

Tutorial Letter 202/1/2018

Linear Mathematical Programming

DSC2605

Semester 1

Department of Decision Sciences

This tutorial letter contains solutions to Assignment 2

Bar code

Question 1

Discussion on myUnisa. No solution is provided.

[7]

Question 2

(a) Decision variables

Let

- X_A be the number of model A carriages,
- X_B be the number of model B carriages,
- X_{C_c} be the number of model C carriages used for coal,
- X_{C_l} be the number of model B carriages used for lumber.

(b) LP model

- Objective function: Minimise $z = 6X_A + 4X_B + 8(X_{C_c} + X_{C_l})$
- Functional constraints:
 1. Fleet capacity of coal: $25X_A + 18X_{C_c} \geq 3400$
 2. Fleet capacity of lumber: $8X_B + 10X_{C_l} \geq 1800$ or $4X_B + 5X_{C_l} \geq 900$.
 3. At least a quarter of the carriages must be able to transport coal, that means

$$X_A + X_{C_c} \geq \frac{1}{4}(X_A + X_B + X_{C_c} + X_{C_l}),$$

or equivalently

$$3X_A - X_B + 3X_{C_c} - X_{C_l} \geq 0.$$

The LP model is then given by:

$$\text{Minimize } z = 6X_A + 4X_B + 8X_{C_c} + 8X_{C_l}$$

Subject to

$$25X_A + 18X_{C_c} \geq 3400$$

$$4X_B + 5X_{C_l} \geq 900$$

$$3X_A - X_B + 3X_{C_c} - X_{C_l} \geq 0.$$

and

$$X_A, X_B, X_{C_c}, X_{C_l} \geq 0.$$

(c) Lingo codes and solution report

MODEL:

