2 = 4 - 4x_3 - 3x_4$$

i.e.

$$x_2 = 4 - 4s - 3t.$$

Note that the 1st and 2nd columns have leading 1s and hence x_1 and x_2 are leading variables. The 3rd and 4th columns do not have leading 1s, thus x_3 and x_4 are free variables. The general solution is then given by

$$\begin{aligned} x_1 &= -7+8s+7t \\ x_2 &= 4-4s-3t \\ x_3 &= s \\ x_4 &= t, \end{aligned}$$

where t, $s \in \mathbb{R}$.

The solution set is

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8s + 7t - 7 \\ -4s - 3t + 4 \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 8 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 4 \\ 0 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

We have learnt how to solve linear systems of equations. In all examples so far, we have dealt with inhomogeneous systems and have learnt how to recognize different types of solutions by applying Theorem 4.2. Now let's discuss a particular case where in the linear system $\mathbf{Ax} = \mathbf{b}$, the vector $\mathbf{b}$ is a zero vector denoted by $\mathbf{0}$. In this case we are dealing with a homogeneous system.

Definition 4.3 Trivial and nontrivial solutions.
The homogeneous system $\mathbf{Ax} = \mathbf{0}$ *always has the solution* $\mathbf{x} = \mathbf{0}$ *which is called the trivial solution. Any other nonzero solutions of* $\mathbf{Ax} = \mathbf{0}$*, if they exist, are called nontrivial solutions.*

We note that Definition 4.3 above can be supported by Theorem 4.2. In fact, for the system $\mathbf{Ax} = \mathbf{0}$, we have the augmented matrix given by

$$[\mathbf{A}|\mathbf{0}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right].$$

If the reduced row echelon form of the augmented matrix $[\mathbf{A}|\mathbf{0}]$, after deleting all zero rows, is $[\mathbf{I}_n|\mathbf{0}]$, i.e.

$$[\mathbf{I}_n|\mathbf{0}] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 1 & \cdots & 1 & 0 \end{array}\right]$$

then the system has the unique solution

$$x_1 = 0,\ x_2 = 0, \ldots, x_n = 0,$$

i.e. the trivial solution.

If the reduced echelon form of $[\mathbf{A}|\mathbf{0}]$ is not $[\mathbf{I}_n|\mathbf{0}]$, then we have infinitely many solutions which includes the trivial solution and nontrivial solutions.

Example 4.8

Let's use the same system as in Example 4.7 but with $\mathbf{b} = 0$, that is the system

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & - & x_4 & = & 0 \\ -2x_1 & - & 3x_2 & + & 4x_3 & + & 5x_4 & = & 0 \\ 2x_1 & + & 4x_2 & & & - & 2x_4 & = & 0. \end{array}$$

The augmented matrix $[\mathbf{A}|\mathbf{0}]$ is given by

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array}\right].$$

The reduced row echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding homogeneous system is

$$\begin{array}{rcrcrcr} x_1 & & - & 8x_3 & - & 7x_4 & = & 0 \\ & x_2 & + & 4x_3 & + & 3x_4 & = & 0 \end{array}$$

and the free variables are x_3 and x_4. By setting them as $x_3 = s$ and $x_4 = t$, we have $x_1 = 8s + 7t$, and $x_2 = -4s - 3t$.

The general solution is

$$\begin{aligned} x_1 &= 8s + 7t \\ x_2 &= -4s - 3t \\ x_3 &= s \\ x_4 &= t. \end{aligned}$$

- When $s = 0$ and $t = 0$, we get the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$.
- When $s \neq 0$ or $t \neq 0$, we get nontrivial solutions.

In the next section, we will see how to use homogeneous equations to determine if a set of vectors is linear dependent or independent.

4.3 Linear Dependence and Independence

Definition 4.4 Linear dependence and independence.
Let $S = \{\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_k\}$ *be a set of vectors in a vector space* V. S *is said to be* linearly dependent *if there exist constants* $c_1, c_2, \cdots, c_k$ *(not all zero) such that*

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}. \tag{4.1}$$

If the equation (4.1) holds only for $c_1 = c_2 = \cdots = c_k = 0$, *then* S *is called* ***linearly independent***.

Note: For any set of vectors $S = \{\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_k\}$ equation (4.1) always holds if $c_1, c_2, \cdots, c_k$ are all equal to zero. The important point is whether or not it is possible to find a solution with at least one of the scalars not equal to zero.

The procedure to determine whether $S = \{\mathbf{x}_1; \mathbf{x}_2; \ldots; \mathbf{x}_k\}$ is linearly dependent, or not, is to form equation (4.1) (a homogeneous system). Now,

- if the system has only the trivial solution (all $c_i = 0$), then the set of vectors is linearly independent, and
- if it has at least one nontrivial solution, then the set of vectors is linearly dependent.

Example 4.9

Consider the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$.

To determine whether $S = \{\mathbf{x}_1; \mathbf{x}_2; \mathbf{x}_3\}$ is linearly dependent or not, we need to determinate c_1, c_2 and c_3 such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0},$$

i.e. such that

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \\ 2c_1 & + & 2c_2 & + & 3c_3 & = & 0. \end{array}$$

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right].$$

After performing a series of EROs, we arrive at the following reduced row echelon form (verify this yourself):

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then the only solution is $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$; and hence the set S is linear independent

Example 4.10

Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

In order to determine whether $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is linearly independent or not, we must determine c_1, c_2, c_3 and c_4 such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4 = \mathbf{0}$$

i.e. such that

$$\begin{array}{rcrcrcrcl} c_1 & + & c_2 & - & 3c_3 & + & 2c_4 & = & 0 \\ 2c_1 & - & 2c_2 & + & 2c_3 & & & = & 0 \\ -c_1 & + & c_2 & - & 1c_3 & & & = & 0. \end{array}$$

This system consists of three equations in four unknowns and thus a nontrivial solution exists. The set of vectors is therefore linearly dependent.

For example, we have

- $c_1 = 1; \quad c_2 = 2; \quad c_3 = 1; \quad c_4 = 0$ giving $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$;
- $c_1 = 1; \quad c_2 = 1; \quad c_3 = 0; \quad c_4 = -1$ giving $\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_4 = \mathbf{0}$.

For completeness sake we solve the system.

The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & 1 & -3 & 2 & 0 \\ 2 & -2 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \end{array}\right]$$

and the reduced row echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The corresponding system is

$$\begin{array}{ccccccccc} c_1 & & - & c_3 & + & c_4 & = & 0 & E_1 \\ & c_2 & - & 2c_3 & + & c_4 & = & 0 & E_2 \end{array}$$

This system has infinitely many solutions. To find them, we set the free-variables

$$\begin{aligned} c_4 &= t \\ c_3 &= s \end{aligned}$$

where t, $s \in \mathbb{R}$. Then from E_1, we have

$$c_1 - c_3 + c_4 = 0$$

i.e.

$$c_1 = s - t,$$

and from E_2

$$c_2 - 2c_3 + c_4 = 0$$

i.e.

$$c_2 = 2s - t.$$

The general solution is

$$\begin{aligned} c_1 &= s - t \\ c_2 &= 2s - t \\ c_3 &= s \\ c_4 &= t. \end{aligned}$$

The solution set is

$$S = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \; : \; s, \, t \in \mathbb{R} \right\}.$$

The set of nontrivial solutions is given when $s \neq 0$ or $t \neq 0$. For example, if $s = 1$ and $t = 2$, we get a particular nontrivial solution given by

$$c_1 = -1, \; c_2 = 0, \; c_3 = 1 \text{ and } c_4 = 2.$$

This shows that there exists at least one nontrivial solution, and consequently, the set of vectors is linearly dependent.

Note: If a homogeneous system has less equations than unknowns then the system automatically has infinitely many solutions as there exists at least one free variable.

4.4 Evaluation Exercises

1. Redo Example 4.5 showing all elementary row operations that have been omitted.
2. Solve each of the following systems using the Gauss-Jordan elimination method:

   (a)

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 6x_3 & = & -4 \\ 7x_1 & - & 4x_2 & - & 2x_3 & = & 9 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 1 \end{array}$$

   (b)

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 30 \\ x_1 & - & 2x_2 & + & 3x_3 & - & 4x_4 & = & -10 \\ 3x_1 & + & x_2 & + & 2x_3 & & & = & 1 \end{array}$$

   (c)

$$\begin{array}{rrcrcr} & 2x_1 & + & x_2 & = & 4 \\ & x_1 & - & 3x_2 & = & 16 \\ - & 4x_1 & - & 2x_2 & = & -8 \end{array}$$

(d)

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 4 \\ x_1 & - & 3x_2 & = & 16 \\ 4x_1 & + & 2x_2 & = & 6 \end{array}$$

3. Determine whether the set of vectors $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent or not, where
   (a) $\mathbf{x}_1 = (1, 1, 2, 1),\ \mathbf{x}_2 = (0, 2, 1, 1);\ \mathbf{x}_3 = (3, 1, 2, 0)$
   (b) $\mathbf{x}_1 = (1, 2, 0, 1),\ \mathbf{x}_2 = (2, -1, 1, 0);\ \mathbf{x}_3 = (0, 5, -1, 2)$.

Part 2

Linear Programming

Learning Unit 5

Introducing linear programming models

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Learning objectives

After completing this study unit you should be able to

- identify all the components of an LP model
- write down the general LP model.

Sections from prescribed book

3.1. What Is a Linear Programming Problem? - *p49*

5.1 Types of problems

Mark and Christine, a young, up-and-coming professional couple, are discussing the events of the day. Mark is an engineer whose team has developed a new television projection system. Final tests have just been successfully completed and the results are encouraging. The system will be installed on two models, the $VH200$ and the $SB150$. Although very excited, Mark is also a bit worried. “I wish we had a larger work force, more machine time and better marketing capabilities. I am sure we could make a fat profit. But as it is, we don’t even know how many of each model to manufacture.”

Christine has problems of her own at the Unique Paint Company. A new expensive special purpose paint, Sungold, is becoming very popular. The production manager has asked Christine to see if she can find a combination of two new ingredients, code-named Alpha and Beta, that will result in the same brilliance and tint as the original ingredients; but at a lower cost. Christine feels confident that she can.

Christine does not realise that her problem, a typical *blending* problem, is in many ways equivalent to Mark’s, a typical *product-mix* problem.

Resource allocation problems appear in several forms. Examples are given below.

Product-mix problems

Most manufacturing companies are capable of producing more than one product and have the ability to adjust, to some extent at least, the proportions in which products are made. The objective is to choose the product mix that is most profitable. In making this choice, the firm will be constrained by its resources of equipment and labour, material, finances, etc.

Blending problems

Many products, for example those of the chemical, petroleum, pharmaceutical and processed food industries, contain mixtures of basic ingredients. The finished product must meet certain specifications. Subject to these being met, the manufacturer is free to choose the blend of basic ingredients, being constrained by their availability. The choice made aims at producing a satisfactory product at minimum cost.

Transportation problems

A company, for example, a textbook publisher, may have warehouses at various places in the country. All orders received from bookshops must be supplied by these warehouses. For example, copies of textbooks must be shipped to campus bookstores to meet the demand there. The company will want to minimise the distribution costs while meeting this demand.

Purchasing problems

The purchasing department of a company has access to different raw materials in various quantities and qualities. They can buy these in a number of different combinations. They are also subject to production requirements and budget restrictions. They want to find the combination with the lowest cost.

Portfolio selection problems

An investor must decide how to distribute his investments among alternative assets, such as common stocks and bonds, in order to maximise expected return.

Advertising media mix problems

The marketing department of a company must decide how much to spend on advertising in newspapers and magazines and on radio and television. They are restricted by the given budget for production promotion, and the available media space and time. The objective is to maximise the exposure of the product to potential customers.

Production and inventory scheduling problems

Manufacturing companies face variations in the market demand for their products. As it is generally costly to make changes to production schedules, inventory is carried to help meet fluctuations in demand. The problem is to minimise production and inventory holding costs, while meeting anticipated product demand.

The range of applications is diverse, but we can trace the following two common threads:

- Each of the applications involves optimisation, whether it is to maximise or to minimise something.
- Optimisation is always subject to constraints on what is possible.

Many practical management problems can be characterised as problems of constrained optimisation.

5.2 Two examples

5.2.1 Mark's LP model

The television manufacturing company is in the market to make money – its objective is to maximise profit. A profit of R300 is made on each set of model $VH200$ sold and R250 on each set of model $SB150$ sold.

We can see that the more $VH200$ sets manufactured and sold, the better. However, there are certain limitations which prevent the company from manufacturing and selling thousands of $VH200$ models.

These limitations are as follows:

- there are only 40 hours of labour time available per day for production
- there are only 45 hours of machine time available per day
- there is an inability to sell more than 12 sets of model $VH200$ per day.

To manufacture one set of model $VH200$, two hours of labour time and one hour of machine time is required. To manufacture one set of model $SB150$, one hour of labour time and three hours of machine time is required.

Mark's problem is to determine how many sets of each model to manufacture each day so that the total profit will be as large as possible.

Let us formulate his problem as an LP model.

Decision variables

What decisions must be made?

Mark must decide how many model $VH200$ and how many model $SB150$ television sets to manufacture each day. These decisions can be represented by the following *decision variables*:

$$\begin{aligned} VH &= \text{number of model } VH200 \text{ sets to manufacture daily,} \\ SB &= \text{number of model } SB150 \text{ sets to manufacture daily.} \end{aligned}$$

The decision variables used when formulating models should completely describe the decisions to be made.

Objective function

The objective is to maximise profit.

This objective can be expressed in the form of a function of the decision variables and is then called an objective function.

Proportionality is assumed, and this means that since the profit contribution of one set of model $VH200$ is R300, then the profit contribution of VH units of this model is $300 \times VH$.

Similarly, the profit resulting from the manufacture of SB units of model $SB150$ is $250 \times SB$.

The total profit is therefore $300VH + 250SB$. The objective function is then

$$\text{Maximise } PROFIT = 300VH + 250SB.$$

Constraints

What limitations, or restrictions, are there on the problem?

The labour and machine time is limited and there are restricted marketing capabilities. This means that there are limited resources. These resources restrict the number of television sets that can be manufactured. These restrictions are called *constraints*.

Labour constraint

One model $VH200$ set requires two hours of labour time and the unknown quantity VH requires $2 \times VH$ hours. Similarly $1 \times SB$ hours of labour time is required to manufacture the model $SB150$ sets.

The total labour time required is therefore $2VH + SB$.

There are 40 labour hours available for manufacturing the television sets.

The labour time used may not be more than the available labour hours. This can be expressed as

$$\begin{aligned} \text{labour hours required} &\leq \text{labour hours available} \\ \text{labour hours } VH200 + \text{labour hours } SB150 &\leq \text{labour hours available} \\ 2VH + SB &\leq 40. \end{aligned}$$

Note the $\leq$ sign. The total number of labour hours available, 40, need not necessarily all be used.

Machine constraint

The machine constraint can be deduced in a similar way as

$$VH + 3SB \leq 45.$$

Marketing constraint

The marketing capabilities are restricted and the result of this is that it is impossible to sell more than 12 sets of model $VH200$ daily. This constraint is represented as

$$VH \leq 12.$$

Sign restrictions

It is impossible to manufacture a negative number of television sets and so VH and SB must be nonnegative. These constraints are expressed as

$$VH \geq 0;\, SB \geq 0.$$

The LP model for Mark's problem is

$$\begin{array}{llcll}
\text{Maximise} & \multicolumn{4}{l}{PROFIT = 300VH + 250SB} \\
\text{subject to} & & & & \\
 & 2VH + SB & \leq & 40 & \text{(Labour time)} \\
 & VH + 3SB & \leq & 45 & \text{(Machine time)} \\
 & VH & \leq & 12 & \text{(Marketing)} \\
\text{and} & \multicolumn{4}{l}{VH;\, SB \geq 0.}
\end{array}$$

NOTE: Since production continues day after day, it is not necessary to complete all sets at the end of the day. A fractional number of sets is permissible (e.g. $5,7$ sets). However, had it been necessary to complete all sets by the end of the day, additional constraints limiting VH and SB to whole numbers would have been added. Such an addition changes the problem to one of integer programming, which will not be discussed in this module.

Note that one can use any symbols to represent the decision variables as long as one has defined what the symbols represent. In this example VH and SB were used to represent, respectively, the numbers of model VH200 sets to be manufactured daily and SB the number of SB150 sets to be manufactured daily. We could also have used, for example x_1 and x_2 to represent these two decision variables. The choice of symbols is up to you!

5.2.2 Christine's LP model

In preparing Sungold paint it is required that the paint has a brilliance rating of at least 300 degrees and a tint level of at least 250 degrees. Brilliance and tint levels are determined by the two ingredients, Alpha and Beta.

Each gram of Alpha produces one degree of brilliance in a tin of paint. Likewise for Beta. However, the tint is controlled entirely by the amount of Alpha, one gram of it producing three degrees of tint in one tin of paint.

The cost of Alpha is 45 cents per gram and the cost of Beta is 12 cents per gram.

Assuming that the objective is to minimise the cost of the ingredients, the problem is to find the quantity of Alpha and Beta to be included in the preparation of each tin of paint. An optimal answer for one tin will remain optimal for any number of tins as long as the relationships are linear. The total quantity of paint to be produced is of course more than one tin, and it is determined mainly by the demand and the manufacturing technology formulation.

Decision variables

The decision variables are

$$\begin{aligned} \alpha &= \text{quantity (in grams) of Alpha in each tin of paint,} \\ \beta &= \text{quantity (in grams) of Beta in each tin of paint.} \end{aligned}$$

Objective function

The objective is to minimise the total cost of the ingredients.

Since the cost of Alpha is 45 cents per gram and since α grams are to be used in each tin, the cost per tin is $45 \times \alpha$. Similarly for Beta, the cost is $12 \times \beta$.

The total cost is $45\alpha + 12\beta$, and the objective function is given by

$$\text{Minimise}\, Z = 45\alpha + 12\beta,$$

where Z represents the cost.

Constraints

Brilliance constraint

Each gram of Alpha produces one degree of brilliance in a tin of paint and so α grams of Alpha produce $1 \times \alpha$ degrees of brilliance. Similarly β grams of Beta produce $1 \times \beta$ degrees of brilliance in a tin of paint.

The total brilliance produced in a tin of paint by the two ingredients is then

$$\alpha + \beta.$$

A brilliance rating of at least 300 degrees per tin is required.

The brilliance produced must be at least the brilliance required. This can be expressed as

$$\begin{aligned} \text{brilliance produced} &\geq \text{brilliance required} \\ \text{brilliance from Alpha} + \text{brilliance from Beta} &\geq \text{brilliance required} \\ \alpha + \beta &\geq 300. \end{aligned}$$

Tint constraint

The tint constraint can be deduced in a similar way as

$$\begin{aligned} \text{tint produced} &\geq \text{tint required} \\ \text{tint from Alpha} + \text{tint from Beta} &\geq \text{tint required} \\ 3 \times \alpha + 0 \times \beta &\geq 250 \\ 3\alpha &\geq 250. \end{aligned}$$

Sign restrictions

It is impossible to use negative amounts of the ingredients and so

$$\alpha \geq 0;\ \beta \geq 0.$$

The LP model for Christine's problem is

$$\begin{array}{lllllll}
\text{Minimise} & Z = 45\alpha + 12\beta & & & & & \\
\text{subject to} & & & & & & \\
 & \alpha & + & \beta & \geq & 300 & \text{(Brilliance)} \\
 & 3\alpha & & & \geq & 250 & \text{(Hue)} \\
\text{and} & \alpha;\ \beta \geq 0. & & & & &
\end{array}$$

5.3 Components of an LP model

Formulation is actually more of an art than a science – and there is no foolproof "recipe" that can be followed to formulate a problem as an LP model. However, by formally defining the components of a model and following some broad directives, the process can be made easier. So let us take another look at the components of a model and use Mark and Christine's problems as a reference.

An LP model consists of the components as given below:

Decision variables

The sole purpose of formulating an LP model is to get answers to a problem. Decision variables are, as the name indicates, those variables that represent the decisions that have to be made. When the LP model is solved, the values obtained for the decision variables will be the answer to the problem.

To make sure that we know exactly what it is we have to decide about and what the answers will actually mean, we have to define the decision variables as clearly and completely as possible.

Mark's problem is solved when he can tell the company how many of each model television set to manufacture each day to maximise profit. The decision variables are therefore

$$\begin{aligned}
VH &= \text{the number of units of model } VH200 \text{ to manufacture daily,} \\
SB &= \text{the number of units of model } SB150 \text{ to manufacture daily.}
\end{aligned}$$

Christine's problem is solved when she can tell the Unique Paint Company how much of each ingredient to put into a tin of Sungold paint. The decision variables for her problem are therefore

$$\begin{aligned}\alpha &= \text{quantity, in grams, of Alpha in each tin of paint,}\\ \beta &= \text{quantity, in grams, of Beta in each tin of paint.}\end{aligned}$$

Objective function

The objective function is a mathematical expression, given as a linear function, that shows the relationship between the decision variables and a single goal (or objective) under consideration. Linear programming attempts to either maximise or minimise the value of the objective function.

Mark's objective function is

$$\text{Maximise } PROFIT = 300VH + 250SB.$$

Christine's objective function is

$$\text{Minimise } Z = 45\alpha + 12\beta,$$

where Z represents the cost.

Profit or cost coefficients

The coefficients in the objective function are either profit or cost coefficients.

In Mark's problem the objective is to maximise profit and the objective function is

$$\text{Maximise } PROFIT = 300VH + 250SB.$$

The 300 and 250 in this objective function are profit coefficients.

In Christine's problem the objective is to minimise cost and the objective function is

$$\text{Minimise } Z = 45\alpha + 12\beta.$$

The 45 and 12 in this objective function are cost coefficients.

Constraints

Optimisation is always performed subject to a set of constraints. Therefore, linear programming can be defined as a constrained optimisation technique. These constraints are expressed in the form of linear inequalities or, sometimes, equalities. They reflect the fact that resources are limited (in Mark's product-mix problem) or they specify the product requirements (in Christine's blending problem).

Activity (or technology) coefficients

The coefficients of the decision variables in the constraints are called the activity coefficients. They indicate the rate at which a given resource is depleted (Mark) or at which a requirement is met (Christine). They appear on the left-hand side of the constraints.

Right-hand elements

On the right-hand side of a constraint either the capacity, or availability, of a resource (as in Mark's problem) or the minimum requirements (as in Christine's problem) are given.

Sign restrictions

Only nonnegative (zero or positive) values of the decision variables are considered. This requirement merely specifies the fact that negative values of physical quantities do not exist.

5.4 General LP model

The general LP model can be presented in the following mathematical terms:

$$\text{Maximise}\quad F(x_1;\, x_2;\, \ldots;\, x_n) = c_1x_1 + c_2x_2 + \cdots + c_jx_j + \cdots + c_nx_n$$

subject to linear constraints

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{ln}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & & & & & \vdots & & \vdots \\
a_{il}x_1 & + & a_{i2}x_2 & + & \cdots & + & a_{in}x_n & \leq & b_i \\
\vdots & & & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m
\end{array}$$

and $x_1;\, x_2;\, \cdots;\, x_n \geq 0.$

Here

$$\begin{aligned}
a_{ij} &= \text{the activity coefficients,} \\
c_j &= \text{the profit (cost) coefficients,} \\
b_i &= \text{the right-hand elements,} \\
x_j &= \text{the decision variables,}
\end{aligned}$$

with

$$\begin{aligned}
i &= 1;\, 2;\, \ldots;\, m, \\
j &= 1;\, 2;\, \ldots;\, n, \\
n &= \text{number of decision variables,} \\
m &= \text{number of constraints.}
\end{aligned}$$

The notation is interpreted as follows:

The entry a_{22} is read as a-two-two. When the index consists of two figures, the first one denotes the row number and the second one the column number.

The entry a_{32} (a-three-two) denotes the coefficient in the third row, second column.

Learning Unit 6

Formulating linear programming models

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Learning objectives

After completing this learning unit you should be able to formulate an LP model to represent a given real life problem.

Sections from prescribed book

3.5. A work-Scheduling Problem - *p72*

3.8. Blending Problems - *p85*

3.9. Production Process Models - *p95*

3.11. Multiperiod Financial Models - *p105*

3.12. Multiperiod Work Scheduling - *p109*

6.1 A diet problem

This is a simplification of the classical diet problem.

The objective is to determine the type and amount of foods to be included in a daily diet to meet certain nutritional requirements at minimum cost. The foods we include are milk, tuna fish, wholewheat bread and spinach. The only nutrients we consider are vitamins A, C and D and iron.

The nutritional and cost data are given in the following table:

Nutrient	**Litre of milk**	**250 g tin of tuna fish**	**Loaf of bread**	**500 g packet of spinach**	**RDA**
Vit A (IU)	6 400	237	0	34 000	5 000
Vit C (mg)	40	0	0	71	75
Vit D (IU)	540	0	0	0	400
Iron (mg)	28	7	13	8	12
Cost	R4, 50	R6, 65	R4, 00	R5, 40	

where **RDA** stands for Recommended Daily Allowance.

Decision variables

We have to decide how much of each type of food to include in the daily diet. Let

X_1 = number of litres of milk to be included in the daily diet,
X_2 = number of 250 g tins of tuna fish to be included in the daily diet,
X_3 = number of loaves of bread to be included in the daily diet,
X_4 = number of 500 g packets of spinach to be included in the daily diet.

Objective function

The objective is to minimise cost. The objective function is therefore

$$\text{Minimise } COST = 4,50X_1 + 6,65X_2 + 4,00X_3 + 5,40X_4.$$

Constraints

We want to minimise the cost of the daily diet while meeting the RDA requirements. The constraints are as follows:

$$\begin{array}{rcrcrcrcrl}
6\,400X_1 & + & 237X_2 & & & + & 34\,000X_4 & \geq & 5\,000 & \text{(Vitamin A)} \\
40X_1 & & & & & + & 71X_4 & \geq & 75 & \text{(Vitamin C)} \\
540X_1 & & & & & & & \geq & 400 & \text{(Vitamin D)} \\
28X_1 & + & 7X_2 & + & 13X_3 & + & 8X_4 & \geq & 12 & \text{(Iron)}
\end{array}$$

The LP model is

$$\begin{array}{ll}
\text{Minimise} & COST = 4,50X_1 + 6,65X_2 + 4,00X_3 + 5,40X_4 \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
6\,400X_1 & + & 237X_2 & & & + & 34\,000X_4 & \geq & 5\,000 \\
40X_1 & & & & & + & 71X_4 & \geq & 75 \\
540X_1 & & & & & & & \geq & 400 \\
28X_1 & + & 7X_2 & + & 13X_3 & + & 8X_4 & \geq & 12
\end{array} \\
\text{and} & X_1; X_2; X_3; X_4 \geq 0.
\end{array}$$

6.2 A production/transportation problem

In this example, two production plants located in different parts of the country must produce and distribute a product to three regional warehouses.

The labour and power costs are less at plant 1 where each unit is produced at a cost of R4, 50. Each unit at plant 2 is produced at a cost of R6. Plant 1 has a production capacity of 2 000 units and plant 2 has a production capacity of 3 000 units.

The three warehouses have demands of 500, 2 000 and 900 units of the product respectively.

The cost of shipping, based primarily on distance, is given in the following table:

Cost of shipping one unit (rand)			
	To warehouse		
From plant	**1**	**2**	**3**
1	1,3	1,9	1,8
2	1,7	1,2	1,4

The problem is to meet all demands while minimising the combined cost of production and distribution. The decision to be made concerns the number of units to transport from each plant to each warehouse.

Decision variables

Let

$$\begin{aligned}
P_1W_1 &= \text{number of units shipped from plant 1 to warehouse 1,}\\
P_1W_2 &= \text{number of units shipped from plant 1 to warehouse 2,}\\
P_1W_3 &= \text{number of units shipped from plant 1 to warehouse 3,}\\
P_2W_1 &= \text{number of units shipped from plant 2 to warehouse 1,}\\
P_2W_2 &= \text{number of units shipped from plant 2 to warehouse 2,}\\
P_2W_3 &= \text{number of units shipped from plant 2 to warehouse 3.}
\end{aligned}$$

We can also write

$$P_iW_j = \text{number of units shipped from plant } i \text{ to warehouse } j$$
$$\text{where } i = 1,\ 2 \text{ and } j = 1,\ 2,\ 3.$$

Objective function

Since each unit that is shipped must first be produced, the total cost includes the production and the shipping costs. The objective function is

$$\begin{aligned}\text{Minimise } COSTS &= (4,5+1,3)P_1W_1 + (4,5+1,9)P_1W_2 + (4,5+1,8)P_1W_3\\ &\quad +(6+1,7)P_2W_1 + (6+1,2)P_2W_2 + (6+1,4)P_2W_3.\end{aligned}$$

Constraints

The capacity at plant 1 is 2 000 units. Therefore, no more than 2 000 units can be shipped from plant 1. Likewise, no more than 3 000 units can be shipped from plant 2. The constraints on supply are

$$\begin{aligned}P_1W_1 + P_1W_2 + P_1W_3 &\leq 2\,000\\ P_2W_1 + P_2W_2 + P_2W_3 &\leq 3\,000.\end{aligned}$$

All demands must be met. To meet the demand at warehouse 1, all shipments sent to warehouse 1 must add up to 500. The shipments to warehouses 2 and 3 must add up to 2 000 and 900, respectively. The demand constraints are therefore

$$\begin{aligned}P_1W_1 + P_2W_1 &= 500\\ P_1W_2 + P_2W_2 &= 2\,000\\ P_1W_3 + P_2W_3 &= 900.\end{aligned}$$

The LP model is

Minimise $COSTS = 5,8P_1W_1 + 6,4P_1W_2 + 6,3P_1W_3 + 7,7P_2W_1 + 7,2P_2W_2 + 7,4P_2W_3$

subject to

$$\begin{array}{rcrcrcrl}P_1W_1 & + & P_1W_2 & + & P_1W_3 & \leq & 2\,000 & \text{(Capacity at plant 1)}\\ P_2W_1 & + & P_2W_2 & + & P_2W_3 & \leq & 3\,000 & \text{(Capacity at plant 2)}\\ & & P_1W_1 & + & P_2W_1 & = & 500 & \text{(Demand at warehouse 1)}\\ & & P_1W_2 & + & P_2W_2 & = & 2\,000 & \text{(Demand at warehouse 2)}\\ & & P_1W_3 & + & P_2W_3 & = & 900 & \text{(Demand at warehouse 3)}\end{array}$$

and $P_iW_j \geq 0$ for $i = 1$, 2 and $j = 1$, 2, 3.

6.3 A blending problem

Green Turf Nursery wants to sell its own brand of lawn fertilizer this year. It plans to sell two types of fertilizer, one high in nitrogen content and the other one an all-purpose fertilizer. The fertilizers are mixed from two different compounds that contain nitrogen and phosphorus in different amounts.

The nitrogen and phosphorus content of each compound and its cost per kilogram are given in the following table:

Cost and composition of fertilizer compounds			
Compound	**Cost per kg**	**Nitrogen** (%)	**Phosphorus** (%)
1	R5,00	60	10
2	R6,00	10	40

The estimated demand this season is 5 000 bags of high-nitrogen fertilizer and 7 000 bags of all-purpose fertilizer. A bag weighs 25 kilograms.

The nitrogen content of the high-nitrogen fertilizer must be between 40 and 50 percent. And the phosphorus content of the all-purpose fertilizer must be at most 20 percent.

Green Turf must determine how many kilograms of each compound to purchase in order to satisfy the estimated demand at the minimum cost.

We can visualise this blending problem in Figure 6.1.

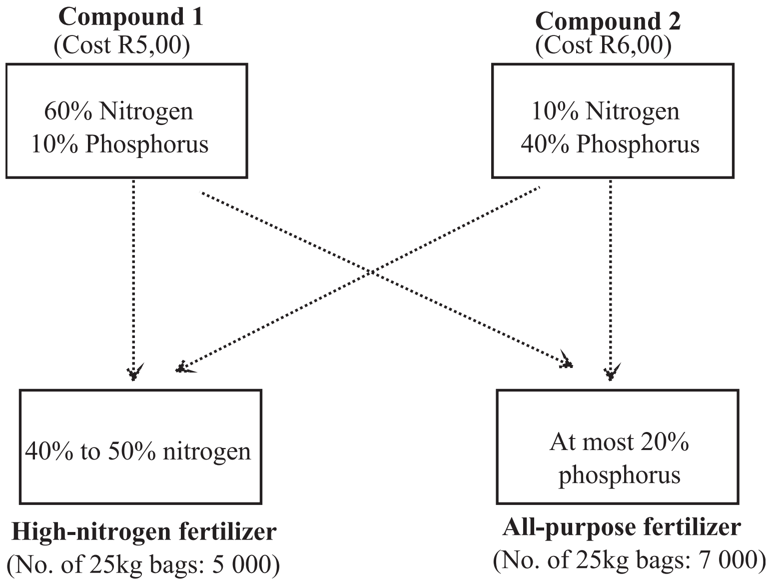

Figure 6.1: Blending problem

Decision variables

Let

$$\begin{aligned}
C1HN &= \text{number of kg of compound 1 purchased for high-nitrogen fertilizer,}\\
C2HN &= \text{number of kg of compound 2 purchased for high-nitrogen fertilizer,}\\
C1AP &= \text{number of kg of compound 1 purchased for all-purpose fertilizer,}\\
C2AP &= \text{number of kg of compound 2 purchased for all-purpose fertilizer.}
\end{aligned}$$

The objective function is

$$\text{Minimise } COST = 5C1HN + 5C1AP + 6C2HN + 6C2AP.$$

Constraints

Demand constraints

Assuming that Green Turf wants to at least meet the estimated demand, the demand constraint for the high-nitrogen fertilizer can be represented as

$$\text{high-nitrogen fertilizer mixed} \geq \text{demand.}$$

The high-nitrogen fertilizer is made up of the two compounds and the total amount mixed is given in kilograms by $C1HN + C2HN$.

The demand for high-nitrogen fertilizer is 5 000 bags, and each bag weighs 25 kilograms. The demand for high-nitrogen fertilizer in kilograms is then 5 000 × 25.

When writing down the constraints of a problem, it is very important that all terms in a particular constraint have the same units. The units on the left-hand side of a constraint must be the same as the units on the right-hand side.

The demand constraint for high-nitrogen fertilizer is

$$\begin{aligned}
\text{high-nitrogen fertilizer mixed} &\geq \text{demand}\\
\text{no. of kg of high-nitrogen fertilizer mixed} &\geq \text{demand in kg}\\
C1HN + C2HN &\geq 5\ 000 \times 25.
\end{aligned}$$

The demand constraint for the all-purpose fertilizer can be deduced in the same way as

$$C1AP + C2AP \geq 7\ 000 \times 25.$$

High-nitrogen fertilizer content constraints

The high-nitrogen fertilizer must contain between 40 and 50 percent nitrogen. This involves two different constraints. Let us consider the lower limit of 40 percent nitrogen first.

The requirement that the high-nitrogen fertilizer must contain at least 40 percent nitrogen means that at least 40 percent of the total amount of high-nitrogen fertilizer must be nitrogen, in other words

$$\begin{aligned} \text{total nitrogen content} &\geq 40\% \text{ of total amount of high-nitrogen fertilizer} \\ \text{total nitrogen content} &\geq 0,4(C1HN + C2HN). \end{aligned}$$

Now we must determine the nitrogen content of the high-nitrogen fertilizer.

Compound 1 contains 60 percent nitrogen, which means that 60 percent of compound 1 is nitrogen. Therefore, the nitrogen contribution of compound 1 is $0,6C1HN$.

Compound 2 contains 10 percent nitrogen and therefore the nitrogen contribution of compound 2 is $0,1C2HN$.

The total nitrogen content of the high-nitrogen fertilizer is therefore

$$0,6C1HN + 0,1C2HN.$$

The lower limit of the high-nitrogen fertilizer content constraint is therefore

$$\begin{aligned} \text{total nitrogen content} &\geq 40\% \text{ of total amount of high-nitrogen fertilizer} \\ 0,6C1HN + 0,1C2HN &\geq 0,4(C1HN + C2HN) \\ 0,2C1HN - 0,3C2HN &\geq 0. \end{aligned}$$

Similarly, the upper limit of 50 percent nitrogen in the high-nitrogen fertilizer content constraint is

$$\begin{aligned} \text{total nitrogen content} &\leq 50\% \text{ of total amount of high-nitrogen fertilizer} \\ 0,6C1HN + 0,1C2HN &\leq 0,5(C1HN + C2HN) \\ 0,1C1HN - 0,4C2HN &\leq 0. \end{aligned}$$

All-purpose fertilizer content constraint

The all-purpose fertilizer must contain at most 20 percent phosphorus. This constraint can be deduced from

$$\begin{aligned} \text{total phosphorus content} &\leq 20\% \text{ of total amount of all-purpose fertilizer} \\ 0,1C1AP + 0,4C2AP &\leq 0,2(C1AP + C2AP) \\ -0,1C1AP + 0,2C2AP &\leq 0. \end{aligned}$$

Sign-restrictions

The number of kilograms of each compound used cannot be negative and so all the decision variables must be nonnegative.

The LP model is

Minimise $COST = 5C1HN + 5C1AP + 6C2HN + 6C2AP$

subject to

$$\begin{array}{rcrcrcrcl} C1HN & + & C2HN & & & & & \geq & 125\,000 \\ & & & & C1AP & + & C2AP & \geq & 175\,000 \\ 0,2C1HN & - & 0,3C2HN & & & & & \geq & 0 \\ 0,1C1HN & - & 0,4C2HN & & & & & \leq & 0 \\ & & & & -0,1C1AP & + & 0,2C2AP & \leq & 0 \end{array}$$

and $C1HN; C2HN; C1AP; C2AP \geq 0.$

6.4 A portfolio selection problem

Portfolio selection is a financial decision problem in which specific investment alternatives must be selected, given a certain budget.

The alternatives generally consist of common stocks, bonds and other securities. The objective is usually to maximise return or minimise risk. When the relationships involved are linear, we can apply linear programming.

The Plus Credit Corporation has R500 000 which has accrued from previous investments. Management would like to reinvest the R500 000 in a portfolio that will maximise return while maintaining an acceptable level of risk.

The investment alternatives, rates of return and risk measurements are as given in the following table:

Investment alternative	**Expected rate of return (%)**	**Risk factor (σ)**
Municipal bonds	7,5	0,2
Government bonds	6,5	0,0
Midland Oil stock	11,9	6,0
Western Coal stock	8,8	4,5
Northern Automative stock	10,0	5,0

Management has decided that the average risk factor (σ) for the entire portfolio should not exceed $3,0$.

It also wants to invest at least 20 percent in bonds, but at least 30 percent of the bond investment must be in government bonds. Its final policy is to invest no more than R250 000 in Western Coal stock.

Decision variables

The decision variables should reflect the basic question that needs to be answered: how much should Plus invest in each of the five alternatives? Let

$$\begin{aligned} MB &= \text{amount invested in municipal bonds,} \\ GB &= \text{amount invested in government bonds,} \\ MOS &= \text{amount invested in Midland Oil stock,} \\ WCS &= \text{amount invested in Western Coal stock,} \\ NAS &= \text{amount invested in Northern Automotive stock.} \end{aligned}$$

Objective function

The objective is to maximise return.

For municipal bonds, the expected rate of return is $7,5$ percent. This means that for every R1 invested, the yield will be $7,5\% \times$ R1 $=$ R$0,075$. The total return on an investment is the amount of the investment plus the yield on the investment and so the total return on the R1 invested in municipal bonds will be R1 $+$ R$0,075 =$ R$1,075$. The total return on the unknown amount MB invested in municipal bonds is then R$1,075 \times MB$.

The total returns on the other investment alternatives can be deduced in the same way. This will result in the following objective function:

$$\text{Maximise } RETURN = 1,075MB + 1,065GB + 1,119MOS + 1,088WCS + 1,1NAS.$$

Constraints

Budget constraint

Plus wants to invest all the available money and therefore the budget constraint is

$$\begin{aligned} \text{amount invested} &= \text{amount available} \\ MB + GB + MOS + WCS + NAS &= 500\,000. \end{aligned}$$

Risk constraint

For municipal bonds the risk factor for every R1 invested is $0,2$ and so the risk for the total investment in municipal bonds is $0,2MB$.

The total risks for the entire portfolio will be $0,2MB + 6,0MOS + 4,5WCS + 5,0NAS$.

The average risk factor for the entire portfolio should not exceed $3,0$. Therefore,

$$\begin{aligned}
\text{average risk factor} &\leq 3,0 \\
\frac{\text{total risks}}{\text{total investment amount}} &\leq 3,0 \\
\frac{0,2MB + 6,0MOS + 4,5WCS + 5,0NAS}{500\,000} &\leq 3,0 \\
0,2MB + 6,0MOS + 4,5WCS + 5,0NAS &\leq 3,0 \times 500\,000.
\end{aligned}$$

Total bond investment constraint

At least 20 percent of the investment must be in bonds. Therefore,

$$\begin{aligned}
\text{investment in bonds} &\geq 20\%\text{ of total investment amount} \\
MB + GB &\geq 0,2 \times 500\,000.
\end{aligned}$$

Government bond investment constraint

The amount invested in Government bonds must be at least 30 percent of the amount invested in bonds. Therefore,

$$\begin{aligned}
GB &\geq 0,3(MB + GB) \\
-0,3MB + 0,7GB &\geq 0.
\end{aligned}$$

Western Coal stock constraint

No more than R250 000 may be invested in Western Coal stock. Therefore,

$$WCS \leq 250\,000.$$

The LP model is

Maximise $RETURN = 1,075MB + 1,065GB + 1,119MOS + 1,088WCS + 1,1NAS$

subject to

$$\begin{array}{rcrcrcrcrcr}
MB & + & GB & + & MOS & + & WCS & + & NAS & = & 500\,000 \\
0,2MB & & & + & 6MOS & + & 4,5WCS & + & 5,0NAS & \leq & 1\,500\,000 \\
MB & + & GB & & & & & & & \geq & 100\,000 \\
-0,3MB & + & 0,7GB & & & & & & & \geq & 0 \\
 & & & & & & WCS & & & \leq & 250\,000
\end{array}$$

and $MB; GB; MOS; WCS; NAS \geq 0.$

6.5 A multi-phase portfolio selection problem

Sekiedo Corporation has $R14,5$ million available for investment now and will have a further $R2,5$ million available next year.

Sekiedo wants as much cash as possible at the end of a three-year period for a take-over bid which has been planned well in advance. There are seven investment alternatives available during this period. Some of these are available only after a year and others after two years.

Information on the investment alternatives is given in the following table:

Investment alternative	**Period until investment**	**Duration of investment**	**Return (%)**	**Type of investment**
Alpha	0	3	100	flats
Beta	0	1	20	loans
Theta	0	2	50	flats
Gamma	1	2	80	shares
Lambda	1	1	20	shares
Kappa	1	2	50	minerals
Delta	2	1	30	minerals

Decision variables

Let

$$\begin{aligned}
\alpha &= \text{amount invested in Alpha,}\\
\beta &= \text{amount invested in Beta,}\\
\theta &= \text{amount invested in Theta,}\\
\gamma &= \text{amount invested in Gamma,}\\
\lambda &= \text{amount invested in Lambda,}\\
\kappa &= \text{amount invested in Kappa,}\\
\delta &= \text{amount invested in Delta.}
\end{aligned}$$

All amounts are in millions of rands.

It is obvious that time will be very important in this system. Money invested in Beta will be available after a year, with interest, for reinvestment by Sekiedo. The amount available for reinvestment after a year will have an effect on the choice to made then, meaning that Gamma, Lambda and Kappa are dependent on Beta.

To simplify these relationships we represent the flow of money over time. From the table we see that the first three investments may be made immediately. After a year it must be decided how the funds then available are to be divided between Gamma, Lambda and Kappa, etc. This can be visualised in Figure 6.2, or in Figure 6.3.

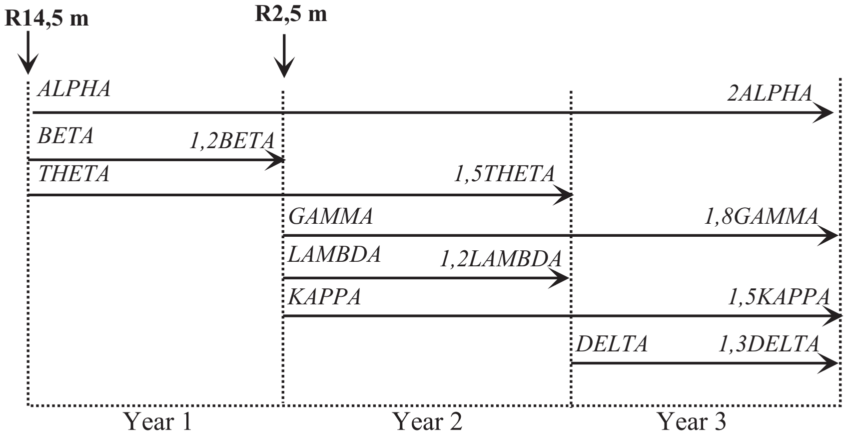

Figure 6.2: Sekiedo time flow of money

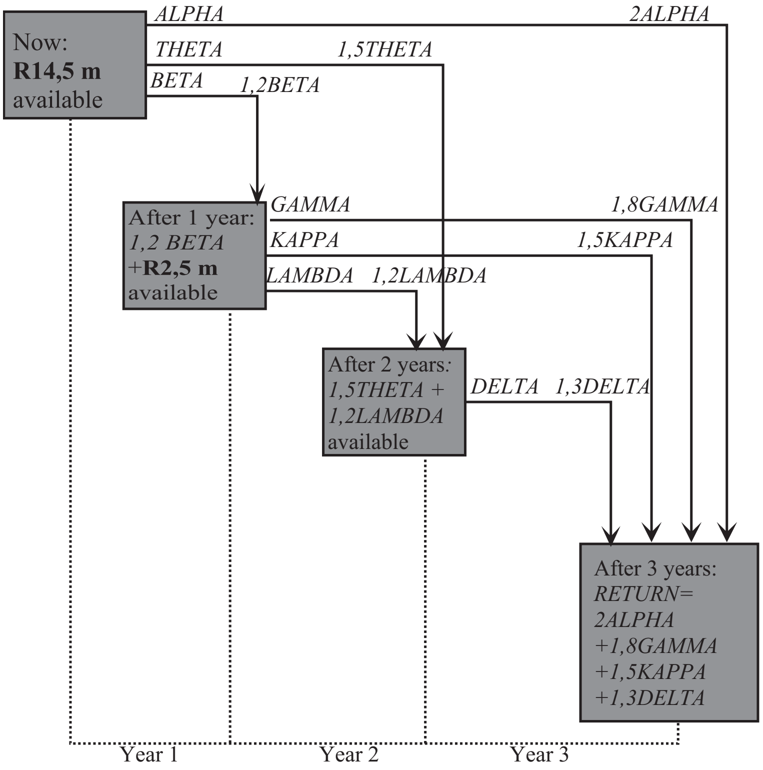

Figure 6.3: Sekiedo ime flow of money

In the graphs above

$ALPHA = \alpha$

$BETA = \beta$

$ALPHA = \alpha$

$GAMMA = \gamma$

$LAMBDA = \lambda$

$DELTA = \delta$

$THETA = \theta$

$KAPPA = \kappa$

Objective function

The goal is to have as much cash as possible at the end of the term. The objective function is therefore

$$\text{Maximise } RETURN = 2\alpha + 1,8\gamma + 1,5\kappa + 1,3\delta.$$

Constraints

Year 1 constraint

At the beginning of year 1 there is R14,5 million available for investment. Alpha, Beta and Theta are the only investment alternatives that can be invested in. The year 1 constraint is then

$$\alpha + \beta + \theta = 14,5.$$

Why the "=" sign and not the "$\leq$" sign?

The latter would mean that we could keep out an amount in cash until a better investment opportunity became available. We would do this only if it were impossible to invest this amount and earn interest on it until the better option became available. Beta is a one-year investment earning 20 percent interest. It is better to invest the amount in Beta for a year than to keep it in cash until the other investments become available after a year. The entire sum is therefore divided between Alpha, Beta and Theta. This argument also applies at the other two time-points when investments are made and each distribution constraint will therefore be an equality.

Year 2 constraint

After a year Beta has grown to $1,2\beta$ and an additional R2,5 million is available for investment. (Remember, at the end of the term the invested amount is paid out plus the interest earned.) At the beginning of year 2 this amount must be divided between Gamma, Lambda and Kappa, the investment alternatives available at this stage. Therefore, the year 2 constraint is

$$\begin{array}{lrcrcrcrcl} & & & \gamma & + & \lambda & + & \kappa & = & 2,5+1,2\beta \\ \text{or} & -1,2\beta & + & \gamma & + & \lambda & + & \kappa & = & 2,5. \end{array}$$

Year 3 Constraint

At the beginning of year 3 an amount of $1,5\theta + 1,2\lambda$ is available for investment and Delta is the only investment alternative that can be invested in. The year 3 constraint is

$$\begin{array}{lrcrcrcl} & & & & & \delta & = & 1,5\theta+1,2\lambda \\ \text{or} & -1,5\theta & - & 1,2\lambda & + & \delta & = & 0. \end{array}$$

The LP model is

Maximise $RETURN = 2\alpha + 1,8\gamma + 1,5\kappa + 1,3\delta$

subject to

$$\begin{array}{rcrcrcrcrcrcrcl} \alpha & + & \beta & + & \theta & & & & & & & & & = & 14,5 \\ & & -1,2\beta & & & + & \gamma & + & \lambda & + & \kappa & & & = & 2,5 \\ & & & & -1,5\theta & & & - & 1,2\lambda & & & + & \delta & = & 0 \end{array}$$

and α; β; θ; γ; λ; κ; $\delta \geq 0$.

6.6 Exercises and Solutions

6.6.1 Exercises

1. Handy-Dandy Company wishes to schedule the production of a kitchen appliance which requires two resources – labour and material. The company is considering three different models. The production department has furnished the following data on the requirements and the profit contribution of the different models:

	Model		
	A	**B**	**C**
Labour (hours/unit)	7	3	6
Material (kg/unit)	4	4	5
Profit (R/unit)	4	2	3

The supply of raw material is restricted to 200 kilograms per day. The daily availability of manpower is 150 hours.

Formulate a linear programming model to determine the daily production rate of the various models in order to maximise the total profit.

2. An advertising company is planning an advertising campaign in three different media – television, radio and magazines. The purpose of the advertising campaign is to reach as many potential customers as possible. Results of a market study are given below (values in thousands):

	Television			
	Day time	**Prime time**	**Radio**	**Magazines**
Cost per advertising unit (in R)	40	75	30	15
Number of potential customers reached per advertising unit	400	900	500	200
Number of female customers reached per advertising unit	300	400	200	100

The company does not want to spend more than R800 000 on advertising.

It further requires that

- at least two million women are reached
- advertising on television is limited to R500 000
- at least three advertising units are bought on daytime television, and at least two units during prime time
- the number of advertising units on radio and magazines should each be between five and ten.

Formulate an LP model for this problem.

3. A company has two grades of inspectors, grade 1 and grade 2, who are to be assigned for a quality control inspection. It is required that at least 1 800 items be inspected per 8-hour day. Each grade 1 inspector can check items at the rate of 25 items per hour, with an accuracy of 98 percent. Each grade 2 inspector can check at the rate of 15 items per hour, with an accuracy of 95 percent.

   The wage rate of a grade 1 inspector is $R40,00$ per hour, while that of a grade 2 inspector is $R30,00$ per hour. Each time an error is made by an inspector, the cost to the company is $R20,00$. The company has eight grade 1 inspectors and ten grade 2 inspectors available. The company wants to determine the optimal assignment of inspectors that will minimise the total cost of the inspection.

   Formulate an LP model for this problem.

4. Top Speed Bicycle Company manufactures and markets a line of 20-speed bicycles nationwide. The firm has assembly plants in Cheapside and Oldprice. Its three warehouses are located near the large market areas of Freetown, Pricey and Expense.

The demand for the next year at the Freetown warehouse is 10 000 bicycles, at the Pricey warehouse 8 000 bicycles and at the Expense warehouse 15 000 bicycles. The assembly plant capacity at each location is limited. The Cheapside plant can produce 20 000 bicycles, while the Oldprice plant can produce 15 000 bicycles per year.

The costs (in rands per bicycle) of shipping bicycles from the assembly plants to the warehouses are as follows:

	To		
From	**Freetown**	**Pricey**	**Expense**
Cheapside	2	3	5
Oldprice	3	1	4

Formulate an LP model to help the company determine a shipping schedule which will minimise its total annual transportation costs if all demands are met.

6.6.2 Solutions to exercises

1. Let

$$\begin{aligned} NA &= \text{number of model A produced daily,} \\ NB &= \text{number of model B produced daily,} \\ NC &= \text{number of model C produced daily.} \end{aligned}$$

The LP model is

$$\begin{array}{ll} \text{Maximise} & PROFIT = 4NA + 2NB + 3NC \\ \text{subject to} & \\ & \begin{array}{rcrcrcr} 7NA & + & 3NB & + & 6NC & \leq & 150 \\ 4NA & + & 4NB & + & 5NC & \leq & 200 \end{array} \\ \text{and} & NA;\ NB;\ NC \geq 0. \end{array}$$

2. Let

$$\begin{aligned} DTT &= \text{number of advertising units bought on daytime television,} \\ PTT &= \text{number of advertising units bought on prime-time television,} \\ RAD &= \text{number of advertising units bought on radio,} \\ MAG &= \text{number of advertising units bought in magazines.} \end{aligned}$$

The objective is to maximise the total number of potential customers reached.

The LP model is

$$\text{Maximise} \quad NUMBER = 400DTT + 900PTT + 500RAD + 200MAG$$

subject to

$$\begin{array}{rcrcrcrcr}
40DTT & + & 75PTT & + & 30RAD & + & 15MAG & \leq & 800 \\
300DTT & + & 400PTT & + & 200RAD & + & 100MAG & \geq & 2\,000 \\
40DTT & + & 75PTT & & & & & \leq & 500 \\
DTT & & & & & & & \geq & 3 \\
 & & PTT & & & & & \geq & 2 \\
 & & & & RAD & & & \geq & 5 \\
 & & & & RAD & & & \leq & 10 \\
 & & & & & & MAG & \geq & 5 \\
 & & & & & & MAG & \leq & 10.
\end{array}$$

NOTE: Lower limits are included in the constraints. The nonnegativity constraints are therefore not necessary. Monetary values and customer numbers are in thousands.

3. Let

G_1 = number of grade 1 inspectors assigned,

G_2 = number of grade 2 inspectors assigned.

The objective is to minimise the total cost of inspection per day which consists of two types of cost: wages paid to inspectors and the cost of inspection errors.

A grade 1 inspector earns R40 per hour and each error he makes costs R20. Each grade 1 inspector therefore costs the company $40 + 10 = 50$ rand per hour. The error cost of R10 is deduced as follows:

The error rate of a grade 1 inspector is 2 percent. In one hour he checks 25 items and so the number of errors made per hour is $0,02 \times 25 = 0,5$. The cost per error is R20 and therefore the error cost per hour is R20 × number of errors made in an hour

$$C = 20 \times 0,5 = 10.$$

The error cost per hour is R10.

Similarly, each grade 2 inspector costs $30 + 20(15)(0,05) = 45$ rand per hour.

The daily cost must be minimised and therefore the objective function is given by

$$\text{Minimise } COST = 8(50G_1 + 45G_2) = 400G_1 + 360G_2.$$

The number of available inspectors in each grade is limited. So we have the following constraints:

$$\begin{aligned} G_1 &\leq 8 && \text{(Grade 1)} \\ G_2 &\leq 10 && \text{(Grade 2)} \end{aligned}$$

The company requires at least 1 800 items to be inspected per 8-hour day. Each grade 1 inspector can inspect 8 × 25 items and each grade 2 inspector can inspect 8 × 15 items per 8-hour day. Therefore,

$$\begin{array}{lrcrcr} & 200G_1 & + & 120G_2 & \geq & 1\,800 \\ \text{or} & 5G_1 & + & 3G_2 & \geq & 45. \end{array}$$

The LP model is

$$\begin{array}{ll} \text{Minimise} & COST = 400G_1 + 360G_2 \\ \text{subject to} & \\ & \begin{array}{rcrcr} G_1 & & & \leq & 8 \\ & & G_2 & \leq & 10 \\ 5G_1 & + & 3G_2 & \geq & 45 \end{array} \\ \text{and} & G_1;\, G_2 \geq 0. \end{array}$$

4. Let

CF = number of bicycles shipped from Cheapside to Freetown,
CP = number of bicycles shipped from Cheapside to Pricey,
CE = number of bicycles shipped from Cheapside to Expense,
OF = number of bicycles shipped from Oldprice to Freetown,
OP = number of bicycles shipped from Oldprice to Pricey,
OE = number of bicycles shipped from Oldprice to Expense.

The objective is to minimise total transportation costs.

The LP model is

$$\text{Minimise} \quad COST = 2CF + 3CP + 5CE + 3OF + OP + 4OE$$

subject to

$$\begin{array}{rcrcrcrl}
CF & + & CP & + & CE & \leq & 20\,000 & \text{(Cheapside's capacity)} \\
OF & + & OP & + & OE & \leq & 15\,000 & \text{(Oldprice's capacity)} \\
 & & CF & + & OF & = & 10\,000 & \text{(Demand at Freetown)} \\
 & & CP & + & OP & = & 8\,000 & \text{(Demand at Pricey)} \\
 & & CE & + & OE & = & 15\,000 & \text{(Demand at Expense)}
\end{array}$$

and all variables ≥ 0.

6.6.3 Evaluation exercises

Question 1 – *Winston* 3.1, Problem 4, p.55 (*Old Winston* 3.1, Problem 4, p.56)

IGNORE the last sentence of this problem and do the following:

1. Summarise the given information in tabular form.
2. Calculate the time needed to paint one Type 1 truck and the time needed to paint one Type 2 truck.
3. Calculate the time needed to assemble one Type 1 truck and the time needed to assemble one Type 2 truck.
4. Construct a table that will reflect the results calculated in (2) and (3) above.
5. Define the decision variables for this problem.
6. Formulate an LP model to maximise Truckco's profit.

Question 2 – *Winston* 3.8, Problem 3, p.92 (*Old Winston* 3.8, Problem 3, p.91)

Question 3 – *Winston* 3.11, Problem 4, p.108 (*Old Winston* 3.11, Problem 4, p.107)

IGNORE the last sentence of this problem.

The following points will clarify this problem:

- Steve buys and sells the bonds in the current year.
- He wishes to maximise his profit in the current year (profit is his revenue from selling bonds less his payment for buying bonds).
- The cash payments are paid out over the next three years.

- For each bond that Steve buys in the current year, he receives a cash payment in each of the next three years.
- For each bond that Steve sells in the current year, he pays a cash payment in each of the next three years.
- Steve's cash position at the end of year 1 is equal to the cash payments received during year 1.
- Steve's cash position at the end of year i is equal to his cash position at the end of year $i-1$ (discounted) plus the cash payment he receives during year i, where $i = 2, 3$.