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! Assignment 2, Question 2c);
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[OBJECTIVE_FUNCTION] MIN = 6*Xa + 4*Xb + 8*Xcc + 8*Xcl;
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[COAL] 25*Xa + 18*Xcc >= 3400;
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[LUMBER] 4*Xb + 5*Xcl >= 900;
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[COAL_TRANSPORT] 3*Xa - Xb + 3*Xcc - Xcl >= 0;
```

Solution report

Global optimal solution found.

Objective value: 1716.000
 Infeasibilities: 0.000000
 Total solver iterations: 2
 Elapsed runtime seconds: 0.05

Model Class: LP

Total variables: 4
 Nonlinear variables: 0
 Integer variables: 0

Total constraints: 4
 Nonlinear constraints: 0

Total nonzeros: 12
 Nonlinear nonzeros: 0

Variable	Value	Reduced Cost
XA	136.0000	0.000000
XB	225.0000	0.000000
XCC	0.000000	3.680000
XCL	0.000000	3.000000

Row	Slack or Surplus	Dual Price
OBJECTIVE_FUNCTION	1716.000	-1.000000
COAL	0.000000	-0.240000
LUMBER	0.000000	-1.000000
COAL_TRANSPORT	183.0000	0.000000

(d) Optimal solution:

$$z = 1716 \text{ and,}$$

$$x_A = 136, \quad x_B = 225, \quad x_{C_c} = x_{C_l} = 0.$$

Question 3

(a) **LP model**

– Objective function

$$\text{Revenue} = 40(X_{11} + X_{21} + X_{31}) + 50(X_{12} + X_{22} + X_{32})$$

$$\text{Cost} = 6(X_{11} + X_{12}) + 8(X_{21} + X_{22}) + 10(X_{31} + X_{32})$$

It follows that

$$\begin{aligned}\text{Profit} &= \text{Revenue} - \text{Cost} \\ &= 34X_{11} + 44X_{12} + 32X_{21} + 42X_{22} + 30X_{31} + 40X_{32}\end{aligned}$$

– Functional constraints

(1) Availability of Oil 1, 2 and 3.

$$X_{11} + X_{12} \leq 8000 \quad (\text{Availability of Oil 1})$$

$$X_{21} + X_{22} \leq 12000 \quad (\text{Availability of Oil 2})$$

$$X_{31} + X_{32} \leq 16000 \quad (\text{Availability of Oil 3})$$

(2) Constraints on demands of blends.

$$X_{11} + X_{21} + X_{31} \geq 12000 \quad (\text{Demand of Blend 1})$$

$$X_{12} + X_{22} + X_{32} \geq 12000 \quad (\text{Demand of Blend 2})$$

(3) Rate of content of each oil for Blend 1

$$\frac{X_{11}}{X_{11} + X_{21} + X_{31}} \leq 0.4 \quad \text{or} \quad 3X_{11} - 2X_{21} - 2X_{31} \leq 0.$$

$$\frac{X_{21}}{X_{11} + X_{21} + X_{31}} \geq 0.3 \quad \text{or} \quad 3X_{11} - 7X_{21} + 3X_{31} \leq 0.$$

$$\frac{X_{31}}{X_{11} + X_{21} + X_{31}} \leq 0.3 \quad \text{or} \quad 3X_{11} + 3X_{21} - 7X_{31} \geq 0.$$

(4) Rate of content of each oil for Blend 2

$$\frac{X_{12}}{X_{12} + X_{22} + X_{32}} \geq 0.4 \quad \text{or} \quad 3X_{12} - 2X_{22} - 2X_{32} \geq 0.$$

$$\frac{X_{22}}{X_{12} + X_{22} + X_{32}} \leq 0.5 \quad \text{or} \quad X_{12} - X_{22} + X_{32} \geq 0.$$

$$\frac{X_{32}}{X_{12} + X_{22} + X_{32}} \geq 0.3 \quad \text{or} \quad 3X_{12} + 3X_{22} - 7X_{32} \leq 0.$$

– LP model

Maximise $z = 34X_{11} + 44X_{12} + 32X_{21} + 42X_{22} + 30X_{31} + 40X_{32}$
 subject to

$$X_{11} + X_{12} \leq 8000$$

$$X_{21} + X_{22} \leq 12000$$

$$X_{31} + X_{32} \leq 16000$$

$$X_{11} + X_{21} + X_{31} \geq 12000$$

$$X_{12} + X_{22} + X_{32} \geq 12000$$

$$3X_{11} - 2X_{21} - 2X_{31} \leq 0$$

$$3X_{11} - 7X_{21} + 3X_{31} \leq 0$$

$$3X_{11} + 3X_{21} - 7X_{31} \geq 0$$

$$3X_{12} - 2X_{22} - 2X_{32} \geq 0$$

$$X_{12} - X_{22} + X_{32} \geq 0$$

$$3X_{12} + 3X_{22} - 7X_{32} \leq 0$$

and $X_{ij} \geq 0$, ($i = 1, 2, 3$ and $j = 1, 2$).

(b) **LINGO codes and solution report**

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MODEL:

! Assignment 2, Question 3b);

[OBJECTIVE_FUNCTION] MAX= 34*X11 + 44*X12 + 32*X21 + 42*X22 + 30*X31 + 40*X32;

! Functional Constraints;

[AVAILABILITY_OIL_1] X11 + X12 <= 8000;
[AVAILABILITY_OIL_2] X21 + X22 <= 12000;
[AVAILABILITY_OIL_3] X31 + X32 <= 16000;

[DEMAND_BLEND_1] X11 + X21 + X31 >= 12000;
[DEMAND_BLEND_2] X12 + X22 + X32 >= 12000;

[CONTENT_OF_OIL_1_FOR_BLEND_1] 3*X11 - 2*X21 - 2*X31 <= 0;
[CONTENT_OF_OIL_2_FOR_BLEND_1] 3*X11 - 7*X21 + 3*X31 <= 0;
[CONTENT_OF_OIL_3_FOR_BLEND_1] 3*X11 + 3*X21 - 7*X31 >= 0;

[CONTENT_OF_OIL_1_FOR_BLEND_2] 3*X12 - 2*X22 - 2*X32 >= 0;
[CONTENT_OF_OIL_2_FOR_BLEND_2] X12 - X22 + X32 >= 0;
[CONTENT_OF_OIL_3_FOR_BLEND_2] 3*X12 + 3*X22 - 7*X32 >= 0;

```

Solution report

Global optimal solution found.