Learning Unit 7

Graphical representation

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Learning objectives

After completing this study unit you should be able to

- use the graphical approach to determine the optimal solution to an LP model in two variables
- identify infeasibility, unbounded solutions, multiple solutions and degeneracy from a graph
- identify redundant, binding and nonbinding constraints from a graph.

Sections from prescribed book

3.1. Feasible Region and Optimal Solution - *p54*

3.2. The Graphical Solution of Two-Variable Linear Programming Problems - *p56*

3.3. Special Cases - *p63*

7.1 A graphical approach

We have now spent a considerable time "transforming" problems into linear programming models. We will also spend a considerable time solving them using a solution method called the simplex method. However, before doing this, we are going to use a graphical representation of simple LP models to illustrate certain characteristics of LP models. A graphical approach may also be used to solve small problems in two decision variables with only a few constraints.

The nonnegativity constraints of linear programming problems restrict the graphical representation to the first quadrant only. The graphical approach consists of two phases:

- graphical representation of the feasible area
- identifying the optimal solution.

As an illustration we return to Mark's problem. This problem was formulated in Section 5.2.1 as follows:

$$\begin{array}{ll}
\text{Maximise} & PROFIT = 300VH + 250SB \\
\text{subject to} & \\
& \begin{array}{rcrcll} 2VH & + & SB & \leq & 40 & \text{(Labour time)} \\ VH & + & 3SB & \leq & 45 & \text{(Machine time)} \\ VH & & & \leq & 12 & \text{(Marketing)} \end{array} \\
\text{and} & VH;\ SB \ \geq 0.
\end{array}$$

Here

VH = number of sets of model $VH200$ produced daily,
SB = number of sets of model $SB150$ produced daily.

7.2 Finding the feasible area

The feasible area is established by graphing all of the constraints, which are in the form of inequalities and/or equations. The two decision variables under consideration in Mark's problem are VH and SB. It does not matter which axis is used to represent which variable. We arbitrarily choose the horizontal axis to represent VH.

Both VH and SB are $\geq$ 0. We therefore use only the first quadrant for the graphical representation. We have to represent the three constraints, which in this case are inequalities, on the graph. And we start by considering the equality part only (= sign), dropping the less than part (< signs):

$$\begin{array}{rcrcll} 2VH & + & SB & = & 40 & \quad (1) \\ VH & + & 3SB & = & 45 & \quad (2) \\ VH & & & = & 12. & \quad (3) \end{array}$$

In order to draw a line, we need two points. The easiest way to do this is as follows: set the one variable equal to zero and find the value for the other variable. This gives a point on one axis. Repeat the process for the other variable to find a point on the other axis. Connect the points to draw the line.

Consider equation (1): $2VH + SB = 40$.

Let $VH = 0$. Then $SB = 40$. The point on the SB-axis is $(0; 40)$.
Let $SB = 0$. Then $VH = 20$. The point on the VH-axis is $(20; 0)$.

Plot the two points on the axes and draw the line connecting these points.

Consider equation (2): $VH + 3SB = 45$.

Let $VH = 0$. Then $SB = 15$. The point on the SB-axis is $(0; 15)$.
Let $SB = 0$. Then $VH = 45$. The point on the VH-axis is $(45; 0)$.

Plot the two points on the axes and draw the line.

Consider equation (3): $VH = 12$.

Any line consisting of one variable only is a line parallel to the axis representing the other variable. In this case, draw a line parallel to the SB-axis, passing through the point $(12; 0)$.

These three lines are drawn and are given in Figure 7.1.

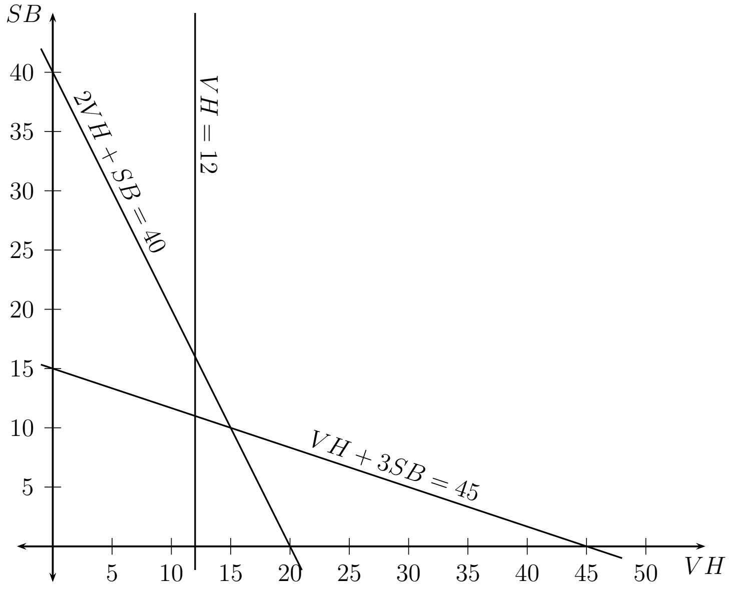

Figure 7.1: Equality constraints

But these lines do not tell us anything about the inequalities.

Consider the first constraint of Mark's model, the inequality $2VH + SB \leq 40$.

This inequality states that the value of $2VH + SB$ must be less than or equal to 40. The equality part (= sign) is easy – all the points on the line $2VH + SB = 40$ will satisfy this requirement. To deal with the "less than" part (< sign), we choose any point not on the line. If this point satisfies the inequality, then all points on the same side of the line will also satisfy the inequality. Conversely, if the point does not satisfy the inequality, then all points on the opposite side of the line will satisfy the inequality.

The easiest way to determine on which side of the line the inequality is satisfied, is to substitute the point $(0; 0)$ into the inequality. Then

$$2VH + SB = 2(0) + 0 = 0 < 40.$$

The point $(0; 0)$ therefore satisfies the inequality and all points on the same side of the line as $(0; 0)$ will satisfy the inequality.

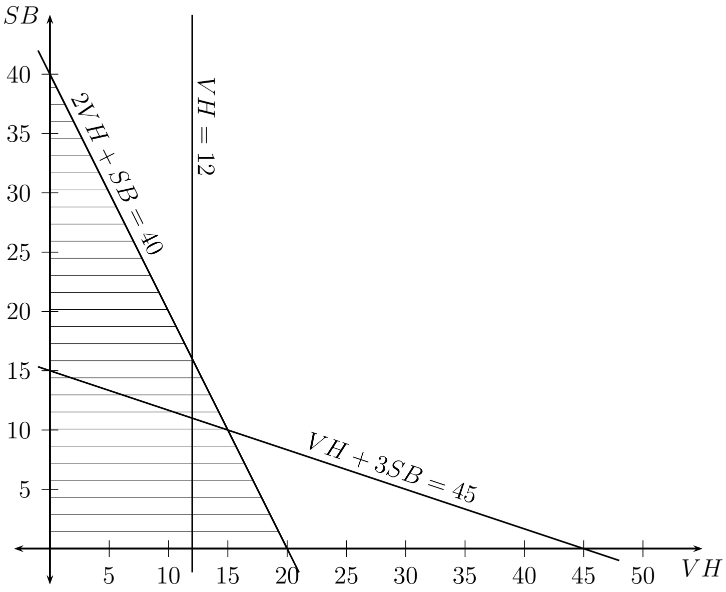

Figure 7.2: Labour time inequality

The area satisfying the inequality is represented by the horizontal lines in Figure 7.2.

All the points in the lined area as well as the points on the line $2VH + SB = 40$ satisfy the inequality $2VH + SB \leq 40$. We call this the solution set of the inequality.

NOTE: The equality sign $(=)$ is part of the inequality $2VH + SB \leq 40$ and therefore the line $2VH+SB = 40$ is represented by a solid line on the graph. If the inequality was in fact a strict inequality, say $2VH+SB < 40$, where the equality sign is *not* part of the inequality, then the points on the line $2VH+SB = 40$ are *not* part of the solution set of the inequality and this fact should be represented on a graph by drawing the line $2VH + SB = 40$ as a dotted line.

Now consider the second constraint of Mark's model, $VH + 3SB \leq 45$, and substitute the point $(0; 0)$ into this inequality. Then

$$VH + 3SB = 0 + 3(0) = 0 < 45.$$

The point $(0; 0)$ satisfies the inequality and all points on the same side of the line satisfy the inequality.

The solution set of this inequality is represented by the solid line $VH + 3SB = 45$ and the positively slanted lines in Figure 7.3.

Now consider the third constraint of Mark's model, $VH \leq 12$.

The points satisfying this inequality lie on the line $VH = 12$ and to the left of this line.

The solution set of this inequality is represented by the solid line $VH = 12$ and the negatively slanted lines in Figure 7.4.

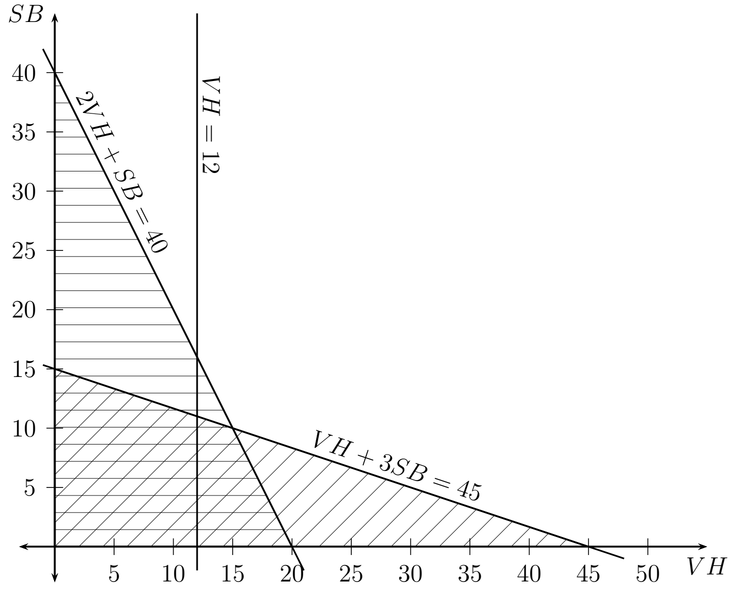

Figure 7.3: Labour and machine time inequalities

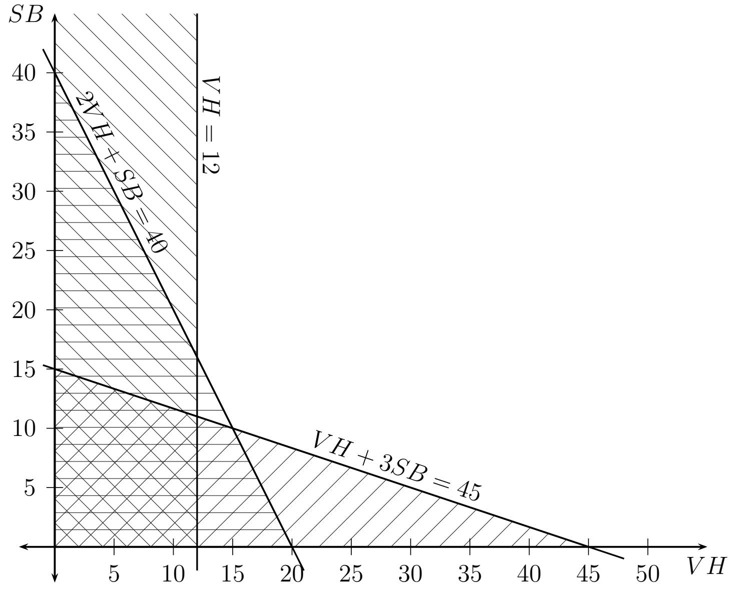

Figure 7.4: Labour and machine time and marketing inequalities

The solution sets of all the inequalities are now determined. In Figure 7.5 we show the area where these solution sets overlap as a shaded area.

This shaded area where all the constraints overlap is called the *feasible area* or *feasible region* of the LP model.

NOTE: The constraint $2VH + SB \leq 40$ makes no contribution to the feasible area, in fact it falls completely outside the feasible area. Any point in the feasible area will also satisfy this constraint. This "non-contributing" constraint is called a *redundant constraint* and can

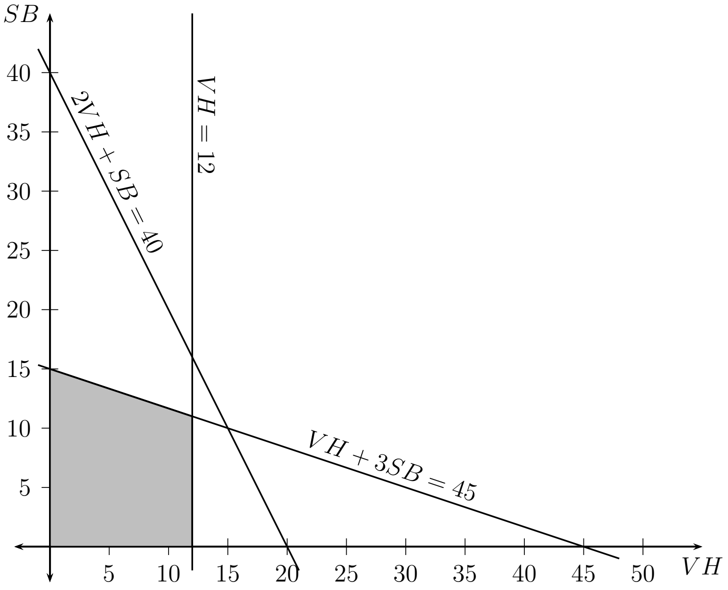

Figure 7.5: Feasible area

be omitted from the model.

7.3 Identifying the optimal solution

Any point in the feasible area satisfies all the constraints and is therefore a feasible solution to the LP model. Since there are an infinite number of points in this area, there are an infinite number of feasible solutions for this problem. To find an optimal solution, it is necessary to identify a solution (point) in the feasible area that maximises the profit (objective function).

How can this be accomplished?

Let us examine the point (6; 3) which is a feasible solution since it lies inside the feasible region. The profit associated with this point is

$$\begin{aligned} PROFIT &= 300VH + 250SB \\ &= 300(6) + 250(3) \\ &= 2\,550. \end{aligned}$$

This means that if the company manufactures six sets of model $VH200$ and three sets of model $SB150$ daily, then the profit will be R2 550.

We now move from the point (6; 3) to the point (12; 3), which is a point on the boundary of the feasible area, but still part of the feasible area. The profit associated with point (12; 3) is R4 350, which is larger than the profit associated with point (6; 3).

Similarly, we move from point (6; 3) to another boundary point (6; 13) and find that the associated profit is R5 050, which is larger than the profit associated with point (6; 3).

Figure 7.6 illustrates this.

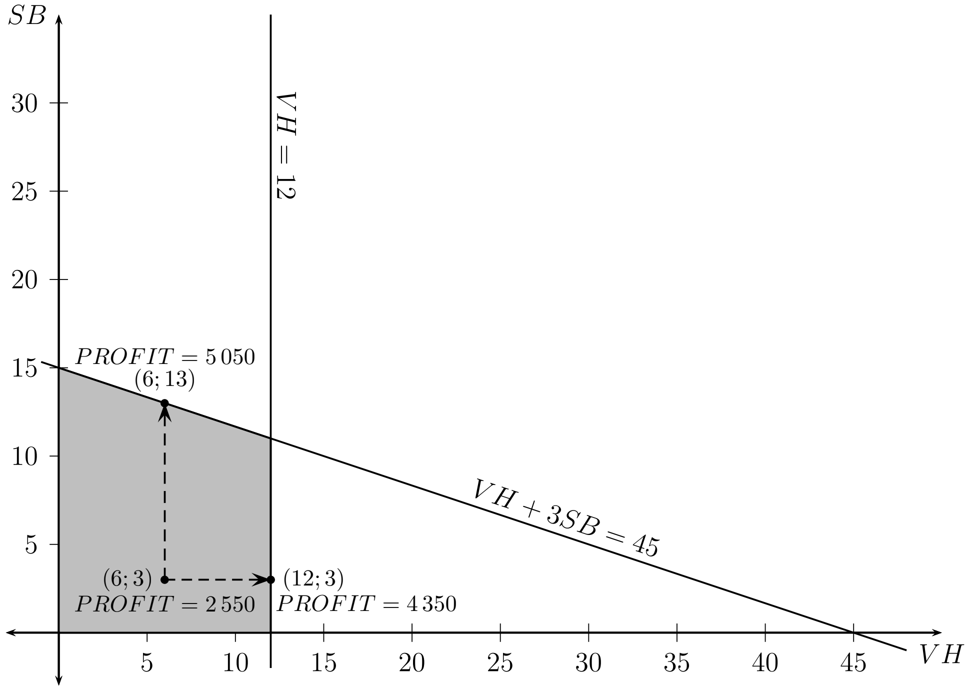

Figure 7.6: Searching for the optimal solution

By repeating this process with other points, we can show that there will be at least one point somewhere on the boundary of the feasible area that is better than the point inside the feasible area. Therefore, *the optimal solution to a linear programming model can never be inside the feasible area, but must be on the boundary.*

If we follow the same argument as before, we can show that *the optimal solution to an LP model will always be at a corner point* (an extreme point) of the feasible area, that is, where two constraints intersect.

The feasible area of Mark's problem has four corner points. The first one is the origin $(0; 0)$, the second one is the intercept on the VH-axis $(12; 0)$ and the third one is the intercept on the SB-axis $(0; 15)$. The fourth corner point is at the intersection of the two constraints. If we substitute $VH = 12$ in $VH + 3SB = 45$, we obtain $SB = 11$. The fourth corner point is $(12; 11)$.

Let us calculate the value of the objective function at the corner points of the feasible region:

Corner points $(VH; SB)$	**Value of objective function** $PROFIT = 300VH + 250SB$
$(0; 0)$	0
$(12; 0)$	3 600
$(12; 11)$	6 350
$(0; 15)$	3 750

From this we see that the point $(12; 11)$ gives the maximum profit. Therefore, the optimal solution is to produce 12 sets of model $VH200$ and 11 sets of model $SB150$ for a profit of R6 350.

This method of calculating the value of the objective function at the corner points is not very efficient and may be a very lengthy exercise.

A more efficient method for obtaining the optimal solution must be found. The optimal solution is the solution that *optimises* (maximises in this case) *the objective function.* Therefore, it is obvious that we should use the objective function to find the optimal solution.

The objective function of Mark's problem is

$$\text{Maximum } PROFIT = 300VH + 250SB.$$

Let us examine a point, say $(5; 5)$ in the feasible area. Here five units of each type of model are produced and the associated profit is

$$PROFIT = 300(5) + 250(5) = 2\,750.$$

The objective function can now be written as

$$300VH + 250SB = 2\,750.$$

This is the equation of a straight line and each of the points on this line will have an objective function value (profit) of R2 750. This line is called an *isoprofit line because all points on it have the same profit* (in Greek "iso" means "same").

We now want to draw this line and we proceed as follows:

We already have one point, namely $(5; 5)$. To find another point, we calculate the intercept on the SB-axis. If $VH = 0$ then

$$\begin{aligned} 300VH + 250SB &= 2\,750 \\ 300(0) + 250SB &= 2\,750 \\ SB &= 11. \end{aligned}$$

We can use the points $(5;5)$ and $(0;11)$ to draw the isoprofit line

$$300VH + 250SB = 2\,750.$$

And this is shown in Figure 7.7 by the dashed line. We use a dashed line so that we can distinguish the isoprofit line from the constraints.

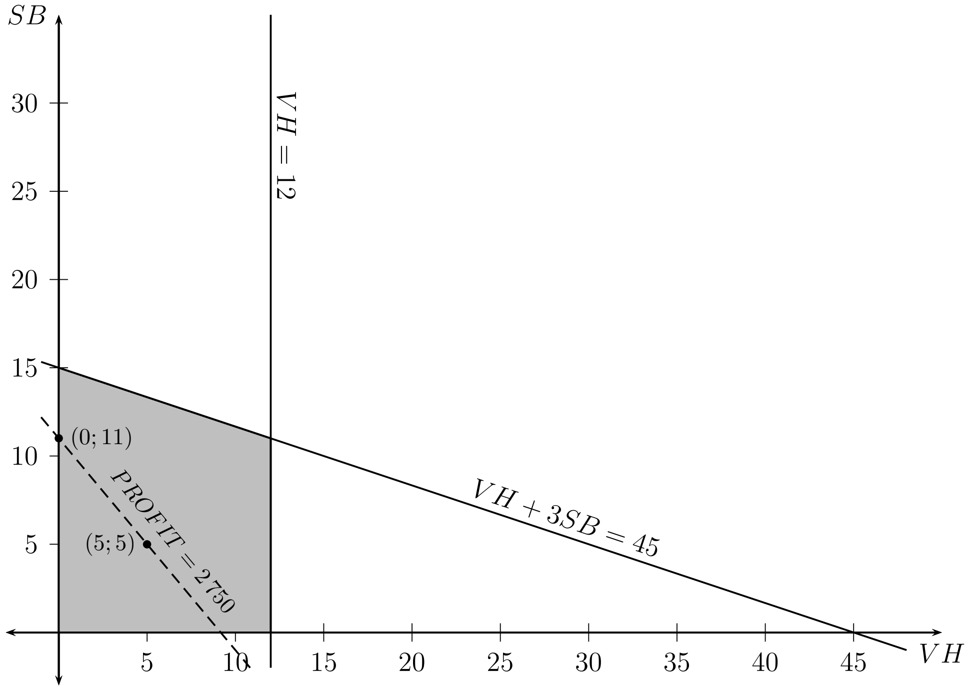

Figure 7.7: Isoprofit line = R2 750

Choose another point in the feasible region, say $(5;10)$. An isoprofit line going through this point will have a corresponding profit of

$$PROFIT = 300(5) + 250(10) = 4\,000.$$

The SB-intercept of $300VH + 250SB = 4\,000$ is at $VH = 0$, and is $SB = 16$.

We can use the points $(5;10)$ and $(0;16)$ to draw the isoprofit line

$$300VH + 250SB = 4\,000.$$

Likewise, points $(10;10)$ and $(0;22)$ can be used to draw the isoprofit line

$$300VH + 250SB = 5\,500.$$

And points $(10;15)$ and $(0;27)$ can be used to draw the isoprofit line $300VH + 250SB = 6\,750$.

The four isoprofit lines can be shown on our graph of the feasible area and appear in Figure 7.8.

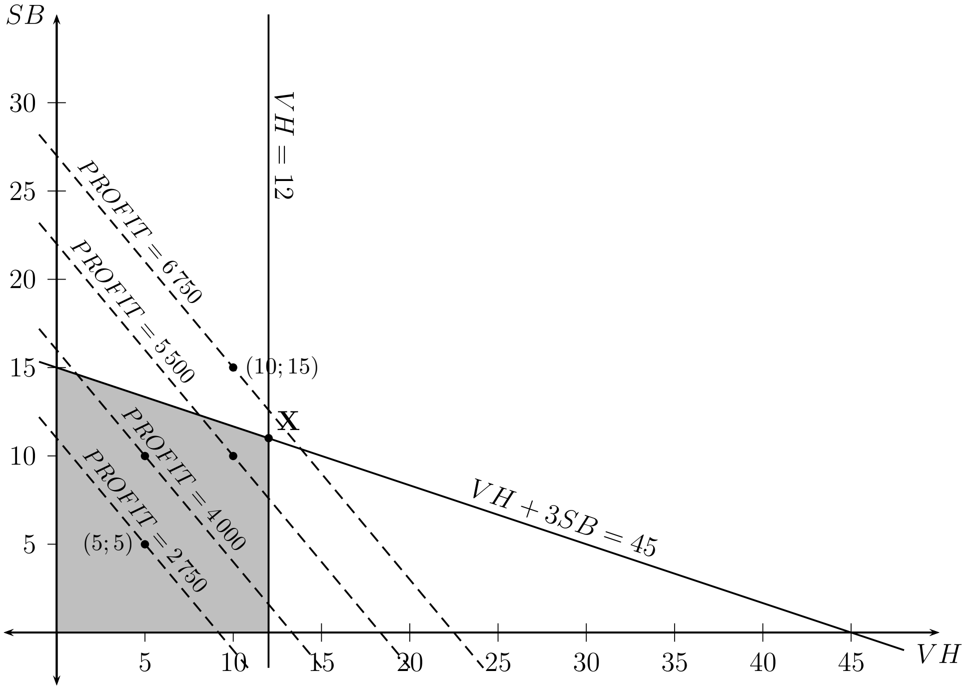

Figure 7.8: Isoprofit lines

We see that the four isoprofit lines are parallel and that the profit gets larger as the isoprofit lines move further away from the origin (point $(0; 0)$) in a direction upwards and to the right. The first three isoprofit lines lie within the feasible area and therefore give feasible solutions. The fourth isoprofit line, $300VH + 250SB = 6\,750$, lies outside the feasible area. Therefore, the points on this line will give infeasible solutions. An infeasible solution is an impossible solution. In Mark's case, this means that that combinations of the two model television sets cannot be produced given the current constraints.

Figure 7.8 shows four isoprofit lines. There are, however, an infinite number of isoprofit lines parallel to these lines. Since the isoprofit lines are parallel, there is no need for us to draw any more lines. The tendency will remain the same, namely, the profit increases as the lines move upwards and to the right.

We are looking for the isoprofit line with the maximum profit that satisfies the constraints. This means that the isoprofit line must be within the feasible area. The isoprofit line with the maximum profit will be the one that is as far as possible to the right of the origin, but still within the feasible area.

The easiest way to determine this maximum isoprofit line is to place a ruler on one of the isoprofit lines and, keeping it parallel to the line, move it upwards and to the right until we find the last isoprofit line that still lies within the feasible area, in other words, the last isoprofit line before the one that lies outside the feasible area.

If we go back to Figure 7.8 and do this, we find that the last isoprofit line is the one that passes through point X. This point $(12; 11)$ is the point of intersection of the lines that represent two boundaries of the feasible area, namely $VH + 3SB = 45$ and $VH = 12$.

This maximum isoprofit line contains just this one point in the feasible area and this point is therefore the solution to the model. The profit associated with this point is

$$\begin{aligned} PROFIT &= 300VH + 250SB \\ &= 300(12) + 250(11) \\ &= 6\,350. \end{aligned}$$

The optimal solution is therefore to produce 12 sets of model $VH200$ and 11 sets of model $SB150$ for a profit of R6 350. This is the same solution as was obtained by means of evaluating the corner points.

In solving Mark's problem we drew four isoprofit lines. This was done to determine the direction in which the isoprofit lines should be moved to increase profit. However, it is unnecessary to draw so many isoprofit lines; in fact, we require just two isoprofit lines.

If we draw two isoprofit lines, we can compare their objective function values to determine the direction of increasing profit. Once we have determined this, we can use a ruler and move it in the required direction until we find the point where the isoprofit line last touches the feasible area. This point will be the optimal solution.

The fact that the isoprofit lines are parallel, and therefore have the same slope, can be applied to derive yet another method of drawing the isoprofit lines.

The objective function can be rewritten as follows:

$$\begin{aligned} 300VH + 250SB &= PROFIT \\ 250SB &= -300VH + PROFIT \\ SB &= -\frac{300}{250}VH + \frac{PROFIT}{250} \\ SB &= -\frac{6}{5}VH + \frac{PROFIT}{250}. \end{aligned}$$

(The function is rewritten by writing SB in terms of VH and $PROFIT$ since SB is represented on the vertical axis of the graph.)

The slope of all the isoprofit lines will then be $-\frac{6}{5}$. The slope is negative, which means that the slant is \, in contrast to a positive slope which has slant /.

To draw the first isoprofit line, we measure six units on the vertical axis and five units on the horizontal axis and draw a line through these points. In other words, we draw a line through points $(0;6)$ and $(5;0)$.

To determine the value of this isoprofit line, we substitute either of these points into the objective function. Then

$$\begin{aligned} PROFIT &= 300VH + 250SB \\ &= 300(0) + 250(6) \\ &= 1\,500. \end{aligned}$$

A second isoprofit line can be drawn by measuring 2×6 units on the vertical axis and 2×5 units on the horizontal axis and drawing a line through these two points, namely $(0;12)$ and $(10;0)$. The corresponding profit value is $PROFIT = 300(10) + 250(0) = 3\,000$.

The graphical representation showing these two isoprofit lines is given in Figure 7.9.

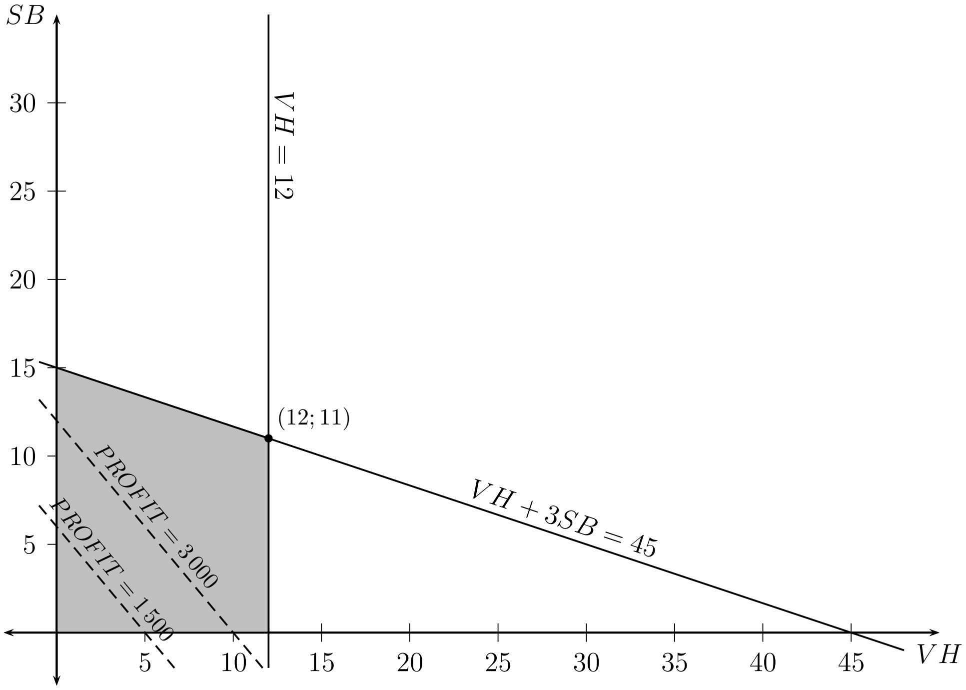

Figure 7.9: Two isoprofit lines

These two isoprofit lines once again show us that the profit increases as we move upwards and towards the right. If we place a ruler on one of these isoprofit lines and continue moving it parallel to the isoprofit lines in the direction of increasing profit, we will find that the maximum profit is obtained when the isoprofit line passes through point $(12;11)$. This is exactly the same result as was obtained before.

7.4 Solving Christine's problem graphically

The graphical approach can also be used for minimisation problems. As an illustration, we return to Christine's LP model (see Section 5.2.2):

$$\begin{array}{ll} \text{Minimise} & Z = 45\alpha + 12\beta \\ \text{subject to} & \\ & \begin{array}{rcrcrl} \alpha & + & \beta & \geq & 300 & \quad\text{(Brilliance)} \\ 3\alpha & & & \geq & 250 & \quad\text{(Tint)} \end{array} \\ \text{and} & \alpha;\ \beta\ \geq 0. \end{array}$$

Here

$$\begin{array}{rcl} \alpha & = & \text{quantity (in grams) of Alpha in each tin of paint,} \\ \beta & = & \text{quantity (in grams) of Beta in each tin of paint.} \end{array}$$

The cost is given in cents per gram.

Let us decide that α will be represented on the horizontal axis and β on the vertical axis.

The first constraint is rewritten as an equality as

$$\alpha + \beta = 300.$$

Let $\alpha = 0$. Then $\beta = 300$. The point on the β-axis is $(0; 300)$.
Let $\beta = 0$. Then $\alpha = 300$. The point on the α-axis is $(300; 0)$.

Plot these two points on the axes and draw the line connecting them.

Now substitute point $(0; 0)$ into the constraint. Then

$$\begin{array}{rcl} \alpha + \beta & \geq & 300 \\ 0 + 0 & \ngeq & 300. \end{array}$$

The point $(0; 0)$ does not satisfy the inequality and so $(0; 0)$ is not part of the feasible area. All points on the opposite side of the line $\alpha + \beta = 300$ will satisfy the inequality.

Now rewrite the second constraint as an equation. Then

$$\begin{array}{rcl} 3\alpha & = & 250 \\ \alpha & = & 83,33. \end{array}$$

This line is a straight line parallel to the β-axis, passing through the point $(83,33; 0)$.

The point $(0;0)$ does not satisfy the inequality $3\alpha \geq 250$ and therefore it does not form part of the feasible area. The objective function is now rewritten as

$$\begin{aligned} Z &= 45\alpha + 12\beta \\ 12\beta &= -45\alpha + Z \\ \beta &= -\frac{45}{12}\alpha + \frac{Z}{12} \\ &= -\frac{15}{4}\alpha + \frac{Z}{12}. \end{aligned}$$

The slope of the isocost lines is $-\frac{15}{4}$. We can draw the first isocost line by measuring say $10 \times 15 = 150$ units on the vertical axis and $10 \times 4 = 40$ units on the horizontal axis, and then drawing a line through these points $(0;150)$ and $(40;0)$.

The cost associated with this isocost line is

$$\begin{aligned} Z &= 45\alpha + 12\beta \\ &= 45(40) + 12(0) \\ &= 1\,800. \end{aligned}$$

A second isocost line can be drawn by measuring $30 \times 15 = 450$ units on the vertical axis and $30 \times 4 = 120$ units on the horizontal axis, and then drawing a line through these points $(0;450)$ and $(120;0)$. The associated cost is R5 400.

The graphical representation of the LP model is given in Figure 7.10.

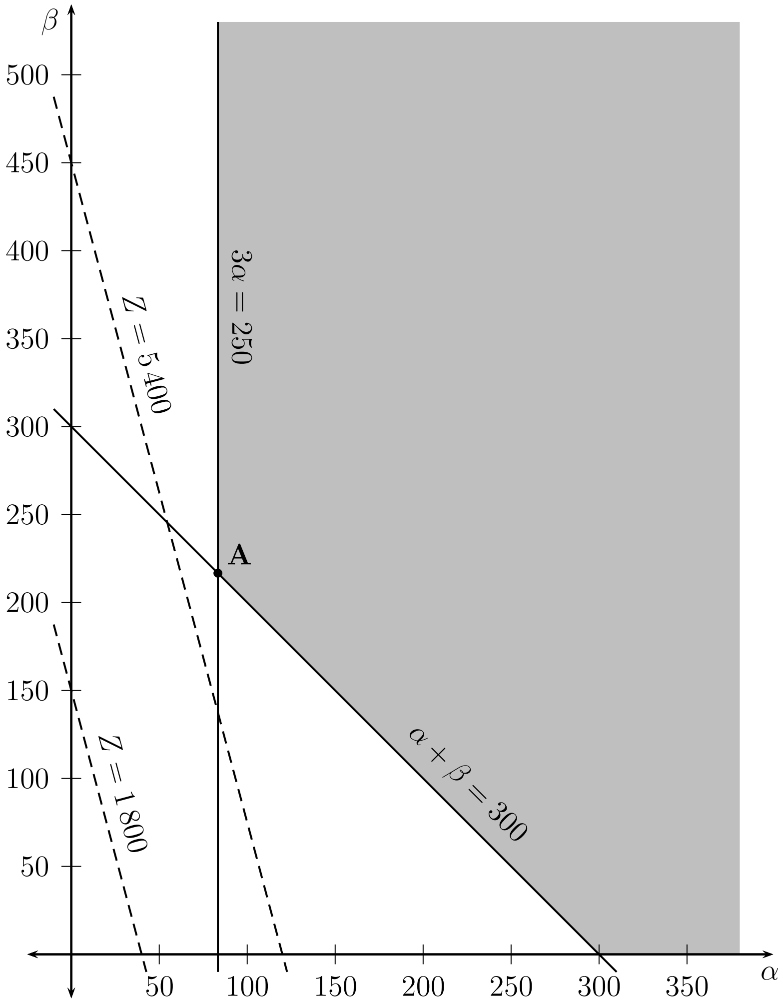

Figure 7.10: Christine's problem

The cost increases as the isocost lines move upwards and to the right. Since the objective is to minimise cost, the optimal solution is found where the isocost line first enters the feasible area. Conversely, the cost decreases as the isocost lines move downwards and to the left. The optimal solution will be where the isocost line just touches the feasible area before it moves out of the feasible area. Therefore, the optimal solution is at point A.

Point A is at the intersection of the two constraints. Here $\alpha = 83,33$ and this can be substituted into the first constraint to obtain the value of β. Then

$$\begin{aligned} \alpha + \beta &= 300 \\ 83,33 + \beta &= 300 \\ \beta &= 216,67. \end{aligned}$$

The optimal solution to Christine's problem is to use 83, 33 grams of Alpha and 216, 67 grams of Beta in each tin of paint.

The minimum cost will be

$$\begin{aligned} Z &= 45\alpha + 12\beta \\ &= 45(83,33) + 12(216,67) \\ &= 6\,350. \end{aligned}$$

The cost is given in cents and therefore the minimum cost is R63, 50 per tin of paint.

7.5 Types of solution using the graphical method

We have now solved Mark and Christine's problems by means of the graphical approach. Each of these models had a single optimal solution. This may not always be the case. It may happen that an LP model has no solution, or even has many optimal solutions. Let us consider the different types of solution that can be obtained for LP models by using the graphical representation method.

7.5.1 Infeasible LPs

Consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & Z = 5x + 3y \\ \text{subject to} & \\ & \begin{array}{rcrcrl} 2x & + & y & \leq & 4 & \quad (1) \\ x & & & \geq & 4 & \quad (2) \\ & & y & \geq & 6 & \quad (3) \end{array} \\ \text{and} & x;\ y \geq 0. \end{array}$$

The graphical representation of the constraints is in Figure 7.11.

The solution sets of the three constraints do not overlap and as such there is no point that satisfies all three constraints simultaneously (at the same time). Therefore, a feasible solution area does not exist, and there is no solution to the problem.

Such a situation where no feasible area exists is called an *infeasible* LP, and it has no solution.

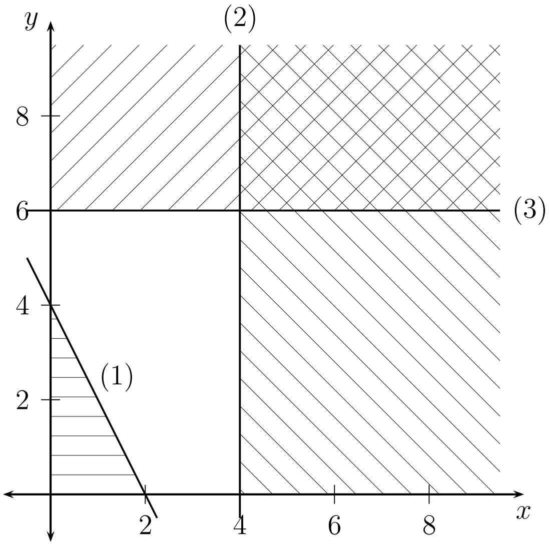

Figure 7.11: Infeasible LP

7.5.2 Unbounded solutions

Consider the following LP model:

$$\begin{array}{lllll} \text{Maximise} & Z = 4M + 2N & & & \\ \text{subject to} & & & & \\ & -M + 2N & \leq & 6 & (1) \\ & -M + N & \leq & 2 & (2) \\ \text{and} & M;\ N & \geq & 0. & \end{array}$$

The graphical representation of this model is in Figure 7.12.

The isoprofit lines increase as they move upwards and to the right across the feasible area. As the feasible area has no boundary on the right, the value of the objective function can increase indefinitely without ever reaching a maximum.

This LP model is said to have an *unbounded solution.*

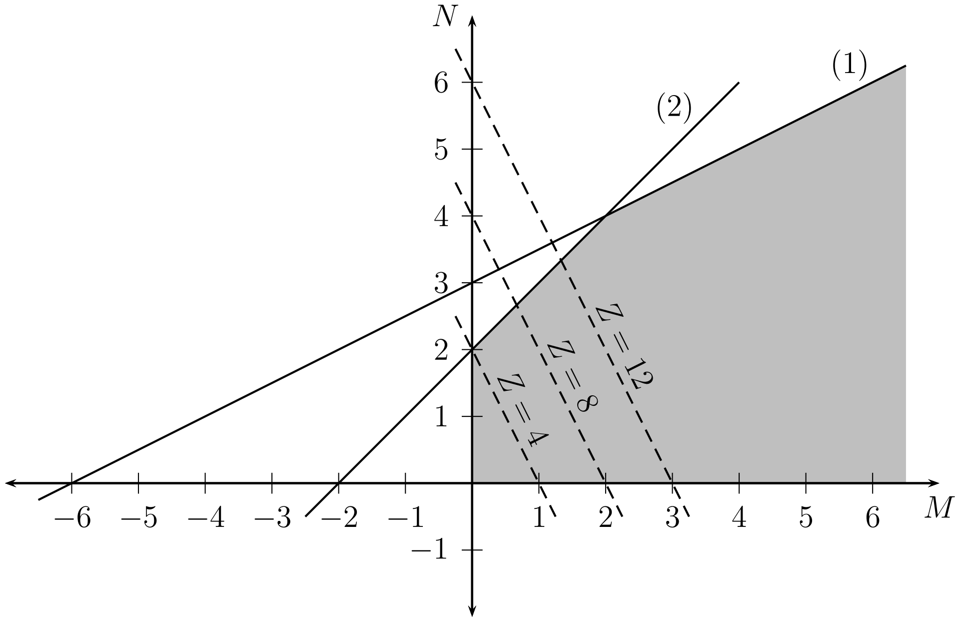

Figure 7.12: Unbounded solution

7.5.3 Multiple optimal solutions

Some of the LP problems might have more than one optimal solutions with the same optimal value of the objective function. This case occurs when the isoprofit (or isocost) lines are parallel to a binding constraint line (see section 7.6). An LP that has multiple optimal solutions can also be recognised by using the corner-point method. In this case at least two corner-points have the same optimal value of the objective function. The general optimal solution is given by any weighted average of two optimal solutions. If we consider an LP problem with decision variable x and y and the two corner-point $A = (a_1, a_2)$ and $B = (b_1, b_2)$ have the same optimal value of the objective function $z = p$, then the general optimal solution is given by $z = p$ and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where the parameter α is the real number taking values in the closed interval [0,1]. This is illustrated in the following example.

Consider the following LP model:

$$
\begin{array}{ll}
\text{Maximise} & P = 2x + 3y \\
\text{subject to} & \\
& \begin{array}{rcrcrl}
2x & + & 3y & \leq & 30 & \quad (1) \\
-x & + & y & \leq & 5 & \quad (2) \\
x & + & y & \geq & 5 & \quad (3) \\
x & & & \leq & 10 & \quad (4)
\end{array} \\
\text{and} & x;\ y\ \geq 0.
\end{array}
$$

The graphical representation of this model is in Figure 7.13.

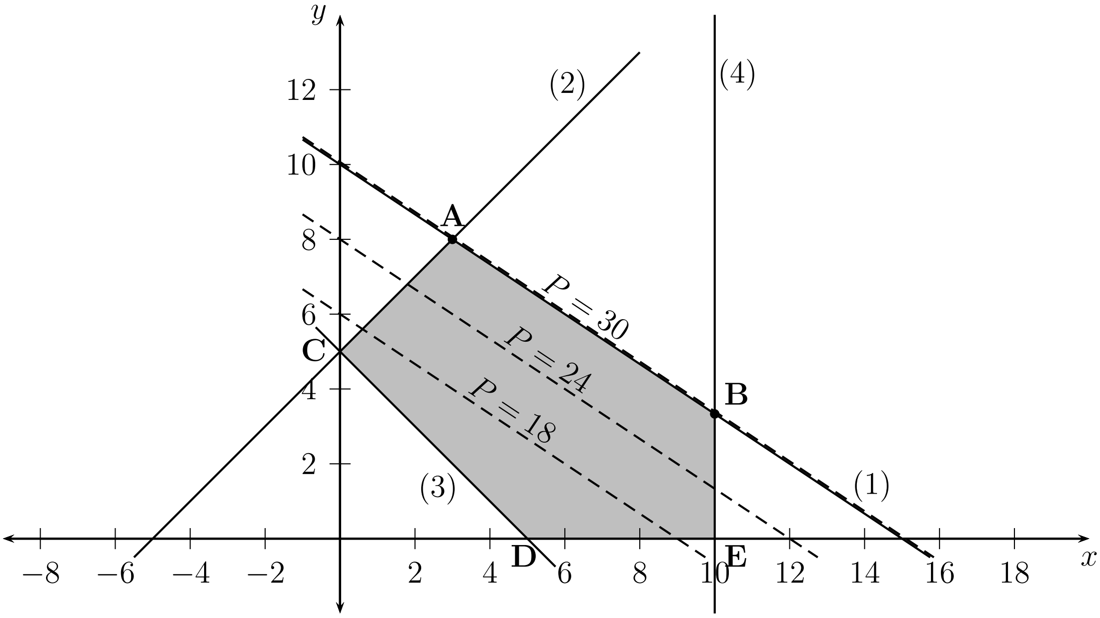

Figure 7.13: Multiple optimal solutions

From the graph it is clear that the isoprofit lines are parallel to constraint (1). Both have a slope of $-\frac{2}{3}$. As we move the isoprofit lines to the right, the whole line segment AB, and not just a single extreme point, will be touched before the isoprofit lines leave the feasible area. Every point on this line segment will result in the same maximum profit, and so every point on this line segment is optimal.

The end points, A and B, are referred to as the alternate end point optimal solutions with the understanding that these points represent the end points of a range of optimal solutions. This LP model is said to have *multiple optimal solutions.*

The corner-points of the feasible region are

$$\begin{aligned}
A &= (3,\ 8)\\
B &= (10,\ \tfrac{10}{3})\\
C &= (0,\ 5)\\
D &= (5,\ 0)\\
E &= (10,\ 0)
\end{aligned}$$

The values of the objective function P at each corner-points are

- At $A = (3,\ 8),\ \ P = 30$
- At $B = (10,\ \frac{10}{3}),\ \ P = 30$
- At $C = (0,\ 5),\ \ P = 15$
- At $D = (5,\ 0),\ \ P = 10$
- At $E = (10,\ 0),\ \ P = 20$

We see that the optimal value of the objective function P occurs at A and B. It follows that the LP model has multiple optimal solutions and the general optimal solution is $P = 30$ and

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \alpha \begin{bmatrix} 3 \\ 8 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 10 \\ \frac{10}{3} \end{bmatrix}\\
&= \begin{bmatrix} 3-10 \\ 8-\frac{10}{3} \end{bmatrix} + \begin{bmatrix} 10 \\ \frac{10}{3} \end{bmatrix}\\
&= \frac{1}{3}\alpha \begin{bmatrix} 21 \\ 14 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 30 \\ 10 \end{bmatrix},
\end{aligned}$$

where $\alpha \in [0,\ 1]$.

7.5.4 Degenerate solutions

Consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & z = 3x + 9y \\ \text{subject to} & \\ & x + 4y \leq 8 \quad (1) \\ & x + 2y \leq 4 \quad (2) \\ \text{and} & x;\ y \geq 0. \end{array}$$

The graphical representation of this model is in Figure 7.14.

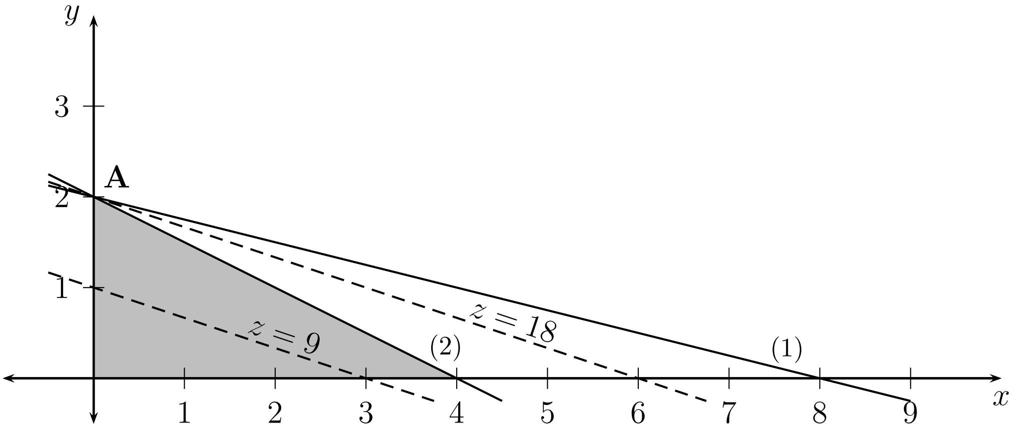

Figure 7.14: Degenerate solution

The optimal solution is at point A. There are three lines going through point A, namely $x = 0,\ x + 4y = 8$ and $x + 2y = 4$.

Only two lines are needed to define a point in a two-dimensional environment. Point A is therefore over determined and one of the lines (constraints) is redundant.

From the graph it is clear that constraint (1) is the culprit. The feasible area will not change if we remove this constraint from the model. A constraint like this is called a *redundant constraint* and can be removed from the model without changing the solution.

This LP model is said to have a *degenerate solution.*

7.6 Types of constraint

7.6.1 Redundant constraints

Let us refer back to Mark's problem. The labour time constraint, $2VH + SB \leq 40$, falls outside the feasible area and makes no contribution to the feasible area. Therefore, it is a *redundant* constraint.

The redundant constraint can be removed from the model. In fact, it should be removed, since it only complicates the model and does not make any positive contribution.

7.6.2 Binding and nonbinding constraints

Consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & PROFIT = 2X + 3Y \\ \text{subject to} & \\ & \begin{array}{rcrcl} 5X & + & 6Y & \leq & 60 \qquad \text{(Labour time)} \\ X & + & 2Y & \leq & 16 \qquad \text{(Machine time)} \\ X & & & \leq & 10 \qquad \text{(Demand)} \end{array} \\ \text{and} & X;\ Y \ \geq 0. \end{array}$$

The graphical representation of this model is given in Figure 7.15.

The optimal solution is found at point G where the labour time and machine time constraints intersect, that is, where $X = 6$ and $Y = 5$.

We now substitute these optimal values into the constraints.

If we substitute $X = 6$ and $Y = 5$ into the labour time constraint, we find

$$\begin{aligned} \text{left-hand side} &= 5X + 6Y \\ &= 5(6) + 6(5) \\ &= 60 \\ &= \text{right-hand side.} \end{aligned}$$

Likewise the left-hand side of the machine time constraint equals the right-hand side at the optimal point $(X; Y) = (6; 5)$.

This means that the available labour time and machine time are completely used up. Constraints like these where the resources are fully utilised are called *binding constraints.*

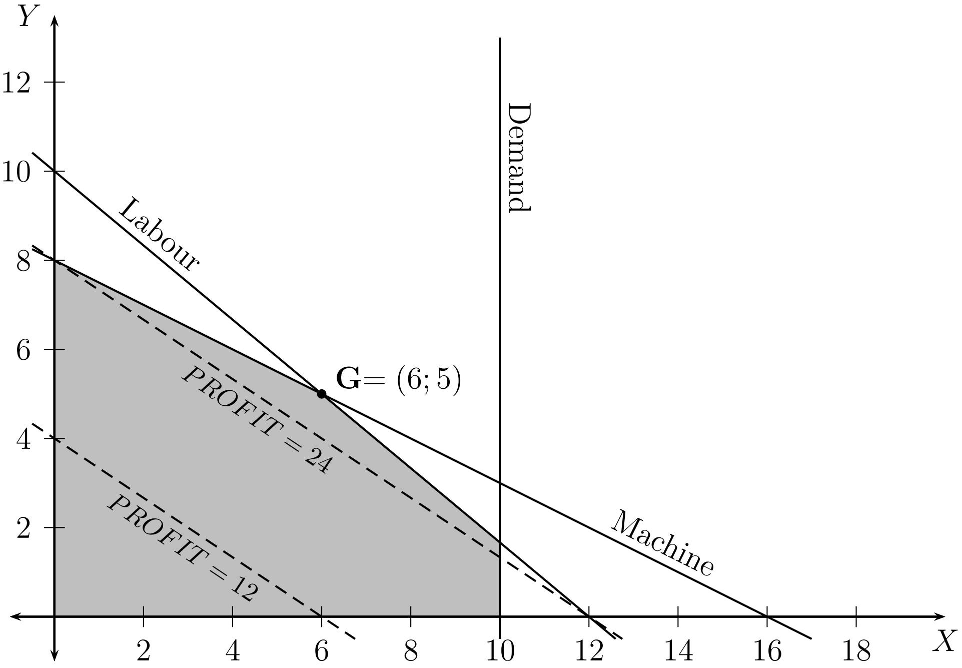

Figure 7.15: Binding and nonbinding constraints

We now substitute the optimal values into the demand constraint and find

$$\text{left-hand side} = X = 6 < 10 = \text{right-hand side.}$$

From this we see that the optimal quantity of X produced is less than the maximum number demanded, that is, it is not optimal to produce the maximum number demanded.

The difference between the left-hand side and the right-hand side of a constraint is called the *surplus* or *slack*, depending on the context of the problem.

A constraint where the resource is not utilised fully, in other words, where there is surplus or slack, is called a *nonbinding constraint.*

From the graph we see that a constraint is binding if the optimal solution falls on the line representing the constraint and a constraint is nonbinding if the optimal solution does not fall on the line representing the constraint.

7.7 Exercises and Solutions

7.7.1 Exercises

1. Holiday Meal Turkey Ranch buys two different brands of turkey feed and blends them to provide a good, low-cost diet for its turkeys. Each brand of feed contains some or all of the three nutritional ingredients essential for fattening turkeys. Each kilogram of brand 1 contains 6 grams of ingredient A, 4 grams of ingredient B and $0,5$ grams of ingredient C. Each kilogram of brand 2 contains 9 grams of ingredient A, 3 grams of ingredient B, but nothing of ingredient C. The brand 1 feed costs the Ranch $75c$ a kilogram, while the brand 2 feed costs R1 a kilogram.

   The minimum monthly requirement per turkey is 90 grams of ingredient A, 48 grams of ingredient B and $1,5$ grams of ingredient C.

   Formulate an LP model to decide how to mix the two brands of turkey feed so that the minimum monthly intake requirement for each nutritional ingredient is met at minimum cost.

   Use the graphical approach to solve this model.

2. Solve the following LP models graphically. For each solution state clearly what type of solution it is and identify the binding, nonbinding or redundant constraints, giving reasons for your answers.

   (a)

$$\text{Maximise} \quad Z = P + Q$$

subject to

$$\begin{array}{rcrcll} P & + & 2Q & \leq & 6 & (1) \\ 2P & + & Q & \leq & 8 & (2) \\ P & & & \geq & 7 & (3) \\ & & Q & \geq & 0. & \end{array}$$

   (b)

$$\begin{array}{llrcrcrl}
\text{Maximise} & Z = M + 3N \\
\text{subject to} \\
& & M & + & N & \leq & 25 & (1) \\
& & 2M & + & N & \leq & 30 & (2) \\
& & & & N & \leq & 35 & (3) \\
\text{and} & M;\ N \geq 0.
\end{array}$$

7.7.2 Solutions to exercises

1. Holiday Meal Turkey Ranch data can be summarised as follows:

Ingredient	**Composition of each kg of feed (grams)**		**Minimum monthly requirement (grams)**
	Brand 1	Brand 2	
A	6	9	90
B	4	3	48
C	0,5	0	1,5
Cost per kg	75c	100c	

Let

ONE = number of kilograms of brand 1 feed bought monthly,
TWO = number of kilograms of brand 2 feed bought monthly.

The LP model is

$$\begin{array}{llrcrcrl}
\text{Minimise} & COST = 75ONE + 100TWO \\
\text{subject to} \\
& & 6ONE & + & 9TWO & \geq & 90 & \text{(Ingredient A)} \\
& & 4ONE & + & 3TWO & \geq & 48 & \text{(Ingredient B)} \\
& & 0,5ONE & & & \geq & 1,5 & \text{(Ingredient C)} \\
\text{and} & ONE;\ TWO \geq 0.
\end{array}$$

Graphical solution is given in Figure 7.16.

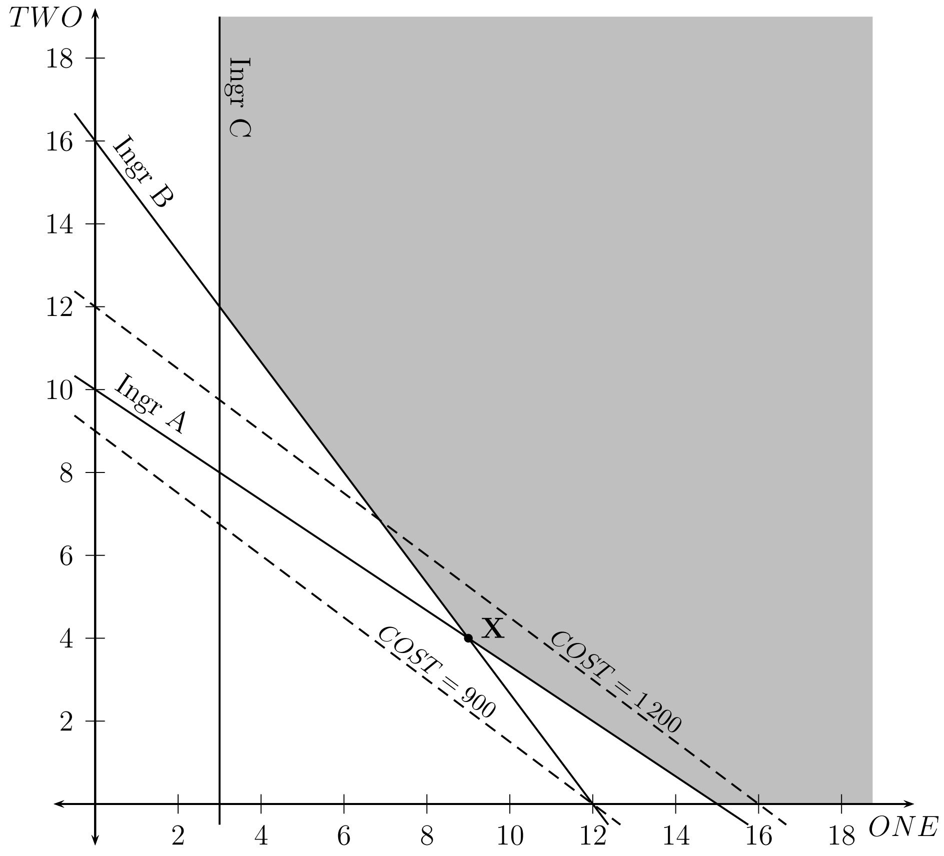

Figure 7.16: Learning Unit 7, Exercise 1

We see that $COST$ decreases as the isocost lines move downwards to the left over the feasible area. The last point that the isocost line touches before leaving the feasible area is point X = (9; 4). This represents the solution resulting in the minimum cost.