Objective value: 1078857.
 Infeasibilities: 0.000000
 Total solver iterations: 5
 Elapsed runtime seconds: 0.06

Model Class: LP

Total variables: 6
 Nonlinear variables: 0
 Integer variables: 0

Total constraints: 12
 Nonlinear constraints: 0

Total nonzeros: 36
 Nonlinear nonzeros: 0

Variable	Value	Reduced Cost
X11	1371.429	0.000000
X12	6628.571	0.000000
X21	7028.571	0.000000
X22	4971.429	0.000000
X31	3600.000	0.000000
X32	4971.429	0.000000

Row	Slack or Surplus	Dual Price
OBJECTIVE_FUNCTION	1078857.	1.000000
AVAILABILITY_OIL_1	0.000000	61.14286
AVAILABILITY_OIL_2	0.000000	59.14286
AVAILABILITY_OIL_3	7428.571	0.000000
DEMAND_BLEND_1	0.000000	-10.00000
DEMAND_BLEND_2	4571.429	0.000000
CONTENT_OF_OIL_1_FOR_BLEND_1	17142.86	0.000000
CONTENT_OF_OIL_2_FOR_BLEND_1	34285.71	0.000000
CONTENT_OF_OIL_3_FOR_BLEND_1	0.000000	-5.714286
CONTENT_OF_OIL_1_FOR_BLEND_2	0.000000	0.000000
CONTENT_OF_OIL_2_FOR_BLEND_2	6628.571	0.000000
CONTENT_OF_OIL_3_FOR_BLEND_2	0.000000	-5.714286

(c) Optimal solution

$$z = 1\,078\,857,$$

$$X_{11} = 1371.429 \quad X_{12} = 6628.571,$$

$$X_{21} = 7028.571 \quad X_{22} = 4971.429,$$

$$X_{31} = 3600.000 \quad X_{32} = 4971.429.$$

Question 4

Minimise $z = x + 2y$

Subject to

$$-2x + 3y \leq 14 \quad (C_1)$$

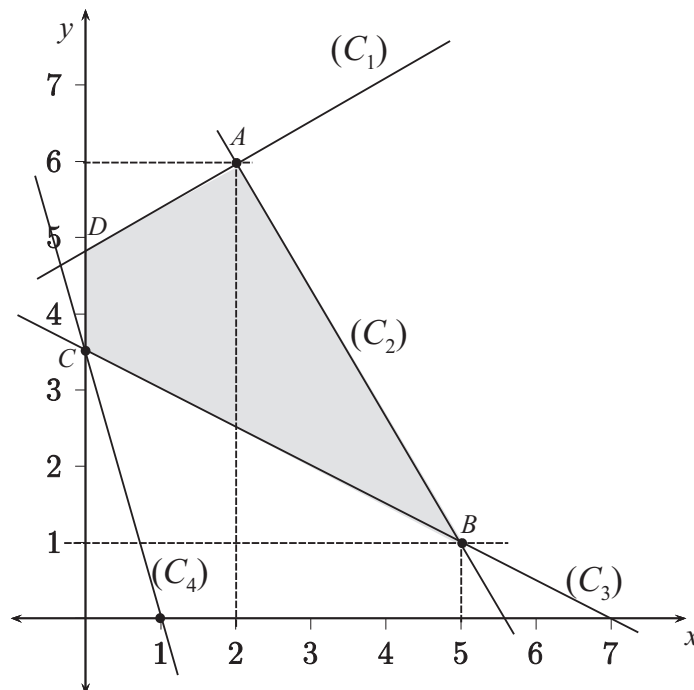
$$5x + 3y \leq 28 \quad (C_2)$$

$$x + 2y \geq 7 \quad (C_3)$$

$$7x + 2y \geq 7 \quad (C_4)$$

and $x, y \geq 0$.

(a) Graphical representation of the LP model



(b) Corner points and values of objective function.

Corner-points	Value of $z = x + 2y$
$A = (2, 6)$	$2 + 2(6) = 14$
$B = (5, 1)$	$5 + 2(1) = 7$
$C = (0, 7/2)$	$0 + 2(7/2) = 7$
$D = (0, 14/3)$	$0 + 2(14/3) = 28/3$

(c) The minimum value of the objective function $z = x + 2y$ occurs at corner-points $B = (5, 1)$ and $C = (0, 7/2)$. Therefore, the LP model has multiple optimal solutions given by

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \alpha \begin{bmatrix} 5 \\ 1 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 0 \\ 7/2 \end{bmatrix}, \\ &= \frac{\alpha}{2} \begin{bmatrix} 10 \\ -5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \quad \alpha \in [0, 1]. \end{aligned}$$

- (d) – Redundant constraint: $7x + 2y \geq 7$, $x \geq 0$.
 – Binding constraints: $5x + 3y \leq 28$, $x + 2y \geq 7$ and $y \geq 0$.
 – Nonbinding constraint: $-2x + 3y \leq 14$.