Therefore, the optimal solution is to buy 9 kilograms of brand 1 feed and 4 kilograms of brand 2 feed monthly and the associated minimum cost per turkey is

$$\begin{aligned} COST &= 75ONE + 100TWO \\ &= 75(9) + 100(4) \\ &= 1\,075. \end{aligned}$$

The minimum cost per turkey is R10, 75.

2. (a) The graphical representation of the constraints is given in Figure 7.17.

   There is no area satisfying all the inequalities simultaneously and so no feasible area exists. We say the LP model (or the problem) is infeasible. There is no solution to this LP model.

   (b) The graphical solution to the LP model is given in Figure 7.18.

   We see that the objective function value Z increases as the isoprofit lines are

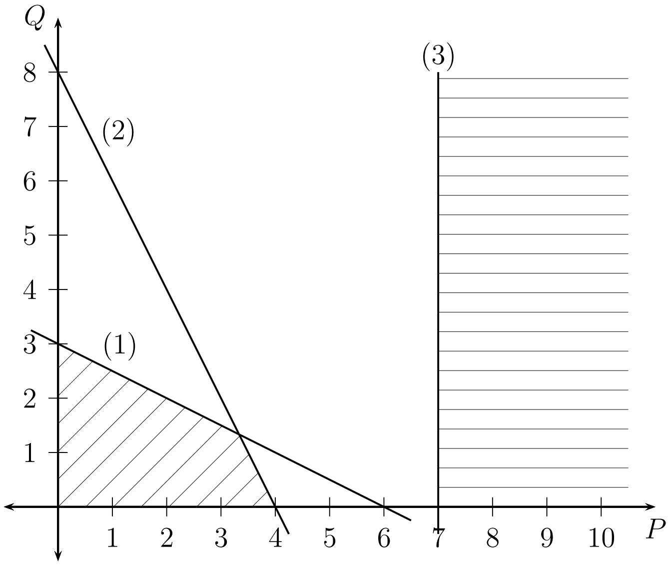

Figure 7.17: Learning Unit 7, Exercise 2(a)

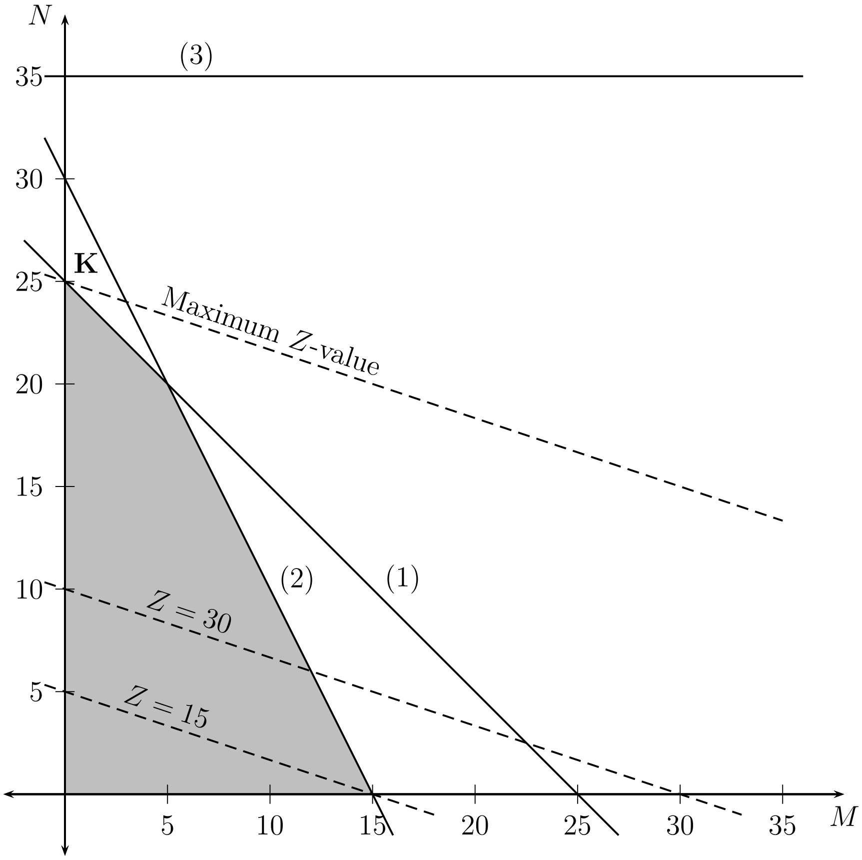

Figure 7.18: Learning Unit 7, Exercise 2(b)

moved upwards and to the right across the feasible region. The last point where an isoprofit line touches the feasible area before moving out of the feasible area is at point K $= (0; 25)$, and this is the optimal solution.

The optimal solution is at $M = 0$ and $N = 25$ and the associated maximum value of the objective function is

$$\begin{aligned} Z &= M + 3N \\ &= 0 + 3(25) \\ &= 75. \end{aligned}$$

The optimal solution lies on constraint (1) and so it is a binding constraint.

The optimal solution does not lie on constraint (2) and so it is a nonbinding constraint.

Constraint (3) does not influence the feasible area and so it is a redundant constraint.

7.7.3 Evaluation Exercises

Question 1 – *Winston* 3.2, Problem 2, p.63 (*Old Winston* 3.2, Problem 2, p.64)

ALSO identify the binding, nonbinding and redundant constraints.

Question 2 – *Winston* 3.2, Problem 4, p.63 (*Old Winston* 3.2, Problem 4, p.64)

Assume the horizontal axis represents x_1 and the vertical axis represents x_2.

Question 3 – *Winston* 3.2, Problem 5, p.63 (*Old Winston* 3.2, Problem 5, p.64)

Learning Unit 8

Simplex and Big M methods

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Learning objectives

After completing this study unit you should be able:

- to use the simplex and Big M methods to solve LP models.
- to recognize the type of solution of an LP model via the simplex or Big M methods.

Sections from prescribed book

4.1. How to Convert an LP to Standard Form - *p127*
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4.3. Direction of Unboundedness - *p134*
4.4. Why Does an LP Have an Optimal bfs? - *p136*
4.5. The Simplex Algorithm - *p140*
4.6. Using the Simplex Algorithm to Solve Minimization Problems - *p149*
4.7. Alternative Optimal Solutions (Multiple Optimal Solutions)- *p152*
4.8. Unbounded LPs - *p154*
4.12. The Big M Method - *p172*

8.1 Introduction to the Simplex Method

Consider the following LP model:

$$\begin{array}{lrcrcll} \text{Maximise} & \multicolumn{5}{l}{PROFIT = 3SCREW + 5BOLT} \\ \text{subject to} \\ & 2SCREW & + & 3BOLT & \leq & 10 & \text{(Manpower)} \\ & SCREW & + & 2BOLT & \leq & 12 & \text{(Capital)} \\ \text{and} & \multicolumn{5}{l}{SCREW;\ BOLT\ \geq 0.} \end{array}$$

Here

$$\begin{aligned} SCREW &= \text{number of screws produced,} \\ BOLT &= \text{number of bolts produced.} \end{aligned}$$

The two constraints must be rewritten into standard form in order to solve the model. We add slack variables, then

$$\begin{array}{rcrcrcrcl} 2SCREW & + & 3BOLT & + & S1 & & & = & 10 \\ SCREW & + & 2BOLT & & & + & S2 & = & 12. \end{array}$$

We now have two equations and four variables. One way of finding solutions to this system of equations is to set two variables at a time equal to zero and to solve for the other two.

(a) If $SCREW = 0$ and $BOLT = 0$, then $S1 = 10$ and $S2 = 12$.
(b) If $SCREW = 0$ and $S1 = 0$, then $BOLT = \frac{10}{3}$ and $S2 = \frac{16}{3}$.
(c) If $SCREW = 0$ and $S2 = 0$, then $BOLT = 6$ and $S1 = -8$.
(d) If $BOLT = 0$ and $S1 = 0$, then $SCREW = 5$ and $S2 = 7$.
(e) If $BOLT = 0$ and $S2 = 0$, then $SCREW = 12$ and $S1 = -14$.
(f) If $S1 = 0$ and $S2 = 0$, then $SCREW = -16$ and $BOLT = 14$.

The variables that we find solutions for are called *basic variables* and the variables that are set equal to zero are called *nonbasic variables.*

However, not all these solutions are feasible! All variables, including slack variables, must be $\geq$ 0. A slack variable represents the part of a resource that is not utilised. It is also a physical quantity and cannot be negative.

The objective function is now used to find the optimal solution. We consider only the feasible solutions.

$$\begin{array}{lrclrclrclrcl} \text{(a)} & \text{SCREW} & = & 0; & BOLT & = & 0; & S1 & = & 10; & S2 & = & 12 \Rightarrow PROFIT = 0. \\ \text{(b)} & SCREW & = & 0; & BOLT & = & \frac{10}{3}; & S1 & = & 0; & S2 & = & \frac{16}{3} \Rightarrow PROFIT = \frac{50}{3}. \\ \text{(c)} & SCREW & = & 5; & BOLT & = & 0; & S1 & = & 0; & S2 & = & 7 \Rightarrow PROFIT = 15. \end{array}$$

(The coefficients of $S1$ and $S2$ in the objective function are zero.)

Any set of values for the variables that satisfies the constraints is called a *solution.* If it further also satisfies the sign restrictions, it is called a *feasible solution.* The feasible solution that also maximises the objective function is called the *optimal solution.* The optimal solution to our model is combination (b) where no screws and $\frac{10}{3}$ bolts are produced for maximum profit of $\frac{50}{3}$.

What we have been doing is very tiresome and is actually an extremely roundabout way of finding an answer to our problem. Fortunately we have a method available, called the simplex method, that does everything we have done previously in a structured way. The simplex method starts off with the easiest solution and then, by exchanging one variable at a time, takes a short cut to the optimal solution.

We rewrite our model into standard form:

$$\begin{array}{ll}
\text{Maximise} & PROFIT = 3SCREW + 5BOLT \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr} 2SCREW & + & 3BOLT & + & S1 & & & = & 10 \\ SCREW & + & 2BOLT & & & + & S2 & = & 12 \end{array} \\
\text{and} & SCREW;\ BOLT;\ S1;\ S2 \geq 0.
\end{array}$$

We start off by writing the model as a system of simultaneous equations. The objective function,

$$PROFIT = 3SCREW + 5BOLT,$$

is rewritten as

$$PROFIT - 3SCREW - 5BOLT = 0,$$

and it is placed at the bottom of the system of equations.

Our model as a system of equations is then

$$\begin{array}{rcrcrcrcrcr} & & 2SCREW & + & 3BOLT & + & S1 & & & = & 10 \\ & & SCREW & + & 2BOLT & & & + & S2 & = & 12 \\ PROFIT & - & 3SCREW & - & 5BOLT & & & & & = & 0. \end{array}$$

If we set $SCREW = 0$ and $BOLT = 0$, we have the following solution:

$$SCREW = 0;\ BOLT = 0;\ S1 = 10;\ S2 = 12 \text{ with } PROFIT = 0.$$

This solution consists of the following two parts:

- $S1$ and $S2$, with values > 0 (the *basic variables* – they form the *basis* of the current solution.)
- $SCREW$ and $BOLT$, with values $= 0$ (the *nonbasic variables*).

The simplex method now chooses one of the nonbasic variables to enter the basis, that is, the set of basic variables, in order to find a next (better) solution.

The variable that will improve the value of the objective function the most is chosen to enter the basis. If we choose $BOLT$, the value of $PROFIT$ will increase by five units while it will increase by only three units if we choose $SCREW$. Remember, we want to reach optimality as soon as possible and therefore we choose $BOLT$ to enter the basis.

The most negative coefficient in the objective function equation corresponds to the most positive coefficient in the objective function itself, and therefore indicates the variable making the best contribution.

The entering variable is the variable with the largest negative coefficient in the objective function equation.

For every variable that enters the basis, one must leave. We have to decide whether $S1$ or $S2$ should leave the basis.

We have chosen $BOLT$ to enter the basis, but its value is unknown. Let us give it a hypothetical value θ (theta). All variables must be ≥ 0, and so $\theta \geq 0$. And it should preferably be > 0 to get improvement in the value of the objective function.

Now

$$\begin{aligned} SCREW &= 0 \\ BOLT &= \theta. \end{aligned}$$

The constraints in standard form,

$$\begin{array}{rcrcrcrcl} 2SCREW & + & 3BOLT & + & S1 & & & = & 10 \\ SCREW & + & 2BOLT & & & + & S2 & = & 12, \end{array}$$

then become

$$\begin{array}{rcrcrcrcl} 3\theta & + & S1 & = & 10 \Rightarrow & S1 & = & 10 - 3\theta \\ 2\theta & + & S2 & = & 12 \Rightarrow & S2 & = & 12 - 2\theta. \end{array}$$

All the variables must be $\geq$ 0. Therefore, we have to choose the value of θ in such a way that no variable becomes negative.

So

$$\begin{array}{rclcccccc} S1 & = & 10 - 3\theta & \geq & 0 & \Rightarrow & \theta & \leq & \frac{10}{3} \\ S2 & = & 12 - 2\theta & \geq & 0 & \Rightarrow & \theta & \leq & 6. \end{array}$$

The inequalities must be satisfied simultaneously and θ must be as large as possible. This can be represented graphically in Figure 8.1.

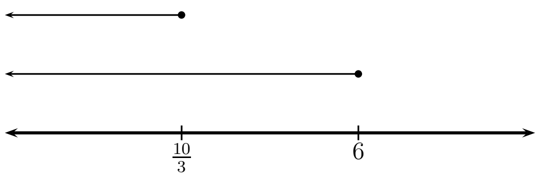

Figure 8.1: Lines representing inequalities

The top line represents $\theta \leq \frac{10}{3}$ and the bottom line represents $\theta \leq 6$. When the two lines overlap, the inequalities are satisfied simultaneously. Therefore, $\theta \leq \frac{10}{3}$ satisfies both inequalities. The largest value of θ is $\frac{10}{3}$.

Now if $\theta = \frac{10}{3}$, then

$$\begin{aligned} S1 &= 10 - 3\theta \\ &= 10 - 3\left(\frac{10}{3}\right) \\ &= 0. \end{aligned}$$

This indicates that $S1$ is now a nonbasic variable. Therefore, it must leave the basis.

This approach is used in the simplex method, but since we are working with a structured system of equations, we can take a shortcut. The leaving variable is identified as follows:

- Mark the entering variable.
- To get the θ-value for each equation, divide the value on the right-hand side of the equation by the coefficient of the entering variable in the same equation. So

$$\theta = \frac{\text{value on right-hand side}}{\text{coefficient of entering variable}}.$$

- Write the θ-value next to the equation in a separate column.
- Choose the minimum $\theta \geq 0$ value. The basic variable in the equation corresponding to this minimum θ-value is the leaving variable.

NOTE: Division by zero is not defined. Ignore zero coefficients when calculating θ-values. Negative θ-values are not allowed as they will not contribute to the improvement of the objective function. Ignore negative coefficients when calculating θ-values.

Let us apply this to our model.

Mark the entering variable, $BOLT$, with an arrow and calculate the θ-value for each equation (excluding the objective function equation). Then

$$\begin{array}{rcrcrcrcrcr|rcl} & & & & & & & & & & & & \theta & \\ & & 2SCREW & + & 3BOLT & + & S1 & & & = & 10 & \frac{10}{3} & = & 3\frac{1}{3} \leftarrow \text{min} \\ & & SCREW & + & 2BOLT & & & + & S2 & = & 12 & \frac{12}{2} & = & 6 \\ PROFIT & - & 3SCREW & - & 5BOLT & & & & & = & 0. & & & \\ & & & & \uparrow & & & & & & & & & \\ & & & & \text{entering variable} & & & & & & & & & \end{array}$$

The θ-value for the first equation is a minimum and $S1$ must leave the basis.

To obtain the solution with $BOLT$ and $S2$ in the basis, we have to apply the one-zero method. We want a system of equations where the coefficients of the basic variables consist of ones and zeros.

The coefficients of the basic variable $S2$ consists of a one in the second equation and zeros in the other equations. And so $S2$ satisfies the requirement.

The basic variable $BOLT$ does not satisfy the requirement and we must apply the one-zero method to it. The coefficient of $BOLT$ in the first equation is the pivot coefficient and must be transformed to one. Divide the first equation by 3 to obtain the pivot equation. Then

$$\begin{array}{rcrcrcrcrcrl} & & \frac{2}{3}SCREW & + & BOLT & + & \frac{1}{3}S1 & & & = & \frac{10}{3} & \leftarrow \text{pivot equation} \\ & & SCREW & + & 2BOLT & & & + & S2 & = & 12 & \\ PROFIT & - & 3SCREW & - & 5BOLT & & & & & = & 0. & \end{array}$$

The coefficients of $BOLT$ in the other equations must become zero.

Multiply the pivot equation by -2 and add to the second equation. Then

$$\begin{array}{rcrcrcrcrcr} & & \frac{2}{3}SCREW & + & BOLT & + & \frac{1}{3}S1 & & & = & \frac{10}{3} \\ & & -\frac{1}{3}SCREW & & & - & \frac{2}{3}S1 & + & S2 & = & \frac{16}{3} \\ PROFIT & - & 3SCREW & - & 5BOLT & & & & & = & 0. \end{array}$$

Multiply the pivot equation by 5 and add to the objective function equation. Then

$$\begin{array}{rcrcrcrcrcr} & & \frac{2}{3}SCREW & + & BOLT & + & \frac{1}{3}S1 & & & = & \frac{10}{3} \\ & & -\frac{1}{3}SCREW & & & - & \frac{2}{3}S1 & + & S2 & = & \frac{16}{3} \\ PROFIT & + & \frac{1}{3}SCREW & & & + & \frac{5}{3}S1 & & & = & \frac{50}{3}. \end{array}$$

We have now completed one iteration of the simplex method. The solution at this stage is as follows:

Basic variables: $BOLT = \frac{10}{3}$; $S2 = \frac{16}{3}$.

Nonbasic variables: $SCREW = 0$; $S1 = 0$.

Objective function value: $PROFIT = \frac{50}{3}$.

Can we improve on this? We know that a negative coefficient in the objective function equation indicates that the value of the objective function can still improve. However, in this case, all the coefficients in the objective function equation are ≥ 0 and no improvement is possible. The optimal solution has therefore been obtained.

The optimal solution is to produce $\frac{10}{3}$ bolts and no screws for a maximum profit of $\frac{50}{3}$; $S1 = 0$ indicates that manpower is fully utilised while $S2 = \frac{16}{3}$ indicates that $\frac{16}{3}$ units of the available capital are not used.

Let us now solve the following *minimising* model:

$$\begin{array}{ll}
\text{Minimise} & z = -5x_1 + 3x_2 - 3x_3 \\
\text{subject to} & \\
 & \begin{array}{rcrcrcr}
 2x_1 & + & x_2 & - & x_3 & \leq & 8 \\
 -x_1 & + & 2x_2 & + & x_3 & \leq & 10 \\
 3x_1 & - & x_2 & + & 3x_3 & \leq & 6
 \end{array} \\
\text{and} & x_1;\ x_2;\ x_3 \geq 0.
\end{array}$$

Since minimisation is the opposite of maximisation, we change the objective function,

$$\text{Minimise } z = -5x_1 + 3x_2 - 3x_3,$$

to

$$\text{Maximise } Z = 5x_1 - 3x_2 + 3x_3$$

by multiplying by -1. Therefore, $Z = -z$.

We now continue with the simplex method as before and obtain

$$\begin{array}{rrrrrrrrrrrrrrr|l}
 & & & & & & & & & & & & & & & \theta \\
 & & 2x_1 & + & x_2 & - & x_3 & + & s_1 & & & & & = & 8 & 4 \\
 & & -x_1 & + & 2x_2 & + & x_3 & & & + & s_2 & & & = & 10 & - \\
 & & 3x_1 & - & x_2 & + & 3x_3 & & & & & + & s_3 & = & 6 & 2 \leftarrow \text{min} \\
Z & - & 5x_1 & + & 3x_2 & - & 3x_3 & & & & & & & = & 0. & \\
 & & \uparrow & & & & & & & & & & & & &
\end{array}$$

entering variable

The variable x_1 has the largest negative coefficient in the objective function equation and must enter the basis. The third equation has the minimum θ-value; so the basic variable in this equation, s_3, must leave the basis.

The variables x_1, s_1 and s_2 are now the basic variables and their coefficients must consist of a one and zeros. The variables s_1 and s_2 satisfy this requirement, but x_1 does not. Therefore, the one-zero method must be applied to x_1.

The coefficient of x_1 in the third equation is the pivot coefficient. Divide the third equation by 3 to obtain the pivot equation. The result is

$$\begin{array}{rrrrrrrrrrrrrrl} & & 2x_1 & + & x_2 & - & x_3 & + & s_1 & & & & & = & 8 \\ & & -x_1 & + & 2x_2 & + & x_3 & & & + & s_2 & & & = & 10 \\ & & x_1 & - & \frac{1}{3}x_2 & + & x_3 & & & & & + & \frac{1}{3}s_3 & = & 2 \quad \leftarrow \text{pivot equation} \\ Z & - & 5x_1 & + & 3x_2 & - & 3x_3 & & & & & & & = & 0. \end{array}$$

Now transform the coefficients of x_1 in the other equations to zero. Multiply the pivot equation by -2 and add it to the first equation. Add the pivot equation to the second equation. Multiply the pivot equation by 5 and add it to the objective function equation. The result is

$$\begin{array}{rrrrrrrrrrrrrl} & & & & \frac{5}{3}x_2 & - & 3x_3 & + & s_1 & & & - & \frac{2}{3}s_3 & = & 4 \\ & & & & \frac{5}{3}x_2 & + & 2x_3 & & & + & s_2 & + & \frac{1}{3}s_3 & = & 12 \\ & & x_1 & - & \frac{1}{3}x_2 & + & x_3 & & & & & + & \frac{1}{3}s_3 & = & 2 \\ Z & & & + & \frac{4}{3}x_2 & + & 2x_3 & & & & & + & \frac{5}{3}s_3 & = & 10. \end{array}$$

All the coefficients in the objective function equation are ≥ 0. This implies that there are no more variables that can improve the value of the objective function. The present solution is optimal and is as follows:

Basic variables: $s_1 = 4;\ s_2 = 12;\ x_1 = 2$.

Nonbasic variables: $x_2 = 0;\ x_3 = 0;\ s_3 = 0$.

Objective function value: $Z = 10,$ which means $z = -10$.

NOTE: Always use multiples of the pivot equation and add it to the other equations, otherwise you may end up with negative values on the right-hand side of the equations. And this is not allowed! (Why not?)

8.2 Examples

Example 8.1

Solving Mark's problem with the simplex method

We return to Mark's problem. The LP model is

$$\begin{array}{ll} \text{Maximise} & PROFIT = 300VH + 250SB \\ \text{subject to} & \\ & \begin{array}{rcrcl} 2VH & + & SB & \leq & 40 \qquad \text{(Labour time)} \\ VH & + & 3SB & \leq & 45 \qquad \text{(Machine time)} \\ VH & & & \leq & 12 \qquad \text{(Marketing)} \end{array} \\ \text{and} & VH;\ SB\ \geq 0. \end{array}$$

This model in standard form is

$$\begin{array}{ll} \text{Maximise} & PROFIT = 300VH + 250SB \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcrcl} 2VH & + & SB & + & S1 & & & & & = & 40 \\ VH & + & 3SB & & & + & S2 & & & = & 45 \\ VH & & & & & & & + & S3 & = & 12 \end{array} \\ \text{and} & VH;\ SB;\ S1;\ S2;\ S3 \geq 0. \end{array}$$

We rewrite the objective function as $PROFIT - 300VH - 250SB = 0$.

The initial system of equations is

$$\begin{array}{rcrcrcrcrcrcr|l} & & & & & & & & & & & & & \theta \\ & & 2VH & + & SB & + & S1 & & & & & = & 40 & 20 \\ & & VH & + & 3SB & & & + & S2 & & & = & 45 & 45 \\ & & VH & & & & & & & + & S3 & = & 12 & 12 \leftarrow \text{min} \\ PROFIT & - & 300VH & - & 250SB & & & & & & & = & 0. & \\ & & \uparrow & & & & & & & & & & & \end{array}$$

VH has the most negative coefficient in the objective function equation and must enter the basis. The third equation has the minimum θ-value and $S3$ must leave the basis.

After the first iteration we have the following system of equations:

$$\begin{array}{rrrrrrrrrrrr|l}
 & & & & & & & & & & & & \theta \\
 & & & SB & + & S1 & & & - & 2S3 & = & 16 & 16 \\
 & & & 3SB & & & + & S2 & - & S3 & = & 33 & 11 \leftarrow \text{min} \\
 & VH & & & & & & & + & S3 & = & 12 & - \\
PROFIT & & - & 250SB & & & & & + & 300S3 & = & 3\,600. & \\
 & & & \uparrow & & & & & & & & &
\end{array}$$

Now, SB enters the basis and $S2$ leaves.

After the second iteration we have the following system of equations:

$$\begin{array}{rrrrrrrrr}
 & & S1 & - & \frac{1}{3}S2 & - & \frac{5}{3}S3 & = & 5 \\
 & SB & & + & \frac{1}{3}S2 & - & \frac{1}{3}S3 & = & 11 \\
 VH & & & & & + & S3 & = & 12 \\
PROFIT & & & + & \frac{250}{3}S2 & + & \frac{650}{3}S3 & = & 6\,350.
\end{array}$$

All the coefficients in the objective function equation are ≥ 0. The present solution is therefore optimal. The optimal solution is as follows:

Basic variables: $SB = 11$; $VH = 12$; $S1 = 5$.

Nonbasic variables: $S2 = 0$; $S3 = 0$.

Objective function value: $PROFIT = 6\,350$.

According to our definition of the decision variables, the company should produce 11 sets of model $SB150$ and 12 sets of model $VH200$ per day to make a maximum daily profit of R6 350.

The slack variable $S1 = 5$ indicates that five hours of labour time are not utilised each day.

The slack variable $S2 = 0$ indicates that no machine time is left over and $S3 = 0$ indicates that the maximum number of sets that can be sold daily are produced.

The values of the slack variables give information on the utilisation of the resources. If the slack is zero, then the entire resource is utilised and the constraint is called a *binding constraint*. When the value of the slack variable is > 0, then the resource is not used completely and the constraint is called a *nonbinding constraint*. ┘

Example 8.2

Use the simplex method to solve the following LP model:

$$\begin{array}{ll}
\text{Minimise} & z = x_1 - x_2 - 3x_3 - 5x_4 \\
\text{subject to} & \\
 & \begin{array}{rrrrrrrrr}
3x_1 & - & 4x_2 & + & x_3 & - & x_4 & \leq & 2 \\
 & & 5x_2 & - & 5x_3 & - & 2x_4 & \leq & 8 \\
2x_1 & - & x_2 & + & 2x_3 & + & x_4 & \leq & 7
\end{array} \\
\text{and} & x_1;\ x_2;\ x_3;\ x_4\ \geq 0.
\end{array}$$

Solution

The LP model in standard form is

$$\begin{array}{lll}
\text{Maximise} & Z = -x_1 + x_2 + 3x_3 + 5x_4 & \qquad\qquad (\text{with } Z = -z) \\
\text{subject to} & & \\
 & \begin{array}{rrrrrrrrrrrrrrr}
3x_1 & - & 4x_2 & + & x_3 & - & x_4 & + & s_1 & & & & & = & 2 \\
 & & 5x_2 & - & 5x_3 & - & 2x_4 & & & + & s_2 & & & = & 8 \\
2x_1 & - & x_2 & + & 2x_3 & + & x_4 & & & & & + & s_3 & = & 7
\end{array} & \\
\text{and} & x_1;\ x_2;\ x_3;\ x_4;\ s_1;\ s_2;\ s_3 \geq 0. &
\end{array}$$

The initial system of equations is

$$\begin{array}{rrrrrrrrrrrrrrrrr|ll}
 & & & & & & & & & & & & & & & & & \theta & \\
 & & 3x_1 & - & 4x_2 & + & x_3 & - & x_4 & + & s_1 & & & & & = & 2 & - & \\
 & & & & 5x_2 & - & 5x_3 & - & 2x_4 & & & + & s_2 & & & = & 8 & - & \\
 & & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & & & & & + & s_3 & = & 7 & 7 & \leftarrow \\
Z & + & x_1 & - & x_2 & - & 3x_3 & - & 5x_4 & & & & & & & = & 0. & & \\
 & & & & & & & & \uparrow & & & & & & & & & &
\end{array}$$

The entering variable is x_4 and the leaving variable is s_3.

After the first iteration we have the following system of equations:

$$\begin{array}{rcrcrcrcrcrcrcrcrcr|c}
 & & & & & & & & & & & & & & & & & \theta \\
 & & 5x_1 & - & 5x_2 & + & 3x_3 & & & + & s_1 & & & + & s_3 & = & 9 & - \\
 & & 4x_1 & + & 3x_2 & - & x_3 & & & & & + & s_2 & + & 2s_3 & = & 22 & \frac{22}{3} \\
 & & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & & & & & + & s_3 & = & 7 & - \\
Z & + & 11x_1 & - & 6x_2 & + & 7x_3 & & & & & & & + & 5s_3 & = & 35. & \\
 & & & & \uparrow & & & & & & & & & & & & &
\end{array}$$

The entering variable is x_2 and the leaving variable is s_2.

After the second iteration we have the following system of equations:

$$\begin{array}{rcrcrcrcrcrcrcrcr}
 & & \frac{35}{3}x_1 & & & + & \frac{4}{3}x_3 & & & + & s_1 & + & \frac{5}{3}s_2 & + & \frac{13}{3}s_3 & = & \frac{137}{3} \\
 & & \frac{4}{3}x_1 & + & x_2 & - & \frac{1}{3}x_3 & & & & & + & \frac{1}{3}s_2 & + & \frac{2}{3}s_3 & = & \frac{22}{3} \\
 & & \frac{10}{3}x_1 & & & + & \frac{5}{3}x_3 & + & x_4 & & & + & \frac{1}{3}s_2 & + & \frac{5}{3}s_3 & = & \frac{43}{3} \\
Z & + & 19x_1 & & & + & 5x_3 & & & & & + & 2s_2 & + & 9s_3 & = & 79.
\end{array}$$

All the coefficients in the objective function equation are ≥ 0 and the present solution is optimal. The optimal solution is as follows:

Basic variables: $x_2 = \frac{22}{3} = 7,33$; $x_4 = \frac{43}{3} = 14,33$; $s_1 = \frac{137}{3} = 45,67$.

Nonbasic variables: $x_1 = 0$; $x_3 = 0$; $s_2 = 0$; $s_3 = 0$.

The optimal value of the objective function is $z = -Z = -79$.

8.3 Comparing the graphical approach to the simplex method

We first solved Mark's problem graphically and then by means of the simplex method. Let us now compare the two methods.

With the simplex method we initially have $S1$, $S2$ and $S3$ in the basis. Here $VH = 0$ and $SB = 0$, which corresponds to the point $(0; 0)$ on the graph.

After the first iteration we have $S1$, $S2$ and VH in the basis. Here $VH = 12$ and $SB = 0$, which corresponds to the point $(12; 0)$ on the graph.

After the second iteration we have reached the optimal solution with $S1$, SB and VH in the basis. Here $VH = 12$ and $SB = 11$, which corresponds to the point $(12; 11)$ on the graph.

In Figure 8.2 the path that the simplex method followed to reach the optimal solution can be seen clearly.

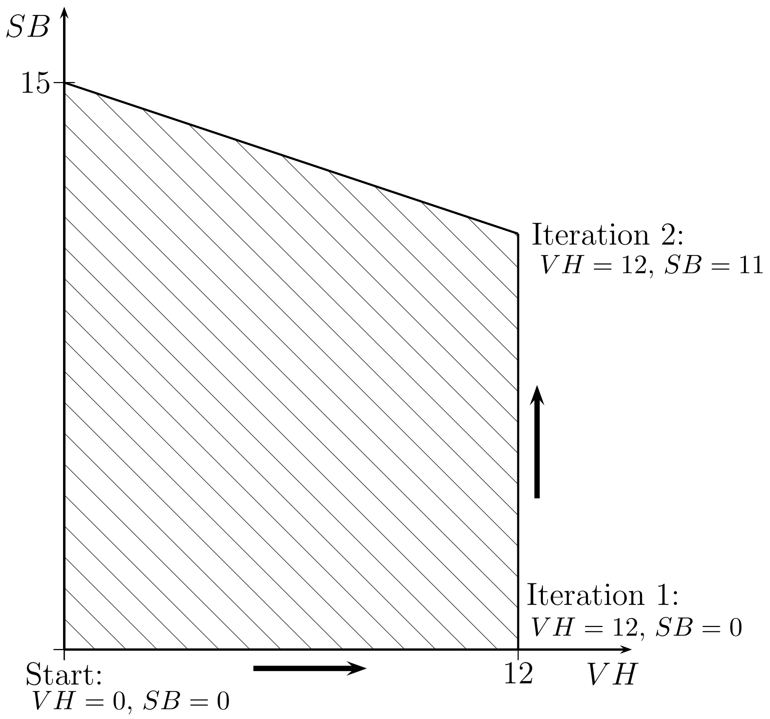

Figure 8.2: Mark's problem – comparison between graphical and simplex methods

8.4 The Simplex Method in a Tabular Form

The algebraic form of the simplex method presented in the previous section is the best one for learning the underlying logic of the method. However, it is not the most convenient form for performing the required calculations. When you need to solve the problem by hand, it might be easier to use the tabular form.

In fact, the tabular form of the simplex method records only essential information, namely:

- the coefficients of all variables,
- the constants on the right-hand sides of the equations that we denote by "rhs",
- the basic variables that we denote by "BV", and
- the ratios denoted by "θ" as previously.

The tabular form of the simplex method uses a simplex tableau to compactly display all information about an iteration of the simplex method. The table below compares the algebraic and the tabular form of the simplex algorithm.

Algebraic Form			**Tabular Form**										
Eq		θ	Row	coefficient of:									
				z	x_1	x_2	x_3	s_1	s_2	s_3	rhs	BV	θ
(1)	$2x_1 + x_2 - x_3 + s_1 = 8$	4	R_1	0	2	1	-1	1	0	0	8	s_1	4
(2)	$-x_1 + 2x_2 + x_3 + s_2 = 10$	-	R_2	0	-1	2	1	0	1	0	10	s_2	-
(3)	$3x_1 - x_2 + 3x_3 + s_3 = 6$	2	R_3	0	3	-1	3	0	0	1	6	s_3	2
(0)	$z - 5x_1 + 3x_2 - 3x_3 = 0$		R_0	1	-5	3	-3	0	0	0	0	z	

To solve an LP problem by using the simplex method in a tabular form, the following steps are required:

(1) Write down the problem in its standard form

(2) Write down the initial simplex tableau (include basic variables in the tableau).

(3) Determine the entering variable. The corresponding column where the entering variable occurs is the **pivot column**.

(4) Determine the leaving variable. The corresponding row where the leaving variable occurs is the **pivot row**. The intersection between the pivot column and the pivot row gives the **pivot element** (or simply **the pivot**)(*).

(5) Apply the one-zero method to the corresponding pivot column and write down the next simplex tableau.

(6) Test optimality.

(7) If the optimal solution is not reached, repeat steps (3), (4), (5) and (6).

A simplex tableau will therefore look like the table below:

Simplex tableau

		Decision variables					**Slack/surplus variables**				**Artificial variables**				Right hand side	Basic variables	Ratios
Row	z	x_1	x_2	x_3	$\ldots$	x_w	s_1	s_2	$\ldots$	s_p	a_{p+1}	a_{p+2}	$\ldots$	a_n	(rhs)	(BV)	(θ)
R_1																	
R_2																	
$\vdots$																	
R_m																	
R_0	1															z	

*

---

[*] In the remaining part of this MO 001, the pivot will always be circled.

Example 8.3

Consider the following LP model

$$
\begin{array}{ll}
\text{Maximise} & z = 3x_1 + 5x_2 \\
\text{subject to} & \\
& \begin{array}{rcrcr} 2x_1 & + & 3x_2 & \leq & 10 \\ x_1 & + & 2x_2 & \leq & 12 \end{array} \\
\text{and} & x_1;\ x_2; \geq 0.
\end{array}
$$

In standard form, we have

$$
\begin{array}{ll}
\text{Maximise} & z = 3x_1 + 5x_2 \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & + & s_1 & & & = & 10 \\ x_1 & + & 2x_2 & & & + & s_2 & = & 12 \end{array} \\
\text{and} & x_1;\ x_2; s_1;\ s_2 \geq 0,
\end{array}
$$

where s_1 and s_2 are slack variables.

Initial simplex tableau

	z	x_1	x_2	s_1	s_2	rhs	BV	θ	
R_1	0	2	③	1	0	10	s_1	$\frac{10}{3}$	→
R_2	0	1	2	0	1	12	s_2	$\frac{12}{2}$	
R_0	1	−3	−5	0	0	0	z		
			↑						

- We determine the entering basic variable by selecting the "most negative" coefficient in row R_0 (also called "objective row"). In our case, the most negative coefficient in row R_0 is -5. The corresponding entering variable is x_2.
- We determine the leaving variable by applying the minimum ratio test θ :

$$\min\ \theta = \min\left\{\frac{10}{3},\ \frac{12}{2}\right\} = \frac{10}{3}.$$

The corresponding basic variable (BV) for $\min\theta$ is s_1. It follows that s_1 is the leaving variable.

Note:

- The column that corresponds to the entering variable is called the pivot column.
- The row that corresponds to the leaving variable is the pivot row, and
- The element (entry) that corresponds to the intersection between the pivot column and the pivot row is simply called the "pivot". In our case the pivot is 3.

- We find the next simplex tableau using the one-zero method in the pivot column by performing elementary row operations (EROs) that make the pivot 1 and transform all nonzero elements in the pivot column (except the pivot itself) to zeros as done in learning unit 4.

  Here the one-zero method will be applied to the x_2's column. The next simplex tableau is then

Tableau 1

Row	z	x_1	x_2	s_1	s_2	rhs	BV
R_1	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{10}{3}$	x_2
R_2	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	1	$\frac{16}{3}$	s_2
R_0	1	$\frac{1}{3}$	0	$\frac{5}{3}$	0	$\frac{50}{3}$	z

In this tableau there is no negative element in row R_0 (i.e. there is no entering variable). The optimal solution has therefore been obtained and it is given by

$$x_1 = 0,\ x_2 = \frac{10}{3} \text{ and } z = \frac{50}{3}.$$

Example 8.4

$$\begin{array}{ll} \text{Minimise} & z = x_1 - x_2 - 3x_3 - 5x_4 \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcl} 3x_1 & - & 4x_2 & + & x_3 & - & x_4 & \leq & 2 \\ & & 5x_2 & - & 5x_3 & - & 2x_4 & \leq & 8 \\ 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & \leq & 7 \end{array} \\ \text{and} & x_1,\ x_2,\ x_3,\ x_4 \geq 0. \end{array}$$

After transforming the problem into its standard form as done in Example 8.2, we get the initial simplex tableau.

Initial simplex tableau

Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs	BV	θ	
R_1	0	3	-4	1	-1	1	0	0	2	s_1	-	
R_2	0	0	5	-5	-2	0	1	0	8	s_2	-	
R_3	0	2	-1	2	①	0	0	1	7	s_3	7	→
R_0	1	1	-1	-3	-5	0	0	0	0	Z		
					↑							

- The most negative entry in row R_0 is -5. Thus the entering variable is x_4.
- $\min\theta = \min\left\{-,\ -,\ \frac{7}{1}\right\} = 7$. The leaving variable is s_3. We apply the one-zero method to the current pivot column (that is the x_2's column) and obtain the next Simplex tableau below.

Tableau 1

Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs	BV	θ	
R_1	0	5	-5	3	0	1	0	1	9	s_1	-	
R_2	0	4	③	-1	0	0	1	2	22	s_2	$\frac{22}{3}$	→
R_3	0	2	-1	2	1	0	0	1	7	x_4	-	
R_0	1	11	-6	7	0	0	0	5	35	Z		
			↑									

- The most negative entry in R_0 is -6. Thus the entering variable is x_2.
- $\min\theta = \min\left\{-,\ \frac{22}{3}, -\right\} = \frac{22}{3}$. The leaving variable is s_2.
- We apply the one-zero method to the current pivot column (that is the x_2's column) and obtain the next simplex tableau.

Tableau 2

Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	rhs	BV
R_1	0	$\frac{35}{3}$	0	$\frac{4}{3}$	0	1	$\frac{5}{3}$	$\frac{13}{3}$	$\frac{137}{3}$	s_1
R_2	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{22}{3}$	x_2
R_3	0	$\frac{10}{3}$	0	$\frac{5}{3}$	1	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{43}{3}$	x_4
R_0	1	19	0	5	0	0	2	9	79	Z

As there is no negative element in row R_0 then we have reached the optimal solution given by

$$x_1 = 0,\ x_2 = \frac{22}{3},\ x_3 = 0,\ x_4 = \frac{43}{3} \text{ and } z = -Z = -79.$$

Example 8.5

$$\begin{aligned} &\text{Maximise} \quad Z = 4x_1 + 4x_2 \\ &\text{subject to} \\ &\qquad\qquad 2x_1 + 7x_2 \leq 21 \\ &\qquad\qquad 7x_1 + 2x_2 \leq 49 \\ &\text{and} \qquad x_1,\ x_2 \geq 0. \end{aligned}$$

After transforming the problem into its standard form as done in Example 8.3, we get the initial simplex tableau.

Initial simplex tableau

Row	Z	x_1	x_2	s_1	s_2	rhs	BV	θ	
R_1	0	2	$\textcircled{7}$	1	0	21	s_1	3	→
R_2	0	7	2	0	1	49	s_2	$\frac{49}{2}$	
R_0	1	-4	-4	0	0	0	Z		
			↑						

- The most negative entry in row R_0 is -4. We can use either x_1 or x_2 as an entering variable. Let's choose x_2.
- $\min\theta = \min\left\{\frac{21}{7},\ \frac{49}{2}\right\} = \frac{21}{7} = 3$. The leaving variable is s_1.

  We apply the one-zero method to the current pivot column (i.e. the x_2's column) and obtain the following tableau:

Tableau 1

Row	Z	x_1	x_2	s_1	s_2	rhs	BV	θ	
R_1	0	$\frac{2}{7}$	1	$\frac{1}{7}$	0	3	x_2	$\frac{21}{2}$	
R_2	0	$\textcircled{\frac{45}{7}}$	0	$-\frac{2}{7}$	1	43	s_2	$\frac{301}{45}$	→
R_0	1	$-\frac{20}{7}$	0	$\frac{4}{7}$	0	12	Z		
		↑							

- The most negative element in R_0 is $-\frac{20}{7}$. Thus the entering variable is x_1.
- min $\theta = \min\left\{\frac{21}{2},\ \frac{301}{45}\right\} = \frac{301}{45}$. The leaving variable is s_2.

  We apply the one-zero method in the current pivot column (that is the x_1's column) and get

Tableau 2

	Z	x_1	x_2	s_1	s_2	rhs	BV
R_1	0	0	1	$\frac{7}{45}$	$-\frac{2}{45}$	$\frac{49}{45}$	x_2
R_2	0	1	0	$-\frac{2}{45}$	$\frac{7}{45}$	$\frac{301}{45}$	x_1
R_0	1	0	0	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{280}{9}$	Z

As there is no negative element in row R_0, we have reached the optimal solution given by

$$x_1 = \frac{301}{45},\ x_2 = \frac{49}{45} \text{ and } Z = \frac{280}{9}.$$

8.5 The Big M Method

In the previous section we have presented the Simplex method under the assumptions that all constraints must have a sign "$\leq$". In this section we investigate the case where the linear programming problem has a "*greater than or equal to*", "*equality*", or "*mixed*" constraints. Roughly speaking the Big M method is a variation of the Simplex method designed for solving problems with "$\geq$" or/and "=" signs in a model. The main idea of the method consists of the following:

(1) Transform the problem into a standard form by adding slack or/and surplus variables where necessary.

(2) Transform the standard form obtained in (1) into an augmented form by adding artificial variables in every constraint where necessary and by adding (for a minimization problem) or subtracting (for a maximization problem) all artificial variables in the objective function with a coefficient that has a large number, let's say M.

(3) Write down the initial simplex tableau with a **new objective row** obtained by eliminating all artificial variables from the **initial objective row** by performing appropriate elementary row operations (EROs).

(4) Carry on with the simplex algorithm.

Let's now examine each case of all augmented model forms.

8.5.1 “*Greater than or equal to*” constraints

We first return to Christine’s problem. Unique Paint, the company for which Christine works, wants to market its Sungold paint as soon as possible since a rival company is promoting a similar paint and they could run into marketing problems if they postpone production too long. Christine must find the combination of the new ingredients, Alpha and Beta, that will result in brilliance and tint equivalent to the original ingredients, but at a lower cost. An LP model for her problem was formulated as follows:

$$\begin{array}{lllll}
\text{Minimise} & Z = 45\alpha + 12\beta & & & \\
\text{subject to} & & & & \\
 & \alpha + \beta & \geq & 300 & \text{(Brilliance)} \\
 & 3\alpha & \geq & 250 & \text{(Tint)} \\
\text{and} & \alpha;\ \beta \geq 0. & & &
\end{array}$$

Here

α = quantity (in grams) of Alpha in each tin of paint,
β = quantity (in grams) of Beta in each tin of paint.

In order to use the simplex method, we have to rewrite this model in standard form. Since we have $\geq$ constraints, we have to subtract surplus variables S_1 and S_2 to balance them. We must also multiply the objective function by -1 to change it to a maximisation model.

The LP model in standard form is now given by

$$\begin{array}{llll}
\text{Maximise} & z = -Z = -45\alpha - 12\beta & & \\
\text{subject to} & & & \\
 & \alpha + \beta - S_1 & = & 300 \\
 & 3\alpha - S_2 & = & 250 \\
\text{and} & \alpha;\ \beta;\ S_1;\ S_2 \geq 0. & &
\end{array}$$

When a solution to this LP model is obtained, the surplus variable S_1 will indicate the amount by which the brilliance obtained by the blend will exceed the minimum brilliance required in Sungold. Similarly, S_2 will indicate the amount by which the tint obtained will exceed the minimum requirement for tint. Since these surplus variables represent physical quantities, it is obvious that they cannot be negative. And so S_1 and S_2 must be ≥ 0.

This implies that the above system of equations cannot be used to determine an initial basic feasible solution. If we set $\alpha = 0$ and $\beta = 0$, we have $S_1 = -300$ and $S_2 = -250$. And we are then violating the nonnegativity restriction.

LP models having no initial basis may be augmented with artificial variables to create a basis.

In Christine's case, we add artificial variables A_1 and A_2 to the constraints, with A_1 and $A_2 \geq 0$. Then

$$\begin{array}{ccccccccccc} \alpha & + & \beta & - & S_1 & & & + & A_1 & & = & 300 \\ 3\alpha & & & & & - & S_2 & & & + & A_2 & = & 250. \end{array}$$

These artificial variables are only used to force an initial basis to start the simplex method with and have no physical meaning.

If we now set $\alpha = 0,\ \beta = 0,\ S_1 = 0$ and $S_2 = 0$, we have $A_1 = 300$ and $A_2 = 250$. This is a feasible solution to the augmented model but not to the original model since A_1 and A_2 are artificial and are not part of that model.

Let us consider the effect of surplus and artificial variables on the objective function.

Surplus variables, like slack variables, have no effect on the objective function in terms of increasing or decreasing its value. In this case cost is determined by the quantity of Alpha and Beta used in each tin of paint and not by the amount by which the brilliance or tint is exceeded. A coefficient of zero is therefore assigned to each surplus (or slack) variable in the objective function. Surplus and slack variables can be part of an optimal basis – it is quite realistic to have an optimal solution where the minimum requirements are exceeded or where a resource is not fully utilised.

If we assign a zero coefficient to an artificial variable in the objective function, this artificial variable could also be part of the optimal basis. Since the artificial variable has no meaning, the optimal solution would also be meaningless. We must therefore make sure that the artificial variables will not be part of the optimal basis.

To do this, we assign very large negative coefficients to the artificial variables in the objective function. We are now maximising (our model being in standard form) and the simplex method will get rid of these variables as soon as possible since these "large costs" are unacceptable.

Instead of choosing specific numerical values for the coefficients of the artificial variables, we use the symbol M and assume that M is a very large number. For this reason, this technique is known as the *Big M method.*

The objective function now becomes

$$\text{Maximise } z = -45\alpha - 12\beta - MA_1 - MA_2.$$

The augmented model corresponding to Christine's model is then

$$\begin{array}{ll}
\text{Maximise} & z = -45\alpha - 12\beta - MA_1 - MA_2 \\
\text{subject to} & \\
 & \begin{array}{rcrcrcrcrcrcl} \alpha & + & \beta & - & S_1 & & & + & A_1 & & & = & 300 \\ 3\alpha & & & & & - & S_2 & & & + & A_2 & = & 250 \end{array} \\
\text{and} & \alpha;\ \beta;\ S_1;\ S_2;\ A_1;\ A_2 \geq 0.
\end{array}$$

To apply the simplex method, the augmented model must be written as a system of equations. Then

$$\begin{array}{rcrcrcrcrcrcrcl}
 & & \alpha & + & \beta & - & S_1 & & & + & A_1 & & & = & 300 \\
 & & 3\alpha & & & & & - & S_2 & & & + & A_2 & = & 250 \\
Z & + & 45\alpha & + & 12\beta & & & & & + & MA_1 & + & MA_2 & = & 0.
\end{array}$$

The initial basic variables are A_1 and A_2, and they have coefficients of M in the objective function equation. Basic variables should have a coefficient of one in one equation and zeros in the other equations. To comply with this, we must multiply each of the first two equations by $-M$ and add to the objective function equation. This leads to the initial simplex tableau.

Initial simplex tableau

	z	α	β	S_1	S_2	A_1	A_2	rhs	BV	θ	
R_1	0	1	1	-1	0	1	0	300	A_1	300	
R_2	0	$\textcircled{3}$	0	0	-1	0	1	250	A_2	$\frac{250}{3}$	→
R_0	1	45	12	0	0	M	M	0	z	–	
R_0'	1	$45-4M$	$12-M$	M	M	0	0	$-550M$	z	–	
		↑									

Where the new objective row R_0' is obtained by eliminating artificial variables A_1 and A_2 from the objective row R_0 using the ERO:

$$R_1^{'} = R_0 - MR_1 - MR_2$$

- The most negative number in R_0' is $45 - 4M$. Thus the entering variable is α.
- min $\theta = \min\left\{300,\ \frac{250}{3}\right\} = \frac{250}{3}$. The leaving variable is then A_2.
- We find the pivot element to be 3, we apply the one-zero method and get the next simplex tableau:

Tableau 1

	z	α	β	S_1	S_2	A_1	A_2	rhs	BV	θ
R_1	0	0	①	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{650}{3}$	A_1	$\frac{650}{3}$ $\rightarrow$
R_2	0	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{250}{3}$	α	-
R_0	1	0	$12-M$	M	$15-\frac{1}{3}M$	0	$-15+\frac{4}{3}M$	$\otimes$	z	
			$\uparrow$							

In the above tableau $\otimes$ represents $-3\,750-\frac{432}{3}M$.

- The most negative number in R_0 is $12-M$. Thus the entering variable is β
- min $\theta=\left\{\frac{650}{3},\ -\right\}=\frac{650}{3}$. The leaving variable is A_1
- The next simplex tableau is

Tableau 2

	z	α	β	S_1	S_2	A_1	A_2	rhs	BV
R_1	0	0	1	-1	$\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{650}{3}$	β
R_2	0	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{250}{3}$	α
R_0	1	0	0	12	11	$-12+M$	$-11+M$	$-6\,350$	Z

In row R_0 all elements are positive and thus we have reached the optimal solution given by

$$\begin{aligned}\alpha &= \frac{250}{3}\\ \beta &= \frac{650}{3}\\ \text{Min Cost } Z=-z &= 6\,350.\end{aligned}$$

8.5.2 "*Equality*" constraints

Let us now study a problem containing equality constraints, that is, constraints of the "=" type. Consider the following LP model:

$$
\begin{array}{ll}
\text{Maximise} & Z = x_1 + 3x_2 + 4x_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcl}
x_1 & + & 2x_2 & + & 3x_3 & = & 15 \\
2x_1 & + & x_2 & + & 5x_3 & = & 20 \\
x_1 & + & 2x_2 & + & x_3 & = & 10
\end{array} \\
\text{and} & x_1;\ x_2;\ x_3 \geq 0.
\end{array}
$$

This model is in standard form but does not contain any variables that can be used as the initial basis with which to start the simplex method. Again artificial variables must be introduced to create an initial basis. The objective function must also be augmented by adding each artificial variable multiplied by a large negative number, say $-M$.

The augmented model is

$$
\begin{array}{ll}
\text{Maximise} & Z = x_1 + 3x_2 + 4x_3 - Ma_1 - Ma_2 - Ma_3 \\
\text{subject to} & \\
& \begin{array}{rcrcrcccccccl}
x_1 & + & 2x_2 & + & 3x_3 & + & a_1 & & & & & = & 15 \\
2x_1 & + & x_2 & + & 5x_3 & & & + & a_2 & & & = & 20 \\
x_1 & + & 2x_2 & + & x_3 & & & & & + & a_3 & = & 10
\end{array} \\
\text{and} & x_1;\ x_2;\ x_3;\ a_1;\ a_2;\ a_3 \geq 0.
\end{array}
$$

Initial simplex tableau

	Z	x_1	x_2	x_3	a_1	a_2	a_3	rhs	BV	θ	
R_1	0	1	2	3	1	0	0	15	a_1	5	
R_2	0	2	1	⑤	0	1	0	20	a_2	4	→
R_3	0	1	2	1	0	0	1	10	a_3	10	
R_0	1	-1	-3	-4	M	M	M	0	z	–	
R_0'	1	$-1-4M$	$-3-5M$	$-4-9M$	0	0	0	$-45M$	z	–	
				↑							

Where R_0' is the new objective row obtained by performing the following elementary row operation (ERO):

$$R_0' = R_0 - (R_1 + R_2 + R_3)M.$$

In this new objective row R_0' all artificial variables are eliminated. The initial basis is $a_1 = 15,\ a_2 = 20$ and $a_3 = 10$ as we can see it in the tableau. We therefore proceed with the simplex algorithm.

- The entering variable is x_3.

- The leaving variable is a_2.
- After applying the one-zero method to the current pivot column (that is the x_3's column) we get the next simplex tableau given by

Tableau 1

	Z	x_1	x_2	x_3	a_1	a_2	a_3	rhs	BV	θ	
R_1	0	$-\frac{1}{5}$	$\boxed{\frac{7}{5}}$	0	1	$-\frac{3}{5}$	0	3	a_1	$\frac{15}{7}$	→
R_2	0	$\frac{2}{5}$	$\frac{1}{5}$	1	0	$\frac{1}{5}$	0	4	x_3	20	
R_3	0	$\frac{3}{5}$	$\frac{9}{5}$	0	0	$-\frac{1}{5}$	1	6	a_3	$\frac{10}{3}$	
R_0	1	$\frac{1}{5}(3-2M)$	$\frac{1}{5}(-11-16M)$	0	0	$\frac{1}{5}(4+9M)$	0	$16-M$			
			↑								

- The entering variable is x_2 because $\frac{1}{5}(-11-16M)$ is the most negative number.
- $\min \theta = \min \left\{\frac{15}{7},\ 20,\ \frac{10}{3}\right\} = \frac{15}{7}$. It follows that the leaving variable is a_1.
- After applying the one-zero method to the current pivot column (that is the x_2's column) we get the following simplex tableau.

Tableau 2

	Z	x_1	x_2	x_3	a_1	a_2	a_3	rhs	BV	θ	
R_1	0	$-\frac{1}{7}$	1	0	$\frac{5}{7}$	$-\frac{3}{7}$	0	$\frac{15}{7}$	x_2	-	
R_2	0	$\frac{3}{7}$	0	1	$-\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{25}{7}$	x_3	$\frac{25}{3}$	
R_3	0	$\boxed{\frac{6}{7}}$	0	0	$-\frac{9}{7}$	$\frac{4}{7}$	1	$\frac{15}{7}$	a_3	$\frac{5}{2}$	→
R_0	1	$\frac{1}{7}(2-6M)$	0	0	$\frac{1}{7}(11+16M)$	$\frac{1}{7}(1+3M)$	0	$\frac{1}{7}(145-15M)$	Z		
		↑									

- The entering variable is x_1 because $\frac{1}{7}(2-6M)$ is the most negative number.
- $\min \theta = \min \left\{-,\ \frac{25}{3},\ \frac{5}{2}\right\} = \frac{5}{2}$. It follows that the leaving variable is a_3.
- After applying the one-zero method, we get the next simplex tableau.

Tableau 3

	Z	x_1	x_2	x_3	a_1	a_2	a_3	rhs	BV
R_1	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{2}$	x_2
R_2	0	0	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{5}{2}$	x_3
R_3	0	1	0	0	$-\frac{3}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{5}{2}$	x_1
R_0	1	0	0	0	$2+M$	$-\frac{1}{3}+M$	$-\frac{1}{3}+M$	20	Z

As there is no negative number in the objective row R_0 then we have reached the optimal solution given by

$$x_2 = \frac{5}{2},\ x_3 = \frac{5}{2},\ x_1 = \frac{5}{2},\ Z = 20$$

8.5.3 "*Mixed*" constraints

We have now considered "≤" constraints with Mark's problem, "≥" constraints with Christine's problem and "=" constraints in the previous paragraph. Most LP models have a combination of these. For example, consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & I = 2D + 2E - F \\ \text{subject to} & \\ & \begin{array}{rcrcrcr} D & + & E & + & F & \leq & 10 \\ D & + & 2E & - & F & \geq & 8 \\ -D & + & E & + & F & = & 2 \end{array} \\ \text{and} & D;\ E;\ F \geq 0. \end{array}$$

Write the constraints in standard form by adding slack variables and subtracting surplus variables where necessary. (Slack and surplus variables are both denoted by S.)

The standard form of the LP model is

$$\begin{array}{ll} \text{Maximise} & I = 2D + 2E - F \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcrcr} D & + & E & + & F & + & S_1 & & & = & 10 \\ D & + & 2E & - & F & & & - & S_2 & = & 8 \\ -D & + & E & + & F & & & & & = & 2 \end{array} \\ \text{and} & D;\ E;\ F;\ S_1;\ S_2 \geq 0. \end{array}$$

The variable S_1 has a coefficient of one in the first equation and a coefficient of zero elsewhere. It can be used in the initial basis. The second and third equations have no candidates for a basis and we add artificial variables to them.