- (e) The general equation of isocost lines are given by all lines parallel the straight line \overline{AC} . That is the equation given by

$$x + 2y = b, \quad b \in \mathbb{R}$$

Question 5

$$\text{Maximise } z = x_1 + 2x_2 + x_3$$

Subject to

$$x_1 + 2x_2 + x_3 = 4$$

$$3x_1 + x_2 - x_3 = 6$$

and $x, x_2, x_3 \geq 0$.

The LP model in augmented form is given by

$$\text{Maximise } z = x_1 + 2x_2 + x_3 - Ma_1 - Ma_2$$

Subject to

$$x_1 + 2x_2 + x_3 + a_1 = 4$$

$$3x_1 + x_2 - x_3 + a_2 = 6$$

and $x, x_2, x_3, a_1, a_2 \geq 0$.

Initial simplex tableau

	z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
R_1	0	1	2	1	1	0	4	a_1	4/1
R_2	0	3	1	-1	0	1	6	a_2	6/3 = 2
R_0	1	-1	-2	-1	M	M	0	z	
R'_0	1	$-1 - 4M$	$-2 - 3M$	-1	0	0	$-10M$	z	

Where R'_0 is the new objective row obtained by performing the following elementary row operation:

$$R'_0 = R_0 - (R_1 + R_2)M.$$

- The entering variable is x_1 because is the most negative number.
- $\min \theta = \min\{4, 2\} = 2$. It follows that the leaving variable is a_2 .
- The pivot is found in x_1 's column and is 3. Therefore, we apply the one-zero method to x_1 's column by making one the pivot and all other elements in the pivot column to zeroes. We then obtain the next simplex tableau given by

Simplex tableau 1

z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
0	0	5/3	4/3	1	-1/3	2	a_1	6/5
0	1	1/3	-1/3	0	1/3	2	x_1	6/1
1	0	$-5/3(1+M)$	$-4/3(1+M)$	0	$\frac{1}{3}$	$2-2M$	z	

- The entering variable is x_2 .
- $\theta = \min\{6/5, 6\} = 6/5$. The leaving variable is a_1 .
- We find the pivot in the x_2 's column and is given by 5/3. We apply the one-zero method and obtain the next simple tableau:

Simplex tableau 2

z	x_1	x_2	x_3	a_1	a_2	rhs	BV	θ
0	0	1	4/5	3/5	-1/5	6/5	x_2	3/2
0	1	0	-3/5	-1/5	3/5	8/5	x_1	–
1	0	0	0	$\frac{5}{3}(1+M)$	M	4	z	

From this tableau, we have reached an optimal solution given by

$$z = 4, x_1 = 8/5, x_2 = 6/5 \text{ and } x_3 = 0.$$

Since the nonbasic variable x_3 has a zero coefficient in the objective row R_0 , this indicates that an alternative solution exists. We reincorporate nonbasic variable x_3 into the basis. It follows that:

- The entering variable is x_3 .
- $\min \theta = \min\{3/2, -\} = 3/2$. This implies that the leaving variables is x_2 .
- We find the pivot in the x_3 's column and is given by 4/5. We apply the one-zero method and obtain the following simple tableau:

Simplex tableau 2'

z	x_1	x_2	x_3	a_1	a_2	rhs	BV
0	0	5/4	1	3/4	-1/4	3/2	x_3
0	1	3/4	0	1/4	9/20	5/2	x_1
1	0	0	0	$\frac{5}{3}(1+M)$	M	4	z

Thus the alternative solution is given by

$$z = 4, \quad x_1 = 2/5, \quad x_2 = 0 \quad \text{and} \quad x_3 = 3/2.$$

The general optimal solution is given by $z = 4$ and

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \alpha \begin{bmatrix} 8/5 \\ 6/5 \\ 0 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 5/2 \\ 0 \\ 3/2 \end{bmatrix}, \quad \alpha \in [0, 1] \\ &= \frac{\alpha}{10} \begin{bmatrix} -9 \\ 12 \\ -15 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}, \quad \alpha \in [0, 1]. \end{aligned}$$