The augmented model is

$$\begin{array}{ll} \text{Maximise} & I = 2D + 2E - F - MA_2 - MA_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcrcrcrcr} D & + & E & + & F & + & S_1 & & & & & & & = & 10 \\ D & + & 2E & - & F & & & - & S_2 & + & A_2 & & & = & 8 \\ -D & + & E & + & F & & & & & & & + & A_3 & = & 2 \end{array} \\ \text{and} & D,\ E,\ F,\ S_1,\ S_2,\ A_2,\ A_3 \geq 0. \end{array}$$

Our initial basis consists of $S_1 = 10,\ A_2 = 8$ and $A_3 = 2$.

Initial simplex tableau

	I	D	E	F	S_1	S_2	A_2	A_3	rhs	BV	θ	
R_1	0	1	1	1	1	0	0	0	10	S_1	10	
R_2	0	1	2	-1	0	-1	1	0	8	A_2	4	
R_3	0	-1	①	1	0	0	0	1	2	A_3	2	→
R_0	1	-2	-2	1	0	0	M	M	0	I	–	
R_0'	1	-2	$-2-3M$	1	0	M	0	0	$-10M$	I	–	
			↑									

where the new objective row is obtained by eliminating artificial variables A_2 and A_3 from the objective row R_0 using the following ERO:

$$R_0^{'} = R_0 - MR_2 - MR_3$$

- The entering variable is E because $-2-3M$ is the most negative coefficient in R_0.
- $\min\theta = \min\{10,\ 4,\ 2\} = 2$. Then the leaving variable is A_3.
- We apply the one-zero method and get the next simplex tableau:

Tableau 1

	I	D	E	F	S_1	S_2	A_2	A_3	rhs	BV	θ	
R_1	0	2	0	0	1	0	0	-1	8	S_1	4	
R_2	0	③	0	-3	0	-1	1	-2	4	A_2	$\frac{4}{3}$	→
R_3	0	-1	1	1	0	0	0	1	2	E	-	
R_0	1	$-4-3M$	0	$(3+3M)$	0	M	0	$2+3M$	$4M+4$			
		↑										

- The entering variable is D because $4-3M$ is the most negative coefficient in R_0.
- $\min\theta = \min\left\{4,\ \frac{4}{3}\right\} = \frac{4}{3}$. The leaving variable is then A_2.

After applying the one-zero method to the current pivot column (i.e. the D column), we get the next simplex tableau:

Tableau 2

	I	D	E	F	S_1	S_2	A_2	A_3	rhs	BV	θ	
R_1	0	0	0	2	1	$\textcircled{\tfrac{2}{3}}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{16}{3}$	S_1	8	→
R_2	0	1	0	-1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{4}{3}$	D	-	
R_3	0	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{10}{3}$	E	-	
R_0	1	0	0	-1	0	$-\frac{4}{3}$	$\frac{4}{3}+M$	$-\frac{2}{3}+M$	$\frac{28}{3}$	I		
						↑						

- The entering variable is S_2.
- $\min\theta = \min\{8, -, -\} = 8$. The leaving variable is S_1.
- The next simplex tableau is given by

Tableau 3

I	D	E	F	S_1	S_2	A_2	A_3	rhs	BV
0	0	0	3	$\frac{3}{2}$	1	-1	$\frac{1}{2}$	8	S_2
0	1	0	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	4	D
0	0	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	6	E
1	0	0	3	2	0	M	M	20	I

The optimal solution has been reached and is

$$D = 4,\ E = 6,\ F = 0 \text{ and } I = 20.$$

Note: At the end of the algorithm all artificial variables **must be zero**. If in the final simplex tableau we obtain a strictly positive artificial variable, then the LP problem is said to be infeasible. More comments are given in the section below.

8.6 Types of solutions using the Simplex or Big M methods

Recall that in the Learning unit 7, Section 7.5 we learnt about different types of solutions using the graphical method. So now let us see what they look like in the simplex or Big M method. We will use some of the same examples but will solve them with the simplex or Big M methods.

8.6.1 Infeasible LPs

If at the final iteration of the Big M method we have at least one artificial variable greater than zero, then the LP is infeasible. For example, let's consider the following LP problem

$$\begin{array}{ll} \text{Maximise} & z = 5x + 3y \\ \text{subject to} & \\ & \begin{array}{rcrcl} 2x & + & y & \leq & 4 \\ x & & & \geq & 4 \\ & & y & \geq & 6 \end{array} \\ \text{and} & x,\ y \geq 0. \end{array}$$

This problem must be solved by the Big M method since the "$\geq$" sign appears in at least one constraint. The model in standard form is

$$\begin{array}{ll} \text{Maximise} & z = 5x + 3y \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcrcl} 2x & + & y & + & s_1 & & & & & = & 4 \\ x & & & & & - & s_2 & & & = & 4 \\ & & y & & & & & - & s_3 & = & 6 \end{array} \\ \text{and} & x,\ y,\ s_1,\ s_2,\ s_3 \geq 0. \end{array}$$

By adding artificial variables, we get the augmented model:

$$\begin{array}{ll} \text{Maximise} & z = 5x + 3y - Ma_2 - Ma_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcrcrcrcl} 2x & + & y & + & s_1 & & & & & & & & & = & 4 \\ x & & & & & - & s_2 & & & + & a_2 & & & = & 4 \\ & & y & & & & & - & s_3 & & & + & a_3 & = & 6 \end{array} \\ \text{and} & x,\ y,\ s_1,\ s_2,\ s_3,\ a_2,\ a_3 \geq 0. \end{array}$$

After eliminating a_2 and a_3 in the objective function, we get the initial simplex tableau

Initial simplex tableau

	z	x	y	s_1	s_2	s_3	a_2	a_3	rhs	BV	θ	
R_1	0	$\textcircled{2}$	1	1	0	0	0	0	4	s_1	2	→
R_2	0	1	0	0	-1	0	1	0	4	a_2	4	
R_3	0	0	1	0	0	-1	0	1	6	a_3	–	
R_0	1	-5	-3	0	0	0	M	M	0	z	–	
$R_0^{'}$	1	$-5-M$	$-3-M$	0	M	M	0	0	$-10M$	z	–	
		↑										

where $R_0^{'} = R_0 - (R_2 + R_3)M$

- The entering variable is x.
- The leaving variable is s_1.
- The next simplex tableau is

Tableau 1

z	x	y	s_1	s_2	s_3	a_2	a_3	rhs	BV	θ	
0	1	$\textcircled{\frac{1}{2}}$	$\frac{1}{2}$	0	0	0	0	2	x	4	→
0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	1	0	2	a_2	-	
0	0	1	0	0	-1	0	1	6	a_3	6	
1	0	$-\frac{1}{2}(1+M)$	$\frac{1}{2}(5+M)$	M	M	0	0	$-8M+10$	z		
		↑									

- The entering variable is y.
- The leaving variable is x
- and the next simplex tableau is

Tableau 2

z	x	y	s_1	s_2	s_3	a_2	a_3	rhs	BV
0	2	1	1	0	0	0	0	4	y
0	1	0	0	-1	0	1	0	4	a_2
0	-2	0	-1	0	-1	0	1	2	a_3
1	$1+M$	0	$3+M$	M	M	0	0	$12-6M$	

At this stage there is no entering variable. We have then reached the final iteration. As the artificial variables $a_2 = 4$ and $a_3 = 2$ are **not zero**, then we have **an infeasible LP and there is no solution**.

8.6.2 Unbounded solutions

If at any iteration of the simplex or Big M algorithm there is an entering variable but not a leaving variable, then the value of the objective function can be increased indefinitely which means the solution is unbounded. Let's illustrate this by the following example.

Consider the following LP model:

$$\begin{array}{ll} \text{Maximise} & Z = 4M + 2N \\ \text{subject to} & \\ & \begin{array}{rcrcl} -M & + & 2N & \leq & 6 \\ -M & + & N & \leq & 2 \end{array} \\ \text{and} & M,\ N \geq 2. \end{array}$$

The standard form of the model is

$$\begin{array}{ll} \text{Maximise} & Z = 4M + 2N \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcl} -M & + & 2N & + & S_1 & & & = & 6 \\ -M & + & N & & & + & S_2 & = & 2 \end{array} \\ \text{and} & M,\ N,\ S_1,\ S_2 \geq 2. \end{array}$$

The initial simplex tableau is

	Z	M	N	S_1	S_2	rhs	BV	θ
R_1	0	-1	2	1	0	6	S_1	-
R_2	0	-1	1	0	1	2	S_2	-
R_0	1	-4	-2	0	0	0	Z	

The entering variable is M because -4 is the most negative coefficient in R_0 but **there is no leaving variable**. We then say the **solution is unbounded**.

8.6.3 Multiple optimal solutions

If in the **final tableau of the simplex or Big M algorithm**, one of **the nonbasic variables** has a **coefficient of** 0 **in the objective row (ie row** R_0**)** then it indicates that **an alternative solution** exists. This nonbasic variable can be incorporated in the basis to obtain another optimal solution. Once two such optimal solutions are obtained, infinite number of optimal solutions can be derived by taking a weighted sum of the two optimal solutions. Mathematically, if an LP problem has $n(n > 1)$ decision variables and if

$$x^{'} = \begin{bmatrix} x_1^{'} \\ x_2^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} \text{ and } x^{''} = \begin{bmatrix} x_1^{''} \\ x_2^{''} \\ \vdots \\ x_n^{''} \end{bmatrix}$$

are two optimal solutions, then the general optimal solution is given by the formula

$$x = \alpha x^{'} + (1 - \alpha)x^{''},$$

where α is a real parameter taking values in the closed interval $[0, 1]$ and where

$$\begin{aligned} x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \alpha \begin{bmatrix} x_1^{'} \\ x_2^{'} \\ \vdots \\ x_n^{'} \end{bmatrix} + (1 - \alpha) \begin{bmatrix} x_1^{''} \\ x_2^{''} \\ \vdots \\ x_n^{''} \end{bmatrix} \\ &= \alpha \begin{bmatrix} x_1^{'} - x_1^{''} \\ x_2^{'} - x_2^{''} \\ \vdots \\ x_n^{'} - x_n^{''} \end{bmatrix} + \begin{bmatrix} x_1^{''} \\ x_2^{''} \\ \vdots \\ x_n^{''} \end{bmatrix}. \end{aligned}$$

Note: The general optimal solution

$$x = \alpha x^{'} + (1 - \alpha)x^{''}$$

is equivalent to the general optimal solution given by

$$x = \alpha x^{''} + (1 - \alpha)x^{'}.$$

Paul's Pottery Company produces two types of terracotta pots; round ones and square ones. Paul wants to maximise his profit and achieving this objective is constrained by the available labour time and raw material, clay.

The LP model for this problem is

$$\begin{array}{ll} \text{Maximise} & P = 8x + 6y \\ \text{subject to} & \\ & \begin{array}{rcrcr} x & + & 2y & \leq & 40 \\ 4x & + & 3y & \leq & 120 \end{array} \\ \text{and} & x,\ y \geq 0. \end{array}$$

Where

x: number of round pots produced per day

y: number of square pots produced per day

The standard form of this problem is

$$\begin{array}{ll} \text{Maximise} & P = 8x + 6y \\ \text{subject to} & \\ & \begin{array}{rcrcrcrcr} x & + & 2y & + & s_1 & & & = & 40 \\ 4x & + & 3y & & & + & s_2 & = & 120 \end{array} \\ \text{and} & x,\ y,\ s_1,\ s_2 \geq 0. \end{array}$$

Initial simplex tableau

	P	x	y	s_1	s_2	rhs	BV	θ	
R_1	0	1	2	1	0	40	s_1	40	
R_2	0	④	3	0	1	120	s_2	$\frac{120}{4}$	→
R_0	1	-8	-6	0	0	0	P		
		↑							

- The entering variable is x because -8 is the most negative coefficient in R_0.
- The leaving variable is s_2 because $\min\theta$ corresponds to s_2.
- The next simplex tableau is given by

Tableau 1

	P	x	y	s_1	s_2	rhs	BV	θ
R_1	0	0	$\frac{5}{4}$	1	$-\frac{1}{4}$	10	s_1	8
R_2	0	1	$\frac{3}{4}$	0	$\frac{1}{4}$	30	x	40
R_0	1	0	0	0	2	240		

At this tableau, we have reached an optimal solution that is

$$P = 240,\ x = 30 \text{ and } y = 0.$$

Since, in this final tableau, the nonbasic variable y has a coefficient of 0 in the objective row R_0, then this means that an alternative solution exists. We incorporate the nonbasic variable y in the basis.

Tableau 2

	P	x	y	s_1	s_2	rhs	BV	θ	
R_1	0	0	$\enclose{circle}{\frac{5}{4}}$	1	$-\frac{1}{4}$	10	s_1	8	→
R_2	0	1	$\frac{3}{4}$	0	$\frac{1}{4}$	30	x	40	
R_0	1	0	0	0	2	240			
			↑						

- The entering variable is y,
- min $\theta = \min\{8,\ 40\} = 8$. Then the leaving variable is s_1.
- The next optimal simplex tableau is

Tableau 2’

	P	x	y	s_1	s_2	rhs	BV
R_1	0	0	1	$\frac{4}{5}$	$-\frac{1}{5}$	8	y
R_2	0	1	0	$-\frac{3}{5}$	$\frac{2}{4}$	24	x
R_0	1	0	0	0	2	240	

Thus the second alternative optimal solution is

$$P = 240,\ x = 24 \text{ and } y = 8.$$

Thus, the two optimal solutions are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$$

with the same maximum profit $P = 240$. The general optimal solutions is then given by $P = 240$ and

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= \alpha \begin{bmatrix} 30 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 24 \\ 8 \end{bmatrix} \\ &= \alpha \begin{bmatrix} 6 \\ -8 \end{bmatrix} + \begin{bmatrix} 24 \\ 8 \end{bmatrix},\end{aligned}$$

for all $\alpha \in [0, 1]$.

Here

$$\begin{aligned} x &= 6\alpha + 24 \\ y &= -8\alpha + 8. \end{aligned}$$

Hence

$$\begin{aligned} P &= 8x + 6y \\ &= 8(6\alpha + 24) + 6(-8\alpha + 8) \\ &= 240. \end{aligned}$$

This shows that the above solutions give the same optimal profit for all $\alpha \in [0,\ 1]$ (if algebraic form has been used to solve the LP).

We conclude that, when using the simplex method or Big M method, multiple optimal solutions can be identified by a **nonbasic variable having a zero coefficient in the objective function** of the final simplex tableau.

8.6.4 Degenerate solutions

The LP model considered in Section 7.5.4 of Learning unit 7 is

$$\begin{array}{ll} \text{Maximise} & z = 3x + 9y \\ \text{subject to} & \\ & \begin{array}{rcrcrl} x & + & 4y & \leq & 8 & \quad (1) \\ x & + & 2y & \leq & 4 & \quad (2) \end{array} \\ \text{and} & x;\ y\ \geq 0. \end{array}$$

The graph representing this model is given in Figure 8.3.

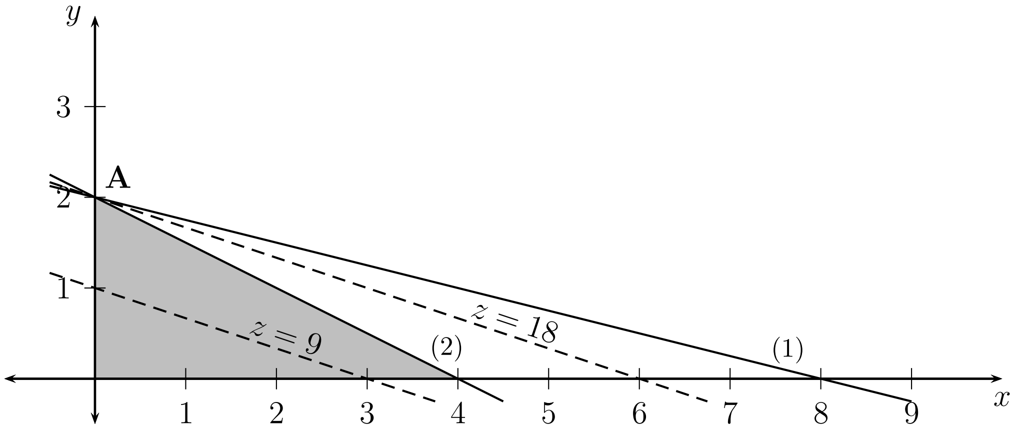

Figure 8.3: Graph – degenerate solution

We now solve the LP model by using the simplex method.

Initial simplex tableau

z	x	y	s_1	s_2	rhs	BV	θ	
0	1	$\textcircled{4}$	1	0	8	s_1	$\frac{8}{4} = 2$	$\rightarrow$
0	1	2	0	1	4	s_2	$\frac{4}{2} = 2$	
1	-3	-9	0	0	0	z		
		$\uparrow$						

From this it follows that y is the entering variable and we have a choice between s_1 and s_2 as the leaving variable. We choose the leaving variable s_1 randomly. We apply the one-zero method and obtain the following simplex tableau :

Tableau 1

z	x	y	s_1	s_2	rhs	BV	θ	
0	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2	y	8	
0	$\textcircled{\frac{1}{2}}$	0	$-\frac{1}{2}$	1	0	s_2	0	$\rightarrow$
1	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18	z		
	$\uparrow$							

From Tableau 1 above, it follows that the solution is:

Basic variables: $y = 2;\ s_2 = 0$.

Nonbasic variables: $x = 0;\ s_1 = 0$.

Objective function value: $z = 18$.

If we apply the one-zero method to the pivot column of Tableau 1, we obtain:

Tableau 2

z	x	y	s_1	s_2	rhs	BV	θ
0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2	y	-
0	1	0	-1	2	0	x	-
1	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18	z	

From Tableau 2 it follows that the solution is:

Basic variables: $x = 0; y = 2$.

Nonbasic variables: $s_1 = 0; s_2 = 0$.

Objective function value: $z = 18$.

There is no improvement in the value of the objective function from the first solution to the optimal solution. The two solutions differ in one aspect only – their basic variables are not the same. The rest of the solution is exactly the same. Therefore,

- the variables have the same values
- the value of the objective function stays the same.

This kind of solution is a *degenerate solution.*

8.7 Evaluation Exercises

Question 1

Use the simplex method to solve the following LP problems:

(a)

$$\begin{array}{ll} \text{Maximise} & Z = 5x_1 + 3x_2 + x_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & \leq & 6 \\ 5x_1 & + & 3x_2 & + & 6x_3 & \leq & 15 \end{array} \\ \text{and} & x_1;\ x_2;\ x_3 \geq 0. \end{array}$$

(b)

$$\begin{array}{ll} \text{Minimise} & C = -4x_1 + x_2 \\ \text{subject to} & \\ & \begin{array}{rcrcl} 3x_1 & + & x_2 & \leq & 6 \\ -x_1 & + & 2x_2 & \leq & 0 \end{array} \\ \text{and} & x_1;\ x_2 \geq 0. \end{array}$$

(c)

$$\begin{array}{ll} \text{Maximise} & Z = 2x_1 - x_2 + x_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcl} 3x_1 & + & x_2 & + & x_3 & \geq & 60 \\ x_1 & - & x_2 & + & 2x_3 & \geq & 10 \\ x_1 & + & x_2 & - & x_3 & \geq & 20 \end{array} \\ \text{and} & x_1;\ x_2;\ x_3 \leq 0. \end{array}$$

(d)

$$\begin{array}{ll} \text{Maximise} & Z = x_1 + 2x_2 \\ \text{subject to} & \\ & \begin{array}{rcrcl} -x_1 & + & x_2 & \leq & 2 \\ -2x_1 & + & x_2 & \leq & 1 \end{array} \\ \text{and} & x_1;\ x_2 \geq 0. \end{array}$$

Question 2

Use the Big M method to solve the following LP problems:

(a)

$$\begin{array}{ll} \text{Minimise} & C = 4x_1 + 4x_2 + x_3 \\ \text{subject to} & \\ & \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \leq & 2 \\ 2x_1 & + & x_2 & & & \leq & 3 \\ 2x_1 & + & x_2 & + & 3x_3 & \geq & 3 \end{array} \\ \text{and} & x_1;\ x_2;\ x_3 \geq 0. \end{array}$$

(b)

$$\begin{array}{ll} \text{Maximise} & Z = 3x_1 + x_2 \\ \text{subject to} & \\ & \begin{array}{rcrcl} x_1 & + & x_2 & \geq & 3 \\ -2x_1 & + & x_2 & \leq & 4 \\ x_1 & + & x_2 & = & 3 \end{array} \\ \text{and} & x_1;\ x_2 \geq 0. \end{array}$$

(c)

$$\begin{array}{ll} \text{Minimise} & Z = 3x_1 \\ \text{subject to} & \\ & \begin{array}{rcrcl} 2x_1 & + & x_2 & \geq & 6 \\ 3x_1 & + & 2x_2 & = & 4 \end{array} \\ \text{and} & x_1;\ x_2 \geq 0. \end{array}$$

Learning Unit 9

Computer solutions

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Learning objectives

After completing this study unit you should be able to

- use the computer packages LINDO and LINGO to solve LP models.
- generate simplex tableaux with LINDO.

Sections from prescribed book

4.9. The LINDO Computer Package - *p158*

4.10. Matrix Generators, LINGO, and Scaling of LPs - *p163*

9.1 Introduction

If you have worked through all the examples and exercises of the previous learning units, you will have a good understanding of the simplex method. You will also know that it takes a lot of time to solve a fairly easy problem and will probably shudder at the thought of solving a complicated problem by hand. Therefore, you will be extremely glad to hear that, from now on, you are going to use the computer to solve problems.

The computer packages, LINDO and LINGO, contained in the prescribed book *Wayne Winston: Operations Research, Applications and Algorithms* will be used to solve our problems.

9.2 Using LINDO

We want to solve LP models with LINDO and therefore we must know how to use this computer package. For instructions on the use of LINDO, please turn to *Winston*. Refer to Tutorial Letter 101 for the exact page references.

The following hints will help you when keying in a model and will ensure that the output is easy to read and interpret:

1. Each symbol must have a unique name consisting of a maximum of eight characters.
2. The objective is keyed in by typing “max” or “min”, followed directly by the equation; therefore, omitting the objective name and the “=” sign.
3. The objective line must be followed by typing in “subject to”, “s.t.” or “st”.
4. Decimals are keyed in by typing a point “.” and not a comma “,”.
5. Inequalities may be keyed in by using “<” or “>”, as LINDO thereby assumes strict inequalities $\leq$ and $\geq$.

6. LINDO assumes that all symbols are nonnegative, so we need not specify this.
7. To identify a model in the output, it may be given a name by typing in "Title", followed by a chosen name. The title must be activated by selecting the TITLE option from the FILE menu.
8. To identify the lines, other than the objective line, in the output, each line may be given a unique name consisting of a maximum of eight characters. The name must be typed at the beginning of a line followed by a round closing bracket ")" and then by the equation/inequality.
9. To clarify information in the model, comments may be included anywhere as long as they are preceded by an exclamation mark "!".
10. Lines, other than the objective line, must be entered in the specific order of symbols on the left-hand side of the sign (>, < or =) and only constants on the right-hand side.

Example 9.1

We now return to Mark's problem. The by now familiar LP model is

$$\begin{array}{lllllll} \text{Maximise} & PROFIT = 300VH + 250SB & & & & & \\ \text{subject to} & & & & & & \\ & 2VH & + & SB & \leq & 40 & \text{(Labour time)} \\ & VH & + & 3SB & \leq & 45 & \text{(Machine time)} \\ & VH & & & \leq & 12 & \text{(Marketing)} \\ \text{and} & VH;\ SB \geq 0. & & & & & \end{array}$$

Here we have

VH = number of sets of model $VH200$ to be produced daily,

SB = number of sets of model $SB150$ to be produced daily.

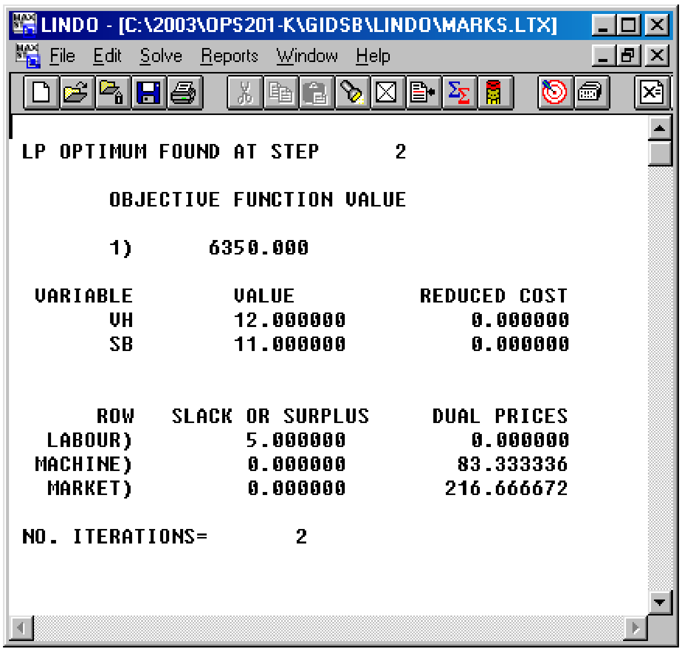
Mark's model is keyed into LINDO and is given in Figure 9.1.

You should be sitting at your computer and have the LINDO file created in Figure 9.1 on your screen. Activate the title by selecting the TITLE command from the FILE menu. Then solve the model by selecting the SOLVE command from the SOLVE menu. A dialog box appears with the question: "Do Range (Sensitivity) Analysis?" Choose NO as answer to this question and CLOSE the dialog box. The solution appears in the Reports Window which lies behind the window containing the model. To see the solution, select "2. Reports Window" from the WINDOW menu.

```
Title Mark's problem
max 300VH+250SB
subject to
Labour) 2VH+SB<40
Machine) VH+3SB<45
Market) VH<12
```

Figure 9.1: Mark's LINDO model, Example 9.1

The solution to Mark's model is given in Figure 9.2.



```
 LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      6350.000

  VARIABLE        VALUE          REDUCED COST
        VH        12.000000          0.000000
        SB        11.000000          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
   LABOUR)         5.000000          0.000000
  MACHINE)         0.000000         83.333336
   MARKET)         0.000000        216.666672

 NO. ITERATIONS=       2
```

Figure 9.2: Mark's LINDO solution report, Example 9.1

In the following section we study this solution in detail.

9.3 Computer solution

9.3.1 The objective function

The *optimal value of the objective function* is given in the first part of the output:

```
LP OPTIMUM FOUND AT STEP      2
```

```
        OBJECTIVE FUNCTION VALUE

        1)      6350.000
```

In Mark's case, the maximum profit that can be made is R6 350.

9.3.2 The variables

The *variables and their values* are given in the second part of the output:

```
  VARIABLE        VALUE          REDUCED COST
        VH        12.000000          0.000000
        SB        11.000000          0.000000
```

The decision variables are given in the first column and their optimal value in the second column. The third column "Reduced Cost" will not be discussed as it is outside the scope of this module.

The optimal product mix for Mark's problem is to produce 12 model $VH200$ and 11 model $SB150$ television sets daily.

9.3.3 The constraints

The *constraints and the value of their slack or surplus* are given in the third part of the output:

```
       ROW   SLACK OR SURPLUS     DUAL PRICES
   LABOUR)         5.000000          0.000000
  MACHINE)         0.000000         83.333336
   MARKET)         0.000000        216.666672

 NO. ITERATIONS=       2
```

The constraints are given in the first column of the output. It is always advisable to give names to the constraints, as these constraint names will then be printed here. If you do not give names to the constraints, row numbers instead of constraint names will be printed in this first column. This makes it rather difficult to interpret the output.

The second column gives the slack or surplus on constraints. Slacks are associated with $\leq$ constraints and surpluses with $\geq$ constraints.

The third column "Dual Prices" will not be discussed as it is outside the scope of this module.

In Mark's case, the labour constraint is a ≤ constraint and therefore the 5 in the second column is a slack. This means that five labour hours are not utilised each day. The labour constraint does not use up all the available resource and is a *nonbinding constraint.*

The machine constraint has a slack of zero indicating that no machine hours are left over and all the hours are utilised. The market constraint also has a slack of zero indicating that the maximum number of sets that can be sold daily are produced. The latter two constraints with no slack are *binding constraints.*

Binding constraints are called "binding" as they "bind" the solution to a problem, that is, they determine the optimal values of the decision variables and the objective function. In Mark's case, they prevent the objective function value, profit, from increasing. The factors that determine the machine and market constraints must be examined if a larger profit is required.

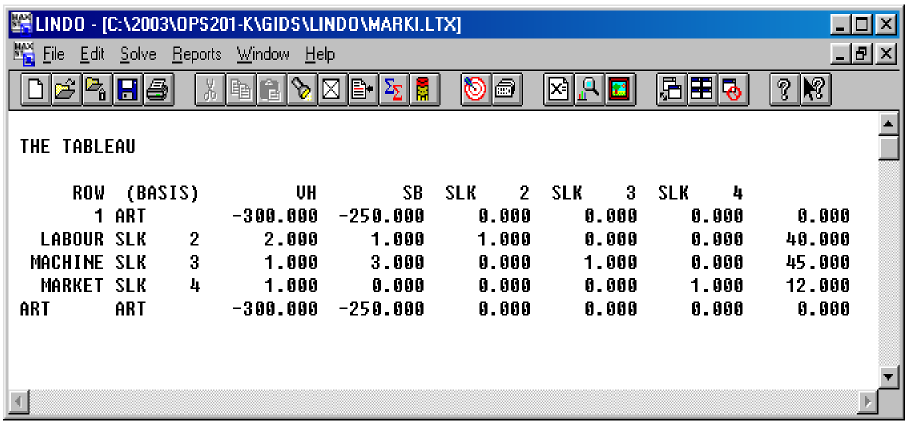
9.4 Simplex tableaux

The computer solution (see Figure 9.2) is obtained by means of the simplex method. If we want to study the simplex tableaux, we will have to generate them with LINDO.

To obtain the initial simplex tableaux, proceed as follows:

Close all the LINDO windows so that you have a blank window on your computer screen. Now open the file containing Mark's model. (Hopefully you saved this model, otherwise you will have to key it in again!) Make sure that the active window on your computer screen is now identical to Figure 9.1.

Now activate the REPORTS menu and select TABLEAU from it. The initial simplex tableau appears in the Reports Window and is given in Figure 9.3.



```
THE TABLEAU

     ROW  (BASIS)         VH         SB   SLK     2   SLK     3   SLK     4
       1 ART        -300.000   -250.000      0.000      0.000      0.000      0.000
  LABOUR SLK     2     2.000      1.000      1.000      0.000      0.000     40.000
 MACHINE SLK     3     1.000      3.000      0.000      1.000      0.000     45.000
  MARKET SLK     4     1.000      0.000      0.000      0.000      1.000     12.000
ART      ART        -300.000   -250.000      0.000      0.000      0.000      0.000
```

Figure 9.3: Mark's initial simplex tableau, Example 9.1

The objective function coefficients are given in the row numbered 1 of this table, and the value of the objective function is the last entry in this row. In this case, it is zero. Exactly the same information is given in the last row.

The first constraint is given in row 2 and therefore uses $SLK2$ as its slack variable. This corresponds to the slack variable $S1$ which we used previously as the slack variable for the first constraint.

Constraints 2 and 3 are given in rows 3 and 4 of the table and the accompanying slack variables are $SLK3$ and $SLK4$ respectively. These variables are equivalent to $S2$ and $S3$ in our previous notation.

The basic variables are given in the second column "(Basis)" of the table and initially the basic variables are $SLK2$, $SLK3$ and $SLK4$. (In our previous notation $S1$, $S2$ and $S3$.)

The optimal values of the basic variables are given in the last column. In this case

$$SLK2 = 40;\ SLK3 = 45;\ SLK4 = 12.$$

The variables that do not appear in the "(Basis)" column are nonbasic variables and they take on zero values. In this case

$$VH = 0;\ SB = 0.$$

Now we want to obtain the final simplex tableau and we proceed as follows:

Close the Reports Window containing the initial simplex tableau. Make sure that Mark's model (Figure 9.1) is in the active window. Solve this model as before. The solution is given in the Reports Window. Go back to Mark's model by calling it up on the active window. Then select TABLEAU from the REPORTS menu and the final simplex tableau is printed after the solution in the Reports Window, and is given in Figure 9.4.

From the final simplex tableau in Figure 9.4, we read off the basic variables and their values as

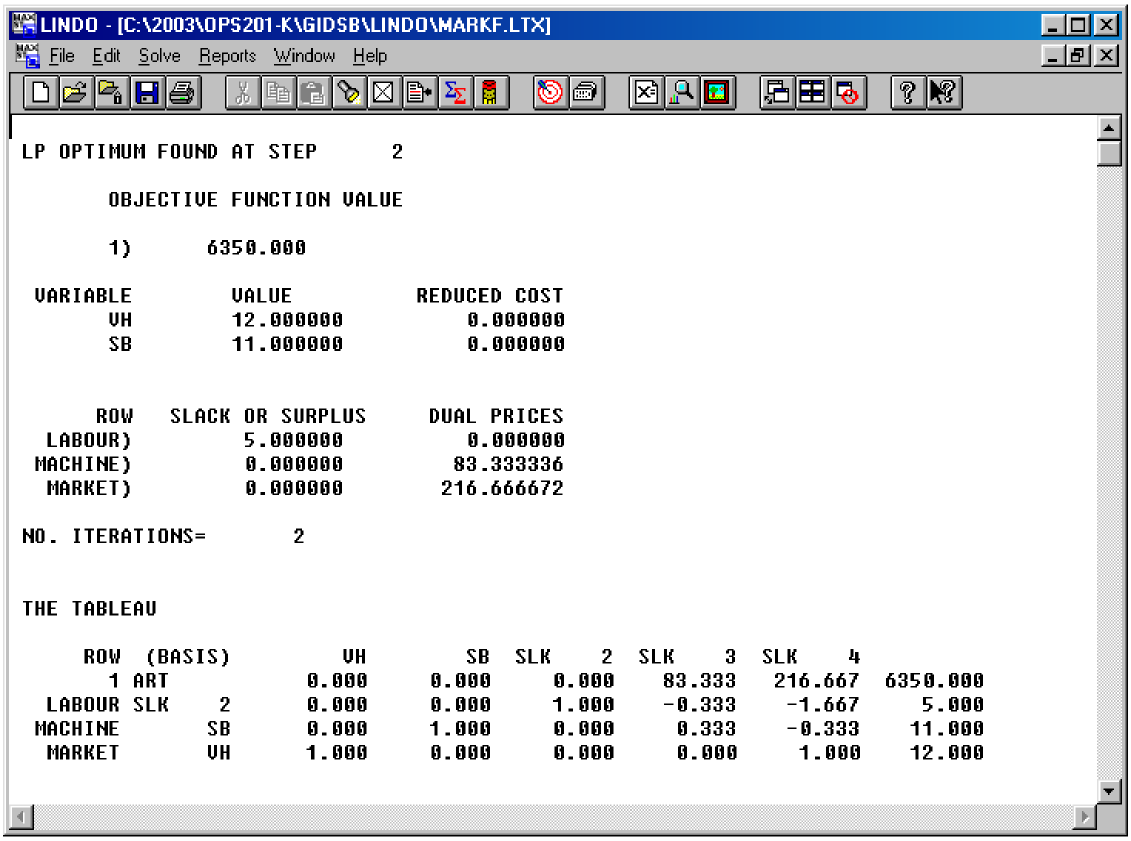
$$SLK2 = S1 = 5;\ SB = 11;\ VH = 12.$$

The value of the objective function is the first entry in the last column of the table. In this case, it is 6 350.

The nonbasic variables are the ones that are not listed in the "(Basis)" column and are therefore

$$SLK3 = S2 = 0;\ SLK4 = S3 = 0.$$

The *status of variables*, that is, whether they are basic or nonbasic variables, can therefore be read from a simplex tableau. Note that if we need to know the status of variables, we cannot read it from the computer solution (as given in Figure 9.2), but will have to generate the necessary simplex tableau.



```
LP OPTIMUM FOUND AT STEP      2

       OBJECTIVE FUNCTION VALUE

       1)      6350.000

 VARIABLE        VALUE          REDUCED COST
       VH        12.000000          0.000000
       SB        11.000000          0.000000


      ROW   SLACK OR SURPLUS     DUAL PRICES
  LABOUR)         5.000000          0.000000
 MACHINE)         0.000000         83.333336
  MARKET)         0.000000        216.666672

NO. ITERATIONS=       2


THE TABLEAU

     ROW  (BASIS)         VH         SB  SLK     2  SLK     3  SLK     4
       1 ART           0.000      0.000      0.000    83.333   216.667  6350.000
  LABOUR SLK    2      0.000      0.000      1.000    -0.333    -1.667     5.000
 MACHINE       SB      0.000      1.000      0.000     0.333    -0.333    11.000
  MARKET       VH      1.000      0.000      0.000     0.000     1.000    12.000
```

Figure 9.4: Mark's solution report and final simplex tableau, Example 9.1

9.5 Using LINGO

Now we want to learn how to use the computer package LINGO. Turn to *Winston* for instructions on the use of LINGO. Refer to Tutorial Letter 101 for the exact page references.

The following hints will help you when keying a model into LINGO:

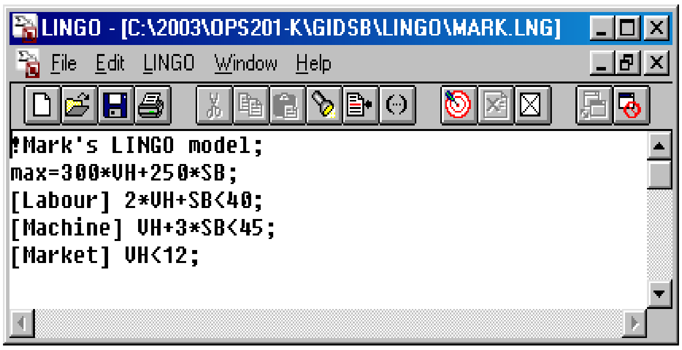
1. The objective function is keyed in by typing "max" or "min", followed by the "=" sign and then the equation.
2. LINGO constraints do not have to be preceded by "subject to", "s.t." or "st".
3. Mathematical operators must be added and an asterisk " * " is used for multiplication.
4. All LINGO statements must end with a semicolon " ; ".
5. A constraint name may be given at the beginning of the line containing the constraint and the name must be enclosed in square brackets "[ ]".
6. Variable and constraint names may consist of a maximum of 32 characters.

7. A title cannot be given, but comments may be included anywhere in the model provided they are preceded by an exclamation mark “ ! ” and end with a semicolon “ ; ”.

Example 9.2

Solving Mark’s problem with LINGO

You must be sitting at your computer with the LINGO computer package open. Key Mark’s model into the LINGO “untitled” window and it should appear as shown in Figure 9.5.



```
!Mark's LINGO model;
max=300*VH+250*SB;
[Labour] 2*VH+SB<40;
[Machine] VH+3*SB<45;
[Market] VH<12;
```

Figure 9.5: Mark’s LINGO codes, Example 9.2

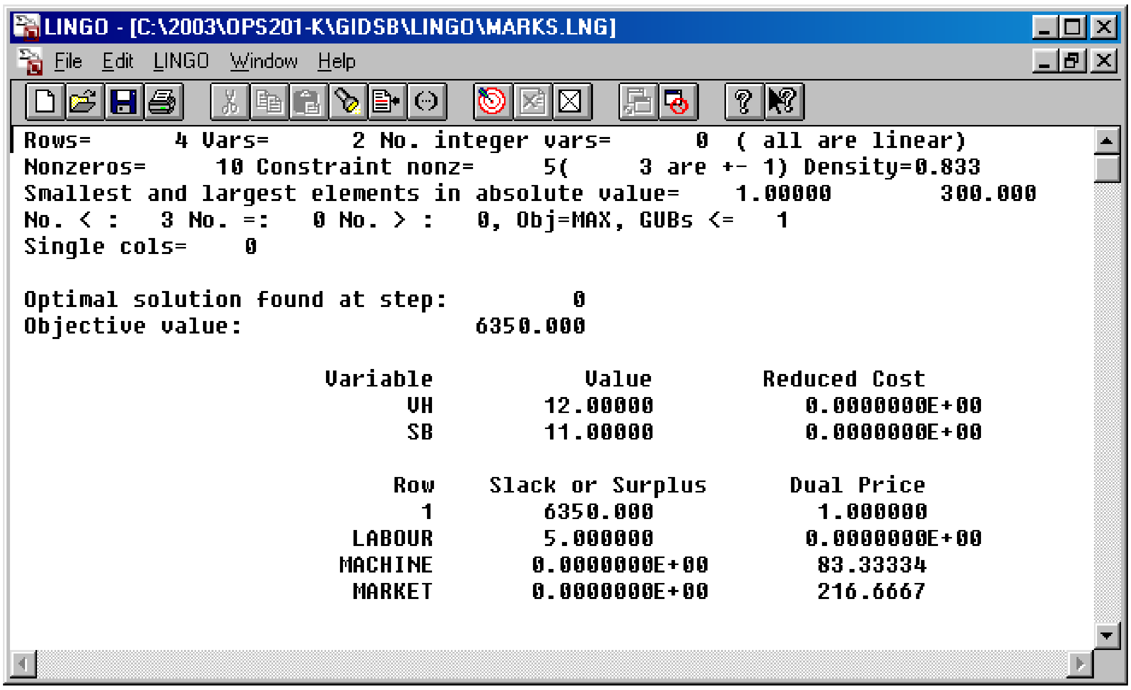
To solve the LINGO model, select the SOLVE command from the LINGO menu. A dialog box appears which should be closed. The solution then appears in the Reports Window, which lies behind the window containing the model. Select the option “2. Reports Window” from the WINDOW menu to see the solution. It should appear as given in Figure 9.6.

Exactly the same optimal solution results as before.

The columns containing “Reduced Cost” and “Dual Price” in the LINGO output are ignored as this is outside the scope of this module.

Example 9.3

Solve the following LP model with LINDO and LINGO:



```
Rows=       4 Vars=       2 No. integer vars=       0  ( all are linear)
Nonzeros=     10 Constraint nonz=     5(      3 are +- 1) Density=0.833
Smallest and largest elements in absolute value=    1.00000        300.000
No. < :   3 No. =:   0 No. > :   0, Obj=MAX, GUBs <=    1
Single cols=    0

Optimal solution found at step:         0
Objective value:                 6350.000

                      Variable           Value        Reduced Cost
                            VH        12.00000        0.0000000E+00
                            SB        11.00000        0.0000000E+00

                           Row    Slack or Surplus      Dual Price
                             1        6350.000          1.000000
                        LABOUR        5.000000        0.0000000E+00
                       MACHINE     0.0000000E+00        83.33334
                        MARKET     0.0000000E+00        216.6667
```

Figure 9.6: Mark's LINGO solution report, Example 9.2

$$\begin{array}{ll}
\text{Minimise} & z = x_1 - x_2 - 3x_3 - 5x_4 \\
\text{subject to} & \\
& \begin{array}{rcrcrcrcr}
3x_1 & - & 4x_2 & + & x_3 & - & x_4 & \leq & 2 \\
 & & 5x_2 & - & 5x_3 & - & 2x_4 & \leq & 8 \\
x_1 & - & x_2 & + & 2x_3 & + & x_4 & \leq & 7
\end{array} \\
\text{and} & x_1;\ x_2;\ x_3;\ x_4 \geq 0.
\end{array}$$

Solution

The LINDO model and its solution are given in Figure 9.7 and Figure 9.8 respectively.

The optimal solution following from LINDO is

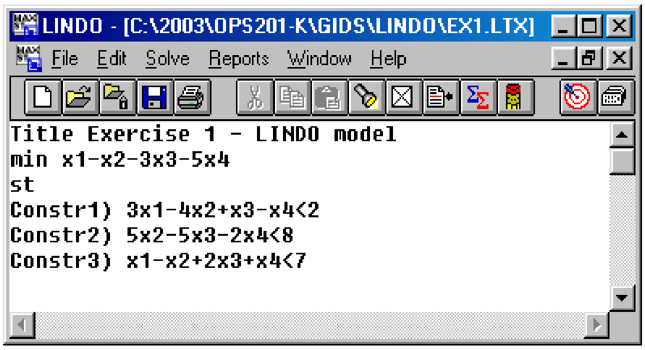
$$x_1 = 0;\ x_2 = 7,333;\ x_3 = 0;\ x_4 = 14,333;\ z = -79.$$

The first constraint has a slack of 45,667 and is a nonbinding constraint.

The second and third constraints have no slack and are therefore binding constraints.

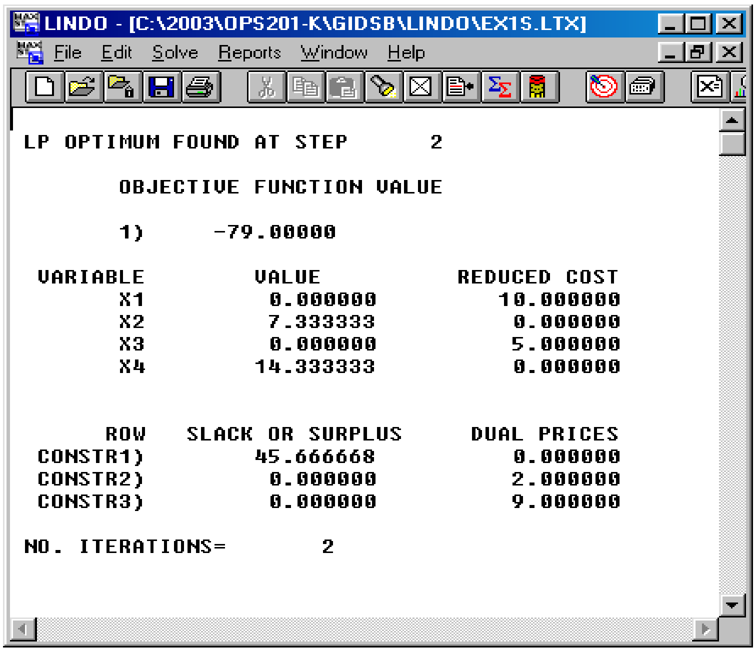
The LINGO model and its solution are given in Figure 9.9 and Figure 9.10 respectively.

The optimal solution that results from LINGO is exactly the same as the solution that results from LINDO.



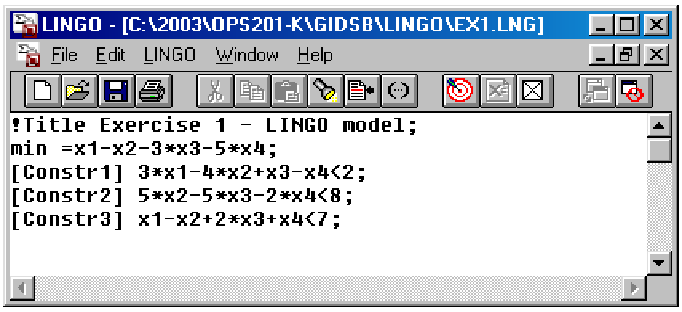
```
Title Exercise 1 - LINDO model
min x1-x2-3x3-5x4
st
Constr1) 3x1-4x2+x3-x4<2
Constr2) 5x2-5x3-2x4<8
Constr3) x1-x2+2x3+x4<7
```

Figure 9.7: LINDO codes, Example 9.3



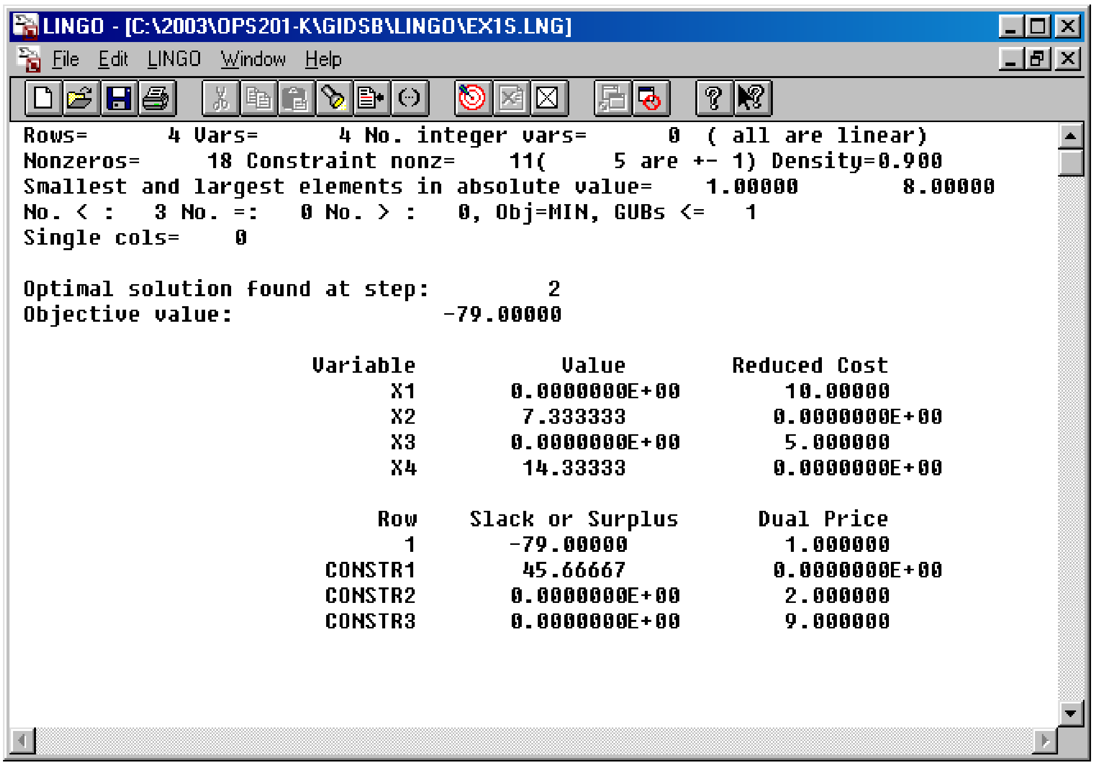
```
 LP OPTIMUM FOUND AT STEP      2

        OBJECTIVE FUNCTION VALUE

        1)      -79.00000

  VARIABLE        VALUE          REDUCED COST
        X1         0.000000         10.000000
        X2         7.333333          0.000000
        X3         0.000000          5.000000
        X4        14.333333          0.000000


       ROW   SLACK OR SURPLUS     DUAL PRICES
  CONSTR1)        45.666668          0.000000
  CONSTR2)         0.000000          2.000000
  CONSTR3)         0.000000          9.000000

 NO. ITERATIONS=       2
```

Figure 9.8: LINDO solution report, Example 9.3



```
!Title Exercise 1 - LINGO model;
min =x1-x2-3*x3-5*x4;
[Constr1] 3*x1-4*x2+x3-x4<2;
[Constr2] 5*x2-5*x3-2*x4<8;
[Constr3] x1-x2+2*x3+x4<7;
```

Figure 9.9: LINGO codes, Example 9.3



```
Rows=       4 Vars=       4 No. integer vars=       0  ( all are linear)
Nonzeros=      18 Constraint nonz=     11(      5 are +- 1) Density=0.900
Smallest and largest elements in absolute value=     1.00000          8.00000
No. < :    3 No. =:    0 No. > :    0, Obj=MIN, GUBs <=    1
Single cols=     0

Optimal solution found at step:         2
Objective value:              -79.00000

                     Variable           Value        Reduced Cost
                           X1    0.0000000E+00        10.00000
                           X2        7.333333    0.0000000E+00
                           X3    0.0000000E+00        5.000000
                           X4        14.33333    0.0000000E+00

                          Row    Slack or Surplus      Dual Price
                            1       -79.00000        1.000000
                      CONSTR1        45.66667    0.0000000E+00
                      CONSTR2    0.0000000E+00        2.000000
                      CONSTR3    0.0000000E+00        9.000000
```

Figure 9.10: LINGO solution report, Example 9.3

Example 9.4

Use LINDO to generate the initial and final simplex tableaux for the LP model given in Example 9.3. For each tableau, indicate the basic and nonbasic variables and their values.

Solution

The initial and final simplex tableaux generated by LINDO are given in Figure 9.11.

From the initial simplex tableau we can read off the following:

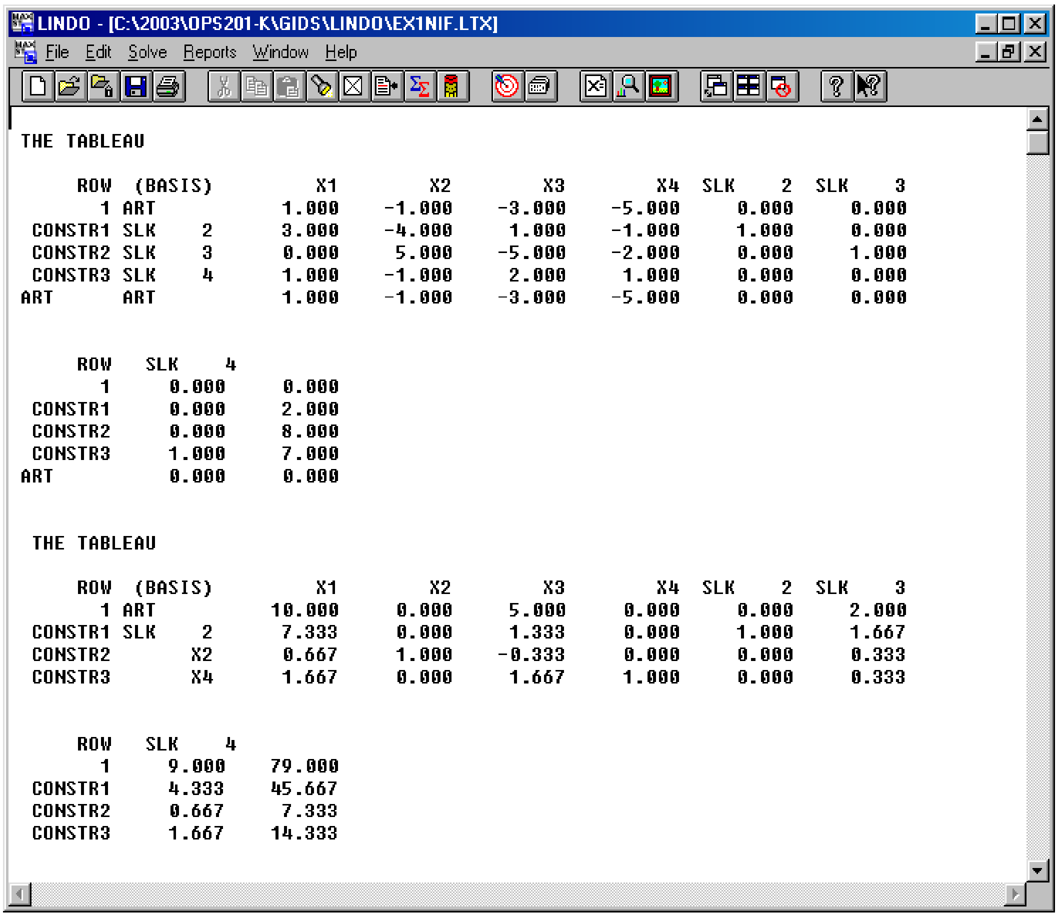
Basic variables:

$$\begin{aligned} SLK2 &= s_1 &= 2 \\ SLK3 &= s_2 &= 8 \\ SLK4 &= s_3 &= 7. \end{aligned}$$

Since the variables x_1, x_2, x_3 and x_4 do not appear in the "(Basis)" column, we deduce that they are nonbasic variables and they assume zero values.

Nonbasic variables:

$$x_1 = x_2 = x_3 = x_4 = 0.$$



```
THE TABLEAU

     ROW  (BASIS)          X1          X2          X3          X4  SLK     2  SLK     3
       1 ART            1.000      -1.000      -3.000      -5.000       0.000       0.000
 CONSTR1 SLK      2     3.000      -4.000       1.000      -1.000       1.000       0.000
 CONSTR2 SLK      3     0.000       5.000      -5.000      -2.000       0.000       1.000
 CONSTR3 SLK      4     1.000      -1.000       2.000       1.000       0.000       0.000
ART      ART            1.000      -1.000      -3.000      -5.000       0.000       0.000


     ROW   SLK     4
       1      0.000       0.000
 CONSTR1      0.000       2.000
 CONSTR2      0.000       8.000
 CONSTR3      1.000       7.000
ART           0.000       0.000


 THE TABLEAU

     ROW  (BASIS)          X1          X2          X3          X4  SLK     2  SLK     3
       1 ART           10.000       0.000       5.000       0.000       0.000       2.000
 CONSTR1 SLK      2     7.333       0.000       1.333       0.000       1.000       1.667
 CONSTR2         X2     0.667       1.000      -0.333       0.000       0.000       0.333
 CONSTR3         X4     1.667       0.000       1.667       1.000       0.000       0.333


     ROW   SLK     4
       1      9.000      79.000
 CONSTR1      4.333      45.667
 CONSTR2      0.667       7.333
 CONSTR3      1.667      14.333
```

Figure 9.11: Initial and final simplex tableaux, Example 9.4

From the final simplex tableau we can read off the following:

Basic variables:

$$\begin{aligned} SLK2 &= s_1 = 45,667 \\ x_2 &= 7,333 \\ x_4 &= 14,333. \end{aligned}$$

Since the other variables do not appear in the "(Basis)" column, we deduce the following:

Nonbasic variables:

$$\begin{aligned} x_1 &= x_3 = 0 \\ SLK3 &= s_2 = 0 \\ SLK4 &= s_3 = 0. \end{aligned}$$

Note that the simplex method is applied by LINDO to maximisation problems. The given minimisation model was therefore transformed to a maximisation model by the transformation $Z = -z$. The optimal value of the objective function that is read from the final simplex tableau is $Z = 79$. The optimal value for the minimisation problem is then $z = -Z = -79$.

9.6 Evaluation exercises

Question 1 – *Winston* 3.1, Example 1, pp.49-52 (same reference in *Old Winston*)

Use **LINDO** to solve Giapetto's problem.

Question 2 – *Winston* 4.5, Problem 3, p.149 (*Old Winston* 4.3, Problem 3, p.143)

Consider the given LP model and do the following:

1. Solve the model using **LINDO**.
2. Solve the model using **LINGO**.

Question 3 – *Winston* 3.1, Problem 4, p.55 (*Old Winston* 3.1, Problem 4, p.56)

1. Solve this model using **LINDO**.
2. Solve this model using **LINGO**.

Question 4 – *Winston* 3.8, Problem 3, p.92 (*Old Winston* 3.8, Problem 3, p.91)

1. Solve this model using **LINDO**.
2. Solve this model using **LINGO**.

Bibliography

[Hillier and Lieberman(2015)] Frederick S Hillier and Gerald J Lieberman. *Introduction to Operations Research*. McGraw-Hill Education, 2015.

[Kolman and Hill(2005)] Bernard Kolman and David R Hill. *INTRODUCTORY LINEAR ALGEBRA. AN APPLIED FIRST COURSE*. Pearson Prentice Hall, 2005.

[Winston and Goldberg(2004)] Wayne L Winston and Jeffrey B Goldberg. *Operations research: applications and algorithms*, volume 3. Duxbury press Belmont, CA, 2004.